A Note on Equilibrium Pricing as Convex Optimization

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Abstract. We study equilibrium computation for exchange markets. We show that the market equilibrium of either of the following two markets:

1. The Fisher market with several classes of concave non-homogeneous utility functions;
2. A mixed Fisher and Arrow-Debreu market with homogeneous and log-concave utility functions

can be computed as convex programming and by interior-point algorithms in polynomial time.

1 Introduction

The study of market equilibria occupies a central place in mathematical economics. This study was formally started by Walras [12] over a hundred years ago. In this problem everyone in a population of n players has an initial endowment of divisible goods and a utility function for consuming all goods—their own and others. Every player sells the entire initial endowment and then uses the revenue to buy a bundle of goods such that his or her utility function is maximized. Walras asked whether prices could be set for everyone's good such that this is possible. An answer was given by Arrow and Debreu in 1954 [1] who showed that such an equilibrium would exist, under very mild conditions, if the utility functions were concave. Their proof was non-constructive and did not offer any algorithm to find such equilibrium prices.

Fisher was the first to consider an algorithm to compute equilibrium prices for a related and different model where players are divided into two sets: producers and consumers; see Brainard and Scarf [2, 11]. Consumers spend money only to buy goods and maximize their individual utility functions of goods; producers sell their goods only for money. The price equilibrium is an assignment of prices to goods so that when every consumer buys a maximal bundle of goods then the market clears, meaning that all the money is spent and all the goods are sold.

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Fisher's model is a special case of Walras' model when money is also considered a good so that Arrow and Debreu's result applies.

In a remarkable piece of work, Eisenberg and Gale [6, 9] give a convex programming (or optimization) formulation whose solution yields equilibrium allocations for the Fisher market with linear utility functions, and Eisenberg [7] extended this approach to derive a convex program for general concave and homogeneous functions of degree 1. Their program consists of maximizing an aggregate utility function of all consumers over a convex polyhedron defined by supply-demand linear constraints. The Lagrange or dual multipliers of these constraints yield equilibrium prices. Thus, finding a Fisher equilibrium becomes solving a convex optimization problem, and it could be computed by the Ellipsoid method or by efficient interior-point methods in polynomial time. Later, Codenotti et al. [4] rediscovered the convex programming formulation, and Jain et al. [10] generalized Eisenberg and Gale's convex model to handling homothetic and quasi-concave utilities introduced by Friedman [8]. Here, polynomial time means that one can compute an ϵ approximate equilibrium in a number of arithmetic operations bounded by polynomial in n and $\log \frac{1}{\epsilon}$; or, if there is a rational equilibrium solution, one can compute an *exact* equilibrium in a number of arithmetic operations bounded by polynomial in n and L, where L is the bit-length of the input data. When the utility functions are linear, the current best arithmetic operations complexity bound is $O(\sqrt{mn}(m+n)^3 L)$ given by [13].

Little is known on the computational complexity for computing market equilibria with non-homogeneous utility functions and for markets other than the Fisher and Arrow-Debreu settings. This note is to derive convex programs to solve several more general exchange market equilibrium problems. We show that the equilibrium of either of the following two markets:

1. The Fisher market with several classes of concave non-homogeneous utility functions;
2. A mixed Fisher and Arrow-Debreu market with homogeneous and log-concave utility functions

can be computed as convex programming and by interior-point algorithms in polynomial time.

First, a few mathematical notations. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space; $\mathbb{R}^n_+$ denote the subset of $\mathbb{R}^n$ where each coordinate is non-negative. $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real numbers and the set of non-negative real numbers, respectively.

A function $u : \mathbb{R}^n_+ \to \mathbb{R}_+$ is said to be *concave* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_+$ and any $0 \le \alpha \le 1$, we have $u(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \ge \alpha u(\boldsymbol{x}) + (1-\alpha) u(\boldsymbol{y})$. It is *homothetic* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_+$ and any $\alpha > 0$, $u(\boldsymbol{x}) \ge u(\boldsymbol{y})$ iff $u(\alpha \boldsymbol{x}) \ge u(\alpha \boldsymbol{y})$. It is *monotone increasing* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_+$, $\boldsymbol{x} \ge \boldsymbol{y}$ implies that $u(\boldsymbol{x}) \ge u(\boldsymbol{y})$. It is *homogeneous* of degree d if for any $\boldsymbol{x} \in \mathbb{R}^n_+$ and any $\alpha > 0$, $u(\alpha \boldsymbol{x}) = \alpha^d u(\boldsymbol{x})$.

2 Convex optimization for the Fisher market with non-homogeneous utilities

Without loss of generality, assume that there is one unit of good for each type of good $j \in P$ with $|P| = n$. Let consumer $i \in C$ (with $|C| = m$) have an initial money endowment $w_i > 0$ to spend and buy goods to maximize his or her utility function for a given price vector $\boldsymbol{p} \in \mathbb{R}^n_+$:

$$\begin{array}{lll} \text{maximize} & u_i(\boldsymbol{x}_i) & \\ \text{subject to} & \boldsymbol{p}^T \boldsymbol{x}_i & \leq w_i \\ & \boldsymbol{x}_i & \geq \mathbf{0}; \end{array} \tag{1}$$

where variable $\boldsymbol{x}_i = (x_{i1}; ...; x_{in})$ is a column vector whose jth coordinates x_{ij} represents the amount of goods bought from producer j by consumer i, $j = 1, ..., n$. Let $u_i(\boldsymbol{x}_i)$ be concave and monotonically increasing. We also assume that every consumer is interested in buying at least one type of good and every type of good is sought by at least one consumer. Then, a price vector $\boldsymbol{p} \geq \mathbf{0}$, together with vectors $\boldsymbol{x}_i$, $i = 1, ..., m$ is called a Fisher equilibrium if $\boldsymbol{x}_i$ is optimal for (1) for the given $\boldsymbol{p}$, and $\sum_i \boldsymbol{x}_i = \boldsymbol{e}$ (the vector of all ones). The last condition requires that all the goods of the producers are sold.

2.1 Homogeneous and log-concave utilities

If $u_i(\boldsymbol{x}_i)$ is homogeneous of degree 1 (this is without loss of generality since any homogeneous function with a positive degree can be monotonically transformed to a homogeneous function with degree 1) and $\log(u_i(\boldsymbol{x}_i))$ is concave in $\boldsymbol{x}_i \in \mathbb{R}^n_+$, the Fisher equilibrium problem can be solved as an aggregate social convex optimization problem (see Eisenberg and Gale [6, 9, 7]):

$$\begin{array}{lll} \text{maximize} & \sum_i w_i \log(u_i(\boldsymbol{x}_i)) & \\ \text{subject to} & \sum_i \boldsymbol{x}_i & = \boldsymbol{e}, \ \forall j, \\ & \boldsymbol{x}_i & \geq \mathbf{0}, \ \forall i; \end{array} \tag{2}$$

where the objective function may be interpreted as a socially aggregated utility.

These homogeneous and log-concave functions include many classical utilities:

– All constant elasticity functions

$$u_i(\boldsymbol{x}) = \left(\sum_{j=1}^{n} (a_j x_j)^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)}, \ a_j \geq 0, \ 0 < \sigma < \infty;$$

– Piece-wise concave linear function

$$u_i(\boldsymbol{x}) = \min_k \{(\boldsymbol{a}^k)^T \boldsymbol{x}\}, \ \boldsymbol{a}^k \geq \mathbf{0}, \ k = 1, ..., K;$$

– The Cobb-Douglass utility function

$$u_i(\boldsymbol{x}) = \prod_{j=1}^{n} x_j^{a_j},\ a_j \geq 0.$$

Jain et al. [10] showed how to transform a homothetic utility function into an equivalent homogeneous degree 1 and log-concave function. Thus, the Fisher equilibrium problem with homothetic utilities can be also solved as a convex optimization problem. A natural question arises: Does this approach apply to more general *non-homogeneous* utility functions?

2.2 Necessary and sufficient condition for a Fisher equilibrium

Consider the optimality conditions of (1). Besides feasibility, they are

$$\begin{aligned}(\nabla u_i(\boldsymbol{x}_i)^T\boldsymbol{x}_i)\cdot\boldsymbol{p} &\geq w_i\cdot\nabla u_i(\boldsymbol{x}_i),\\ \boldsymbol{p}^T x_i &= w_i,\\ x_i &\geq 0,\end{aligned} \tag{3}$$

where $\nabla u(\boldsymbol{x})$ denotes any sub-gradient vector of $u(\boldsymbol{x})$ at $\boldsymbol{x}$.

Thus, the complete necessary and sufficient conditions for a Fisher equilibrium are the following:

$$\begin{aligned}(\nabla u_i(\boldsymbol{x}_i)^T\boldsymbol{x}_i)\cdot\boldsymbol{p} &\geq w_i\cdot\nabla u_i(\boldsymbol{x}_i),\ \forall i\\ \boldsymbol{p}^T\boldsymbol{x}_i &= w_i,\\ \textstyle\sum_i \boldsymbol{x}_i &\leq \boldsymbol{e},\\ \boldsymbol{p}^T\boldsymbol{e} &\leq \textstyle\sum_i w_i,\\ \boldsymbol{x}_i,\boldsymbol{p} &\geq \boldsymbol{0},\ \forall i.\end{aligned} \tag{4}$$

Note here that the condition $\boldsymbol{p}^T\boldsymbol{x}_i = w_i$ should be implied by the rest of conditions in (4): Multiplying $\boldsymbol{x}_i \geq \boldsymbol{0}$ to both sides of the first inequality in (4), we have $\boldsymbol{p}^T\boldsymbol{x}_i \geq w_i$ for all i, which, together with other inequality conditions in (4), imply

$$\sum_i w_i \geq \boldsymbol{p}^T\boldsymbol{e} \geq \boldsymbol{p}^T\left(\sum_i \boldsymbol{x}_i\right) = \sum_i \boldsymbol{p}^T\boldsymbol{x}_i \geq \sum_i w_i,$$

that is, every inequality in the sequence must be tight which implies $\boldsymbol{p}^T\boldsymbol{x}_i = w_i$ for all i. Thus, the reduced necessary and sufficient Fisher equilibrium conditions become

$$\begin{aligned}(\nabla u_i(\boldsymbol{x}_i)^T\boldsymbol{x}_i)\cdot\boldsymbol{p} &\geq w_i\cdot\nabla u_i(\boldsymbol{x}_i),\ \forall i\\ \textstyle\sum_i \boldsymbol{x}_i &\leq \boldsymbol{e},\\ \boldsymbol{p}^T\boldsymbol{e} &\leq \textstyle\sum_i w_i,\\ \boldsymbol{x}_i,\boldsymbol{p} &\geq \boldsymbol{0},\ \forall i.\end{aligned} \tag{5}$$

The inequalities and equalities in (5) are all linear, except the first

$$(\nabla u_i(\boldsymbol{x}_i)^T\boldsymbol{x}_i)\cdot\boldsymbol{p} \geq w_i\cdot\nabla u_i(\boldsymbol{x}_i).$$

An immediate observation is, if every consumer i is interested in exactly one type of good, that is, $u_i(x_i)$ is a univariate concave function $u_i(x_{i\bar{j}_i})$ for some $\bar{j}_i \in P$, then the above condition becomes a single inequality:

$$(u_i'(x_{i\bar{j}_i}) \cdot x_{i\bar{j}_i}) \cdot p_{\bar{j}_i} \geq w_i \cdot u_i'(x_{i\bar{j}_i}),$$

or simply

$$x_{i\bar{j}_i} \cdot p_{\bar{j}_i} \geq w_i.$$

One can transfer this non-linear inequality to

$$\log(x_{i\bar{j}_i}) + \log(p_{\bar{j}_i}) \geq \log(w_i)$$

which is a convex inequality (meaning that the set of feasible solutions is convex). Thus, the Fisher equilibrium set is convex and can be found by solving a convex optimization problem. It turns out that this simple trick works for other utilities as well, as we shall present in the next subsection.

2.3 Concave and non-homogeneous utilities

Consider $u_i(\boldsymbol{x}_i)$ in the following additive or separable form:

$$\begin{array}{c} u_i(\boldsymbol{x}_i) = \sum_{j=1}^n a_{ij}(x_{ij} + b_{ij})^{d_{ij}}, \\ \text{or} \\ u_i(\boldsymbol{x}_i) = \sum_{j=1}^n a_{ij} \log(x_{ij} + b_{ij}), \end{array} \tag{6}$$

where $a_{ij}, b_{ij} \geq 0$, and $0 < d_{ij} \leq 1$, for all i and j, are given, and variable x_{ij} represents the amount of goods bought from good j by consumer i, $j = 1, ..., n$. One can see that $u_i(\boldsymbol{x}_i)$ is a concave and monotone increasing function in $\boldsymbol{x}_i = (x_{i1}; ...; x_{in}) \geq \mathbf{0}$.

This utility function (6) includes as special case several popular utilities:

- linear utility functions: $d_{ij} = 1$ for all j in the first form;
- certain constant elasticity functions: $b_{ij} = 0$ and $d_{ij} = d$, $0 \leq d \leq 1$, for all j in the first form;
- the Cobb-Douglass utility function: $b_{ij} = 0$ in the second form;
- a non-homogeneous Cobb-Douglass utility functions given by [3]: the second form.

Note that $u_i(\boldsymbol{x}_i)$ (6),can be non-homothetic; see, for example, $u(x, y) = \sqrt{x} + y$. Chen at al. [3] developed approximation algorithm with running time polynomial in n and $\frac{1}{\epsilon}$ for the utility function in the second form of (6).

Lemma 1. *Given $u_i(\boldsymbol{x}_i)$ in the forms of (6), $(\nabla u_i(\boldsymbol{x}_i)^T \boldsymbol{x}_i)$ is concave, and $\log(\nabla_j u_i(\boldsymbol{x}_i))$ is convex for every j, in $\boldsymbol{x}_i \in \mathbb{R}^n_+$.*

Proof. For simplicity, let us omit index i, so that

$$u(\boldsymbol{x}) = \sum_{j=1}^{n} a_j (x_j + b_j)^{d_j}$$

or

$$u(\boldsymbol{x}) = \sum_{j=1}^{n} a_j \log(x_j + b_j).$$

Thus, for the first form

$$\nabla u(\boldsymbol{x}) = (..., a_j d_j (x_j + b_j)^{d_j - 1}, ...),$$

so that

$$\nabla u(\boldsymbol{x})^T \boldsymbol{x} = \sum_j a_j d_j (x_j + b_j)^{d_j - 1} x_j.$$

It is easily see that each $(x_j + b_j)^{d_j - 1} x_j$ is concave in $x_j \geq 0$ since $0 \leq d_j \leq 1$; therefore, so is the sum: $\sum_j a_j d_j (x_j + b_j)^{d_j - 1} x_j$.

Furhtermore,

$$\log(\nabla_j u(\boldsymbol{x})) = (d_j - 1) \log(x_j + b_j) + \log(a_j d_j)$$

which is convex in $x_j > 0$ for every j.

Similarly, one can prove the lemma for the the second form. This completes the proof. □

Thus, one can rewrite the nonlinear inequality in (5) as

$$\log(\nabla u_i(\boldsymbol{x}_i)^T \boldsymbol{x}_i) + \log(\boldsymbol{p}_j) \geq \log(w_i) + \log(\nabla_j u_i(\boldsymbol{x}_i)), \ \forall j,$$

which is a convex inequality (the set of feasible solutions is convex) by Lemma 1. Thus,

Theorem 1. *If utilities $u_i(\boldsymbol{x}_i)$ are given in the forms of (6), then the Fisher equilibrium set of (5) is convex and can be computed as a convex optimization problem; for example, by using polynomial-time interior-point methods.*

3 Convex optimization for the Fisher market where consumers may retain money

In the classical Fisher market, consumers spend money only to buy goods and maximize their individual utility functions of goods; producers sell their goods only for money. Now consider a market where each consumer can retain certain amount of money from his or her own budget, that is, his or her utility includes the amount of retained money:

$$\begin{array}{ll} \text{maximize} & u_i(\boldsymbol{x}_i, s_i) \\ \text{subject to} & \boldsymbol{p}^T \boldsymbol{x}_i + s_i \leq w_i \\ & \boldsymbol{x}_i, s_i \quad \geq \boldsymbol{0}, \end{array} \tag{7}$$

where again $\boldsymbol{x}_i = (x_{i1}; ...; x_{in})$ and its jth component x_{ij} represents the amount of good j bought by consumer i, and s_i denotes the retained money (e.g., deposited in a bank for a short-time interest gain). We assume that $u_i(\boldsymbol{x}_i, s_i)$ is a monotone increasing and concave function of $(\boldsymbol{x}_i, s_i) \geq \mathbf{0}$. This mixed market has a number of applications in managing supply chains and resource allocations.

3.1 The mixed market Equilibrium

In this mixed market, an equilibrium is defined as a non-negative price vector $\boldsymbol{p} \in \mathbb{R}^n_+$ at which there exist a bundle of goods $(\boldsymbol{x}_i \in \mathbb{R}^n_+, s_i \geq 0)$ for each consumer $i \in C$ such that the following conditions hold:

1. The vector $(\boldsymbol{x}_i; s_i)$ optimizes retailer i's utility (7) given her money budget w_i.
2. For each good j, the total amount available equals the total amount consumed by the consumers, that is, $\sum_{i \in C} x_{ij} = 1$.
3. The sum of the spending and retaining money equals the sum of the money possessed by all consumers, that is, $\sum_{j \in P} p_j + \sum_{i \in C} s_i = \sum_{i \in C} w_i$.

The existence of such an equilibrium is immediately implied by the existence of an Arrow-Debreu equilibrium by treating money as an additional "good". One may attempt to prove the existence using the Fisher equilibrium model. However, in such a Fisher equilibrium model the price for the money "good" (s_i) has to be fixed to 1 (the same as w_i), which is difficult to enforce. Thus, we need to invoke the Arrow-Debreu model by assigning price p_{n+1} to a unit of money. Then, each consumer's problem becomes

$$\begin{array}{ll} \text{maximize} & u_i(\boldsymbol{x}_i, s_i) \\ \text{subject to} & \boldsymbol{p}^T \boldsymbol{x}_i + p_{n+1} s_i \leq p_{n+1} w_i, \\ & \boldsymbol{x}_i, s_i \qquad \geq \mathbf{0}, \end{array}$$

where the total supply of money is $\sum_i w_i$. Therefore, the Arrow-Debreu theorem implies that an equilibrium price vector $(\boldsymbol{p}; p_{n+1}) \in \mathbb{R}^{n+1}_+$ exists. In particular, $p_{n+1} > 0$ at every Arrow-Debreu equilibrium since money has a value at least to every producer. By dividing $(\boldsymbol{p}; p_{n+1})$ by p_{n+1}, we have an equilibrium price for all goods, and the price for the money "good" equals 1:

Corollary 1. *An equilibrium always exists for the Fisher market where consumers may retain money.*

However, it was unknown if the mixed market admits a convex program for computing its equilibrium, or it has to use the more difficult Arrow-Debreu equilibrium framework to compute it, even the utility is homogeneous and log-concave. The computational complexity issue of the mixed market equilibrium problem is important, since there is a fundamental difference between the Fisher and Arrow-Debreu models with respect to computational complexity. For example, when the utility is Leontief

$$u_i(\boldsymbol{x}) = \min \left\{ \frac{x_j}{a_j} : \ a_j > 0 \right\},$$

a homogeneous of degree one and log-concave function, the Fisher market equilibrium can be computed as a convex program in polynomial time while the Arrow-Debreu market equilibrium is NP-hard to decide; see Ye [14] and Codenotti et al. [5].

We settle the computational complexity issue of the mixed market equilibrium problem in the next subsection by showing that any optimal solution to a convex program yields an equilibrium if the utility functions are log-concave and homogeneous of degree one.

3.2 Convex optimization for computing an equilibrium

From (4), the necessary and sufficient conditions for the mixed market equilibrium are

$$\begin{array}{rl} (\boldsymbol{p};\ 1) \geq \frac{w_i}{\nabla u_i(\boldsymbol{x}_i,s_i)^T(x_i;s_i)} \cdot \nabla u_i(\boldsymbol{x}_i,s_i),\ \forall i \\ \sum_i \boldsymbol{x}_i \leq \boldsymbol{e}, \\ \sum_j p_j + \sum_i s_i \leq \sum_i w_i, \\ \boldsymbol{x}_i, \boldsymbol{p} \geq \boldsymbol{0},\ \forall i; \end{array} \tag{8}$$

where one can see that the price for the money good is set to 1.

Let $u_i(\boldsymbol{x}_i, s_i)$ be homogeneous of degree one and $\log(u_i(\boldsymbol{x}_i, s_i))$ be concave in $(\boldsymbol{x}_i; s)i) \in \mathbb{R}^{n+1}_+$. Recall that this function includes all constant elasticity, piecewise concave linear, the Cobb-Douglass utility, and the Leontief utility functions. Now consider the convex optimization problem

$$\begin{array}{lll} \text{maximize} & \sum_i w_i \log(u_i(\boldsymbol{x}_i, s_i)) - s & \\ \text{subject to} & \sum_i x_{ij} & \leq 1,\ \forall j, \\ & \sum_i s_i - s & = 0, \\ & (\boldsymbol{x}_i, s_i) & \geq \boldsymbol{0},\ \forall i. \end{array} \tag{9}$$

The first set of constraint inequalities indicates that the demand does not exceed the supply; the second simply records the total amount of money retained by all consumers as s. Then, the retained amount s is subtracted linearly from the aggregate social utility function. This makes economical sense since this amount has been withdrawn from the exchange market by the consumers so that one should extract them from the aggregated social utility for the exchange market.

We have

Theorem 2. *Let $(\bar{\boldsymbol{x}}_i, \bar{s}_i)$, $i = 1, ..., m$, be an optimal solution for convex program (9), and let p_j be an optimal Lagrange multiplier for each good j in the first constraint set of (9). Then, these solutions form an equilibrium for the mixed market (7).*

Proof. First, the feasible set of the optimization problem (9) is linear, compact and convex, the maximal solution exists and the maximum value is finite. Moreover, the objective function to be maximized is concave. Thus, the first-order optimality conditions are necessary and sufficient for an optimal solution $(\bar{\boldsymbol{x}}_i, \bar{s}_i)$. These optimality conditions can be written (using the fact that the optimal Lagrange multiplier for the second constraint automatically equals 1) as:

$$\begin{array}{r} \frac{w_i}{u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i)}\nabla_{\boldsymbol{x}_i} u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i) \le \boldsymbol{p},\ \forall i \\ \frac{w_i}{u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i)}\partial_{s_i} u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i) \le 1,\ \forall i \\ \frac{w_i}{u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i)}\left(\nabla_{\boldsymbol{x}_i} u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i)^T\bar{\boldsymbol{x}}_i + \partial_{s_i} u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i)\bar{s}_i\right) = \boldsymbol{p}^T\bar{\boldsymbol{x}}_i + \bar{s}_i, \end{array} \tag{10}$$

where $\boldsymbol{p} = (p_1, ..., p_n)$ and p_j is the optimal Lagrange multiplier for each j in the first constraint set of (9). The third equality of condition (10) is called the complementarity condition, which, together with the fact that $u_i(\boldsymbol{x}_i, s_i)$ is homogeneous of degree one, namely $u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i) = \nabla u_i(\bar{\boldsymbol{x}}_i, \bar{s}_i)^T(\bar{\boldsymbol{x}}_i; \bar{s}_i)$, imply

$$\begin{aligned} \boldsymbol{p}^T\bar{\boldsymbol{x}}_i + \bar{s}_i &= \frac{w_i}{u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i)}\left(\nabla_{\boldsymbol{x}_i} u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i)^T\bar{\boldsymbol{x}}_i + \partial_{s_i} u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i)\bar{s}_i\right) \\ &= \frac{w_i}{u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i)}(\nabla u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i)^T(\bar{\boldsymbol{x}}_i;\bar{s}_i)) \\ &= \frac{w_i}{u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i)} \cdot u_i(\bar{\boldsymbol{x}}_i,\bar{s}_i) \\ &= w_i, \end{aligned}$$

so that

$$\sum_i(\boldsymbol{p}^T\bar{\boldsymbol{x}}_i + \bar{s}_i) = \sum_i w_i.$$

Thus, $(\bar{\boldsymbol{x}}_i, \bar{s}_i)$, $i = 1, ..., m$, and $\boldsymbol{p}$ satisfy the equilibrium conditions of (8). □

It is well-known that one can use interior point methods to solve the linearly constrained convex program (9) to yield both primal and dual optimal solutions in polynomial time; see [13]. Therefore, an equilibrium for the mixed market (7) can be found in polynomial time.

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