Computational Economy Equilibrium and its Application: Progresses on computing Arrow-Debreu-Leontief Competitive Equilibria

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Outlines

- The Arrow-Debreu competitive equilibrium problem.
- A pairing Arrow-Debreu economy with Leontief's utilities.
- Classes of Arrow-Debreu-Leontief equilibrium problems solvable in strongly polynomial time, polynomial time, or FPTAS.
- A trade application of the Arrow-Debreu-Leontief equilibrium.
- More questions and problems

Arrow-Debreu competitive market equilibrium

- Each of a population of n agents has an initial endowment of divisible goods and a non-decreasing utility function on goods. Every agent is able to sell the entire initial endowment and then uses the revenue to buy a bundle of goods such that its utility function is maximized.

- Whether or not equilibrium prices could be set for every good such that this is possible? An affirmative answer was given by Arrow and Debreu in 1954, "Existence of an Equilibrium for a Competitive Economy," Econometrica 22, who showed that such equilibrium would exist if the utility functions were concave under mild conditions.

A pairing exchange market

- Each of n traders brings in 1 unit of a distinct good and is equipped with a utility function on all goods;
- They trade/exchange according to market prices and its own rationality; no production is considered.
- Although restrictive, the pairing model captures all computational difficulties and complexity issues of computational economy/market equilibrium.

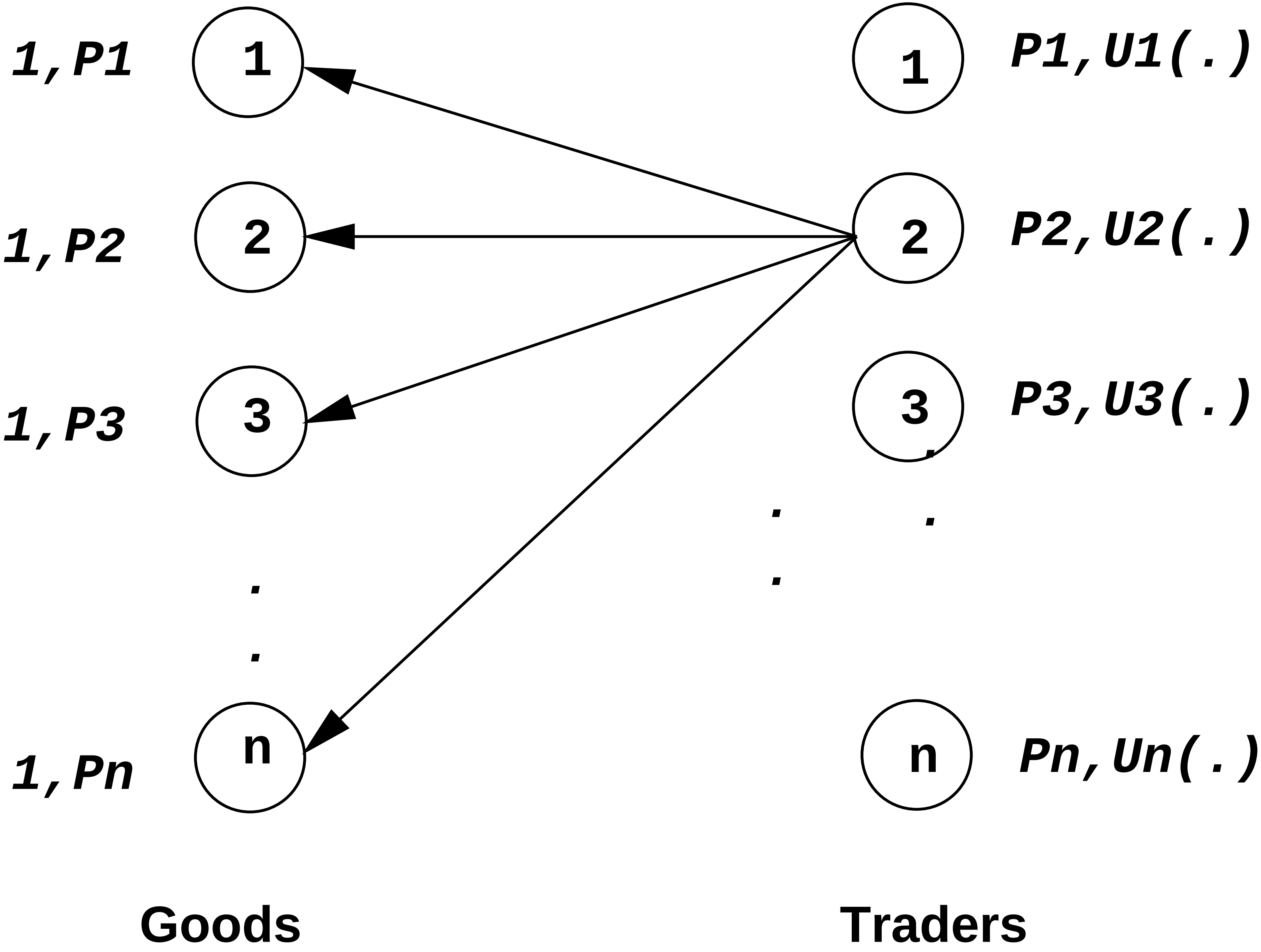

Figure 1: Pairing Exchange Market Model

Computational economy/market equilibrium principles

Let

- p_i be the price for good i, $i = 1, \ldots, n$
- x_{ij} be the amount of good i purchased by trader j

Then, x_{ij}, p_i, $i, j = 1, \ldots, n$, is a market equilibrium if and only if it meets following economic principles.

Market equilibrium principle I

Individual Rationality: For prices p_i, $i = 1, \ldots, n$, and $\mathbf{x}_\cdot$, $\mathbf{x}_j = (x_{1j}, \ldots, x_{nj})$ is a maximal solution to

$$\begin{array}{ll} \text{maximize}_{\mathbf{x}_j} & u^j(\mathbf{x}_j, \bar{\mathbf{x}}_j) \\ \text{subject to} & \sum_i p_i x_{ij} \le p_j, \\ & x_{ij} \ge 0, \quad \forall j; \end{array}$$

where $u^j(\cdot)$ is the utility function of trade j concave in its own decision variable $\mathbf{x}_j$, and externalities $\bar{\mathbf{x}}_j$ represent the purchasing variables of the rest of traders.

Market equilibrium principle II

Physical Constraint: The total purchase volume for good i should not exceed its available physical supply:

$$\sum_j x_{ij} \leq 1; \ \forall i.$$

Or

$$\sum_j \mathbf{x}_j \leq \mathbf{e},$$

where through out this talk, $\mathbf{e}$ is a vector of all ones.

Market equilibrium principle III

Walras Law: Market "Fairness" or "Cruelty"

For every good i,

$$\sum_j x_{ij} < 1 \Rightarrow p_i = 0;$$

so that good i is a "free" good, and this is the only way to clear the market.

The Arrow-Debreu-Leontief economy

Leontief Utility:

$$u^j(\mathbf{x}_j) = \min_i \left\{ \frac{x_{ij}}{a_{ij}} \right\}$$

where a_{ij} represents the demand factor of trader j for good i ($\frac{*}{0} := \infty$).

Let the utility value for trader j be u_j. Then

$$x_{ij} = a_{ij} u_j, \ \forall i.$$

Denote by A the Leontief matrix formed by a_{ij}'s.

Fixed proportion demand on goods

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & \mathbf{a}_{n2} & \dots & a_{nn} \end{pmatrix}$$

Column j: jth trader's good proportion vector.

Given utility value vector $\mathbf{u}$: $A\mathbf{u}$ is the total-demand vector for goods.

Given price value vector $\mathbf{p}$: $A^T\mathbf{p}$ is the unit-cost vector for traders.

Let A have no all-zero column, that is, every trader likes at least one good. Then, does the market has an equilibrium?

The Arrow-Debreu-Leontief equilibrium condition

Since $x_{ij} = a_{ij}u_j$, we must have

- Individual Rationality:

$$p_j = \sum_i p_i x_{ij} = u_j(\mathbf{a}_j^T \mathbf{p}); \text{ or } \quad U^* A^T \mathbf{p} = \mathbf{p}$$

where U is a diagonal matrix whose diagonals are u_js.

- Physical Constraint:

$$\sum_j a_{ij} u_j \leq 1; \text{ or } \quad A\mathbf{u} \leq \mathbf{e}.$$

- Market Fairness: for every good i,
$\sum_j a_{ij} u_j < 1 \Rightarrow p_i = 0$, or $\quad \mathbf{p}^T(\mathbf{e} - A\mathbf{u}) = 0.$

Equilibrium vs quasi-equilibrium

A point (u_j, p_i) satisfying the above three conditions is actually called quasi-equilibrium for the Arrow-Debreu-Leontief competitive economy.

In addition, one needs a no-arbitrage condition, $\mathbf{p}^T\mathbf{a}_j > 0$, for every trader j to make (u_j, p_i) a true equilibrium.

Trader 1: maximize $u_1 := \min\{x_{11}\}$

subject to $p_1 \cdot u_1 \leq p_1;$ and

Trader 2: maximize $u_2 := \min\{x_{12},\ x_{22}\}$

subject to $(p_1 + p_2) \cdot u_2 \leq p_2.$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Here, u_1 is a "self-reliant" trader and u_2 is a dependent trader.

$I: \quad p_1 = 1, \quad p_2 = 0, \quad u_1 = 1, \quad u_2 = 0$; equilibrium.

$II: \quad p_1 = 0, \quad p_2 = 1, \quad u_1 = 0, \quad u_2 = 1$; quasi-equilibrium.

Trader 1: maximize u_1

subject to $0 \cdot u_1 \leq 0;$ and

Trader 2: maximize u_2

subject to $u_2 \leq 1.$

Equilibrium may not exist

While the Arrow-Debreu-Leontief economy always has a quasi-equilibrium, it may not have an equilibrium:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

The only quasi-equilibrium points are $\mathbf{u} = (1,\ 0,\ 0)$ and $\mathbf{u} = (0,\ 1,\ 0)$; and neither of them is an equilibrium.

However, if $A > \mathbf{0}$ or every principal submatrix of A is irreducible, then every quasi-equilibrium is an equilibrium.

Characterization of the Arrow-Debreu-Leontief equilibrium

Most algorithmic research works on the Arrow-Debreu-Leontief economy look for a quasi-equilibrium, which is also difficult to compute, so that we just call it equilibrium.

At an equilibrium $(\mathbf{u}^*, \mathbf{p}^*)$, let the support of $\mathbf{u}^*$ be $B = \text{supp}(\mathbf{u}) = \{j : u_j^* > 0\}$ and the rest be N. Then,

$$\mathbf{u}_B^* > \mathbf{0} \Longrightarrow \mathbf{p}_B^* > \mathbf{0} \Longrightarrow A_{BB}\mathbf{u}_B^* = \mathbf{e},$$

$$\mathbf{u}_N^* = \mathbf{0} \Longrightarrow \mathbf{p}_N^* = \mathbf{0} \Longrightarrow U_B^* A_{BB}^T \mathbf{p}_B^* = \mathbf{p}_B^* > \mathbf{0}.$$

From the physical constraint

$$A_{NB}\mathbf{u}_B^* \leq \mathbf{e}.$$

The Leontief linear complementarity problem

Theorem 1. *(Y 2005) Let* $B \subset \{1, 2, ..., n\}$, $N = \{1, 2, ..., n\} \setminus B$, A_{BB} *be irreducible, and* $\mathbf{u}_B$ *satisfy*

$$A_{BB}\mathbf{u}_B = \mathbf{e}, \quad A_{NB}\mathbf{u}_B \leq \mathbf{e}, \quad \textit{and} \quad \mathbf{u}_B > \mathbf{0}.$$

Then the (right) Perron-Frobenius eigenvector $\mathbf{p}_B$ *of* $U_B A_{BB}^T$ *together with* $\mathbf{u}_B, \mathbf{u}_N = \mathbf{p}_N = 0$ *will be a Arrow-Debreu-Leontief equilibrium; and the converse is also true. Moreover, there is always a rational equilibrium for every such* B, *if the entries of* A *are rational.*

Arrow-Debreu-Leontief equilibrium and LCP

At a Arrow-Debreu-Leontief equilibrium, the utility vector $\mathbf{u}$ is a non-trivial solution of the linear complementarity system (LCP)

$$A\mathbf{u} + \mathbf{v} = \mathbf{e}, \ \mathbf{u}^T\mathbf{v} = 0 \quad \text{or} \quad u_i \cdot v_i = 0 \ \forall i, \ (\mathbf{u} \neq \mathbf{0}, \mathbf{v}) \geq \mathbf{0}.$$

Note that $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = \mathbf{e}$ is a trivial complementary solution.

Is every complementary solution $(\mathbf{u}_B, \mathbf{u}_N = \mathbf{0})$ an equilibrium utility vector? The answer is “no”, since A_{BB} may be reducible so that the price vector $\mathbf{p}_B$ is not strictly positive.

Every complementary solution induces an equilibrium

In the reducible case, let $\mathbf{p}_B$ be any Perron-Frobenius eigenvector with some entries being zeros. Then we must have A^T_{BB} in the reducible form of

$$\begin{pmatrix} A^T_{B'B'} & * \\ \mathbf{0} & A^T_{B''B''} \end{pmatrix} \quad \text{and} \quad U_{B'}A^T_{B'B'}\mathbf{p}_{B'} = \mathbf{p}_{B'} > \mathbf{0}$$

where $B' \subset B$ contains indexes of all positive entries in $\mathbf{p}_B$ and $B'' \subset B$ contains the rest.

Then, simply let $\mathbf{u}'_{B'} = \mathbf{u}_{B'}$ and $\mathbf{u}'_{N'} = \mathbf{0}$ where $N' = N \cup B''$, we have

$$A_{B'B'}\mathbf{u}'_{B'} = \mathbf{e},\ A_{N'B'}\mathbf{u}'_{B'} \le A_{N'B}\mathbf{u}_B \le \mathbf{e},\ \text{and}\ \mathbf{u}'_{B'} > \mathbf{0},\ \mathbf{u}'_{N'} = \mathbf{0}.$$

so that $(\mathbf{u}'_{B'}, \mathbf{u}'_{N'} = \mathbf{0})$ is an equilibrium utility vector.

Theorem 2. *Every non-trivial complementary solution to the LCP induces an equilibrium utility vector whose support is a subset of the original support.*

Note that finding a Perron-Frobenius eigenvector is to solve a system of homogeneous linear equations (one can set a price entry to 1 so that the system becomes non-homogeneous).

Thus, the major computational work of finding an Arrow-Debreu-Leontief equilibrium is to compute a complementary utility solution.

Relation to the Nash bimatrix game

Theorem 3. *(Codenotti, Saberi, Varadarajan and Y 2005) Let (P, Q) denote an arbitrary bimatrix game payoff matrix pair. Let*

$$A = \begin{pmatrix} \mathbf{0} & P \\ Q^T & \mathbf{0} \end{pmatrix}.$$

Then, there is a one-to-one correspondence between the Nash equilibria of the game (P, Q) and the market equilibria of the Arrow-Debreu-Leontief economy described by Leontief matrix A.

Hardness results

- It's NP-Hard to decide whether or not it has a true equilibrium (Codenotti et al. 2005). In addition, the following problems are NP-hard:

  1. Is there more than one equilibrium?
  2. Is there an equilibrium where at least k goods are positively priced?
  3. Is there an equilibrium where at most k goods are positively priced?

- Computing an exact equilibrium is PPAD hard (Chen and Deng 2005, Daskalakis, P. Goldberg, C. Papadimitriou 2005, Codenotti et al. 2005).
- Computing an approximate equilibrium is also PPAD hard (Chen, Deng and Teng 2006, Huang and Teng 2007).

We present a few positive results in the following.

Block triangular matrix

$$A = \begin{pmatrix} A_1 & * & \dots & * \\ \mathbf{0} & A_2 & \dots & * \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & A_k \end{pmatrix}$$

and block A_1 has a dimension no more than k which is fixed.

One can find an equilibrium by ignoring all other blocks but A_1, which is an absorbing or isolated block. This can be done by enumerating all possible LCP solutions in A_1 in strongly polynomial time.

Polynomial time algorithms

- Algorithm for computing an ϵ approximate bimatrix game equilibrium polynomial in $1/\epsilon$ if n is fixed, and quasi-polynomial in n if ϵ is fixed (Lipton, Markakis, Mehta 2003). This leads to a quasi-polynomial algorithm for computing an ϵ-equilibrium and a polynomial time algorithm to compute an exact equilibrium if the rank of the payoff matrices is fixed.
- Polynomial time for computing an ϵ approximate bimatrix game equilibrium when $P+Q$ has a fixed rank k (Kannan and Thorsten 2007).
- Polynomial time algorithms for computing a constant approximation equilibrium (e.g., Tsaknakis and Spirakis 2007).

- Polynomial time algorithm for computing an "exact" equilibrium with fixed number of goods or traders in the non-pairing Arrow-Debreu-Leontief economy by searching through the fixed dimension price or utility vector (Devanur and Kannan 2008).

Most of these exact/approximation algorithms employ linear programming as a subroutine and prove that the total number of linear programs need to be solved is polynomial in dimensions and/or $1/\epsilon$, which lead to an overall polynomial time algorithm.

Leontief matrix with fixed rank

That is, A has a rank no more than k which is fixed.

Theorem 4. *(Basic Equilibrium Theorem) Let the Leontief matrix of an n trader game have rank k. Then, there exists an Arrow-Debreu-Leontief economy equilibrium where the size of support of utility vector $\mathbf{u}$, that is, the number of positive entries in $\mathbf{u}$, is no more than k. Moreover, such an exact equilibrium, both utility and price vectors, can be computed in strongly polynomial time $O(n^{k(k-1)}nk^2)$ arithmetic operations.*

Sketch of proof

Let $\mathbf{u}$ be a non-trivial LCP solution, that is,

$$A\mathbf{u} \le \mathbf{e},\ u_i \cdot (\mathbf{e} - A\mathbf{u})_i = 0\ \forall i,\ \mathbf{u} \ge \mathbf{0}.$$

Hence

$$(A\mathbf{u})_i = 1\ \forall i \in \text{supp}(\mathbf{u}).$$

Then, one can use Carathéodorys theorem to find a basic LCP solution $\bar{\mathbf{u}}$ such that

$$A\bar{\mathbf{u}} = A\mathbf{u} \le \mathbf{e},$$

and at most k entries in $\bar{\mathbf{u}}$ are positive and all from the support of $\mathbf{u}$, and the columns of A associated with positive entries in $\bar{\mathbf{u}}$ are linearly independent. Let $\bar{\mathbf{u}}_B > \mathbf{0}$ and the rest of them be 0. Then, we have $B \subset \text{supp}(\mathbf{u})$ so that $(A\bar{\mathbf{u}})_i = 1$ for all $i \in B$. Therefore

$\bar{\mathbf{u}}$ remains an non-trivial LCP solution. Thus, our existence result follows from Theorem 2, that is, $\bar{\mathbf{u}}$ induces an equilibrium utility vector whose support is a subset of B.

We now turn our attention to compute such a sparse equilibrium. Our algorithm is based on enumerating. First, we select an $1 \le k' \le k$ linear independent columns indexed by B from A, i.e. $A_{.B}$, where A_{BB} is irreducible and the rank of $[A_{BB}, \mathbf{e}]$ is as the same as that of A_{BB}. There are at most

$$\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{k} = O(n^k)$$

many of them.

Then we try to find a solution to

$$A_{BB}\mathbf{u}_B = \mathbf{e}, \quad A_{NB}\mathbf{u}_B \le \mathbf{e}, \quad u_B > \mathbf{0}, \qquad (1)$$

or prove no such solution exists. This can be typically answered using linear programming in polynomial time. However, we can do better. Consider the linear system

$$A_{BB}\mathbf{u}_B = \mathbf{e}, \quad A_{NB}\mathbf{u}_B \le \mathbf{e}, \quad \mathbf{u}_B \ge \mathbf{0}.$$

which has $n + k'$ constraints with k' variables. If feasible, this is a polytope with its vertex given by a basic feasible solution of the system, where a basis contains all linearly independent rows of A_{BB} and the rest from the n inequalities $A_{NB}\mathbf{u}_B \le \mathbf{e}$ and $\mathbf{u}_B \ge \mathbf{0}$. We can find all basic feasible solution by enumerating all basic solutions, and there are at most, again,

$$\binom{n}{k-1} \le n^{k-1}$$

many of them. If no basic solution is feasible, then (1) is infeasible;

otherwise, take the average of all basic feasible solutions, denoted by $\hat{\mathbf{u}}_B$, and check if $\hat{\mathbf{u}}_B > \mathbf{0}$. If not, again (1) is infeasible; otherwise, $\hat{\mathbf{u}}$ is a Arrow-Debreu-Leontief equilibrium utility vector from Theorem 1.

Overall, from the existence part of Theorem 4, a (sparse) Arrow-Debreu-Leontief equilibrium utility vector can be found in $n^{k(k-1)}nk^2$ arithmetic operations, where nk^2 is the arithmetic operation work for checking the linear independency of an $n \times k$ matrix.

FPTAS for symmetric Leontief matrix

That is $A = A^T$: "the demand factor of me from you is as the same as the demand factor of you from me."

Theorem 5. *(Dang, Y, and Zhu 2008) Let A be a real symmetric matrix. Then, it is NP-complete to decide whether or not the LCP has a complementary solution such that $\mathbf{u} \neq \mathbf{0}$.*

The question remains: given symmetric A, is it easy to compute one if the LCP is known to have a complementary solution?

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Three isolated non-trivial complementary solutions.

$$\mathbf{u}^1 = (1/2;\ 0), \quad \mathbf{u}^2 = (0;\ 1/2), \quad \mathbf{u}^3 = (1/3;\ 1/3).$$

A social utility maximization

In the following, we assume that $\mathbf{e}^T A\mathbf{e} > 0$.

We consider a quadratic "social" utility function $\mathbf{u}^T A\mathbf{u}$, and the social maximization problem

$$\max \quad \mathbf{u}^T A\mathbf{u} \quad \text{subject to} \quad \mathbf{e}^T\mathbf{u} = 1,\ \mathbf{u} \geq \mathbf{0}.$$

Theorem 6. *(Dang, Y, and Zhu 2008) Every KKT point of the social maximization problem is a non-trivial complementarity solution (upon to scaling) to the LCP.*

What is the computational complexity to compute such an KKT point? An answer is given based on Y (1998) "On The Complexity of Approximating a KKT Point of Quadratic Programming"

Theorem 7. *(Dang, Y, and Zhu 2008) There is a FPTAS to compute an ϵ-approximate non-trivial complementary solution when A is symmetric and $\mathbf{e}^T A\mathbf{e} > 0$ in $\mathcal{O}(n(\frac{1}{\epsilon})\log(\frac{1}{\epsilon}))$ iterations, and each iteration uses $\mathcal{O}(n^3\log(\log(\frac{1}{\epsilon})))$ arithmetic operations.*

General Leontief matrix?

In this case, even all entries of A being non-negative may not guarantee the existence of a non-trivial complementary solution:

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}.$$

However, we have

Corollary 1. *The LCP always has a non-trivial complementary solution if A has no all-zero column.*

An homotopy interior-point path

Let α be a randomly generated small perturbation vector, and consider

$$\begin{aligned} UA^T\mathbf{p} &= \mu\cdot\mathbf{p}, \\ V\mathbf{p} &= (1-\mu)\cdot\mathbf{e}, \\ A\mathbf{u}+\mathbf{v} &= \mathbf{e}+\mu(1-\mu)\cdot\alpha, \\ (\mathbf{u},\mathbf{v},\mathbf{p}) &\geq \mathbf{0},\ \mathbf{e}^T\mathbf{p}=n. \end{aligned}$$

This system is feasible for any $0 \leq \mu < 1$, and in particular, $\mathbf{u}=\mathbf{0}, \mathbf{v}=\mathbf{p}=\mathbf{e}$ is the unique solution at $\mu = 0$. When $\mu = 1$, its solution is an equilibrium.

A path-following method

Together with Sard's Theorem, one can show (Dang, Y, and Zhu 2008)

Theorem 8. *There exists a unique (interior-point) continuously differentiable path for almost all* α *sufficiently small, which starts from the unique solution* $(\mathbf{0}, \mathrm{e}, \mathrm{e})$ *at* $\mu = 0$ *and leads to an equilibrium at* $\mu = 1$.

We will report preliminary computational experience of the algorithm later.

A Trading Policy Application

(Carlsson, Eberhart, and Y 2008, in preparation)

Let X be a trade volume matrix among n traders where x_{ij} is the amount of good went from trader i to trader j. Then at the equilibrium we have

$$X\mathbf{e} = \mathbf{w}, \quad \text{physical balance}$$

where w_i is the amount good produced by trader i; and

$$X^T\mathbf{p} = P\mathbf{w}, \quad \text{price balance,}$$

where again P is a diagonal matrix whose diagonals are p_is.

Trade among countries

If this is a global trade among n countries, w_i is the amount of aggregate goods produced by country i measured in country i's currency, x_{ij} is the export from country i to country j measured in country i's currency, and $\mathbf{p}$ would be the currency exchange rate to a "global currency".

We could normalize $\mathbf{p}$ such that one country has the rate 1, say dollars, so that it's the global currency. Thus, p_i would be the amount of dollars that one unit of country i's currency can exchange.

A Arrow-Debreu-Leontief economy for global trade

From $X\mathbf{e} = \mathbf{w}$, we can write it as

$$XP^{-1}\mathbf{p} = \mathbf{w}.$$

Thus, $X, \mathbf{p}$ can be viewed as a Arrow-Debreu-Leontief competitive economy equilibrium with the Leontief matrix

$$A = XP^{-1} \quad \text{with the utility vector} \quad \mathbf{u} = \mathbf{p}.$$

There is a justification using the Arrow-Debreu-Leontief competitive economy to model the global trade.

1.0247e+015	187.2400e+009	9.4201e+015	1.3174e+012
1.4497e+012	1.4974e+012	194.8602e+012	72.6610e+009
615.1801e+012	1.5671e+012	6.9719e+018	16.0238e+012
2.7268e+012	24.3430e+009	769.7808e+012	11.2618e+012

Table 1: 2005 trading proportion (A) among China, Germany, Japan and USA

1.0981e+015	181.0792e+009	10.1619e+015	1.5932e+012
1.6026e+012	1.1111e+012	187.3791e+012	74.2923e+009
675.2943e+012	1.3973e+012	6.8338e+018	17.5329e+012
3.3749e+012	23.7370e+009	846.2455e+012	11.9016e+012

Table 2: 2006 trading proportion (A) among China, Germany, Japan and USA

1.1212e+015	164.4630e+009	9.4177e+015	1.7049e+012
1.4940e+012	913.7855e+009	153.6213e+012	68.3210e+009
656.7475e+012	1.1739e+012	5.6723e+018	16.3609e+012
3.4900e+012	22.8998e+009	790.9735e+012	12.4689e+012

Table 3: 2007 trading proportion (A) among China, Germany, Japan and USA

Bilateral trade balance policy

If one needs to maintain the bilateral trade balance policy:

$$x_{ij}p_j = x_{ji}p_i \quad \text{or} \quad \frac{x_{ij}}{p_i} = \frac{x_{ji}}{p_i},$$

Then the Leontief matrix A will be symmetric!

Due to the advance of computational equilibrium algorithms, we are now able to see the equilibrium structure difference when A is symmetric or general.

In particular, we see the difference on the support size of the equilibrium utility vector–the number of traders can benefit from the trade market.

Preliminary computational results

We have applied the path-following algorithm to compute Arrow-Debreu-Leontief equilibria for randomly generate uniform sparse matrix A from two cases: symmetric and non-symmetric.

For each size, we general 10 random problems and record the mean support size, and the maximal support size among the 10 problems.

We observe a significant size difference between the two cases, which indicate that the bilateral balance or symmetric trade policy leads to a much smaller support size, that is, much fewer traders can benefit from the trade market.

n	mean_iter	mean_time	mean_sup	max_sup
100	48.2	0.3	5.3	7
200	53.5	1.2	5.5	6
400	55.1	5.9	5.7	7
800	62.6	33.8	5.8	8
1000	65.0	60.2	6.3	7
1500	71.5	187.2	6.1	8
2000	73.5	411.9	5.9	7
2500	74.6	774.5	6.4	8
3000	78.7	1404.2	6.2	8

Table 4: The Arrow-Debreu-Leontief equilibrium for symmetric uniform matrix

n	mean_iter	mean_time	mean_sup	max_sup
100	149.7	4.0	11.4	22
200	260.4	29.5	20.0	33
300	319.6	99.0	26.5	40
400	398.1	242.2	33.2	55
500	456.4	446.2	40.8	59
600	685.7	999.1	66.0	84
700	603.2	1207.7	75.8	91
800	745.0	2759.5	80.0	109
900	1058.3	3459.5	92.0	129
1000	897.8	4900.8	97.4	134

Table 5: The Arrow-Debreu-Leontief equilibrium for un-symmetric uniform matrix

Conclusions and Challenges

- The pairing Arrow-Debreu-Leontief competitive economy model captures most computational complexity issues for computational economy/market equilibrium, and also provides interesting applications.
- One can embed A into a low rank, $\log(n)/\epsilon^2$, matrix (Johnson and Lindenstrauss 1884)?
- Is there a PTAS to compute an approximate Arrow-Debreu-Leontief equilibrium?
- Look for more applications to show the value of being able to compute equilibria and/or to illustrate equilibrium structures.