Competitive Communication Spectrum Economy and Equilibrium

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Abstract

Consider a competitive "spectrum economy" in communication system where multiple users share a common frequency band and each of them, equipped with an endowed "monetary" budget, will "purchase" its own transmit power spectra (taking others as given) in maximizing its Shannon utility or pay-off function that includes the effects of interference and subjects to its budget constraint. A market equilibrium is a price spectra and a frequency power allocation that independently and simultaneously maximizes each user's utility. Furthermore, under an equilibrium the market clears, meaning that the total power demand equals the power supply for every user and every frequency. We prove that such an equilibrium always exists for a discretized version of the problem, and, under a weak-interference condition or the Frequency Division Multiple Access (FMDA) policy, the equilibrium can be computed in polynomial time. This model may lead to an efficient decentralized method for spectrum allocation management and optimization in achieving both higher social utilization and better individual satisfaction. Furthermore, we consider a trading market among individual users to exchange their endowed power spectra under a price mechanism, and show that the market price equilibrium also exists and it may lead to a more socially desired spectrum allocation.

1 Introduction

Consider a communication system where multiple users share a common frequency band such as cognitive radio (e.g., [15]) or Digital Subscribe Lines (DSL, e.g., [23]), where interference mitigation

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is a major design and management concern. A standard approach to eliminate multi-user interference is to divide the available spectrum into multiple tones (or bands) and pre-assign them to the users on a non-overlapping basis, called Frequency Division Multiple Access (FDMA) policy. Although such approach is well-suited for high speed structured communication in which quality of service is a major concern, it can lead to high system overhead and low bandwidth utilization. With the proliferation of various radio devices and services, multiple wireless systems sharing a common spectrum must coexist [15], and we are naturally led to a situation whereby users can dynamically adjust their transmit power spectral densities over the entire shared spectrum, potentially achieving significantly higher overall throughput and fairness. For such a multi-user system, each user's performance, measured by a Shannon utility function, depends on not only the power allocation (across spectrum) of its own, but also those of other users in the system.

Thus, the dynamic spectrum management problem has recently become a topic of intensive research in the signal processing and digital communication community. From the optimization perspective, the problem solution can be formulated either as a noncooperative Nash game ([7, 28, 25, 18]); or as a cooperative utility maximization problem ([3, 30]). Several algorithms were proposed to compute a Nash equilibrium solution (Iterative Waterfilling method (IWFA) [7, 28]); or globally optimal power allocations (Dual decomposition method ([4, 17, 29]) for the cooperative game. Due to the problems nonconvex nature, these algorithms either lack global convergence or may converge to a poor spectrum sharing strategy.

In an attempt to analyze the performance of the dual decomposition algorithms, Yu and Lui [29] studied the duality gap of the continuous sum-rate maximization problem and showed it to be zero in the general frequency selective case based on engineering intuition. In two recent papers [19, 20], Luo and Zhang presented a systematic study of the dynamic spectrum management problem, covering two key theoretical aspects: complexity and duality. Specifically, they determined the complexity status of the spectrum management problem under various practical settings as well as different choices of system utility functions, and identify subclasses which are polynomial time solvable. In so doing, they clearly delineated the set of computationally tractable problems within the general class of NP-hard spectrum management problems. Furthermore, they rigorously established the zero-duality gap result of Yu and Lui for the continuous formulation when the interference channels are frequency selective. The asymptotic strong zero duality result of [19, 20] suggests that the Lagrangian dual decomposition approach ([4, 17, 29]) may be a viable way to reach approximate optimality for finely discretized spectrum management problems. In fact, when restricted to the FDMA policy, they showed that the Lagrangian dual relaxation, combined with a linear programming scheme, could generate an ϵ-optimal solution for the continuous formulation of the spectrum management problem in polynomial time for any fixed $\epsilon > 0$.

Besides computational difficulty, there remain other issues in the Nash equilibrium or the aggregate social utility maximization model. The Nash equilibrium solution may not achieve social communication economic efficiency; and, on the other hand, an aggregate optimal power allocation may not simultaneously optimizes each user's individual utility. Thus, we naturally turn to a competitive economy equilibrium solution for dynamic spectrum management, where both social economic efficiency and individual optimality could be achieved.

The study of competitive economy equilibria occupies a central place in mathematical economics. This study was formally started by Walras [24] over a hundred years ago. In this problem everyone in a population of n agents has an initial endowment of divisible goods or budget and a utility function for consuming all goods—their own and others. Every agent sells the entire initial endowment and then uses the revenue to buy a bundle of goods such that his or her utility function is maximized (individual optimality) and the market has neither shortage nor surplus (economic efficiency). Walras asked whether prices could be set for every good such that this is possible. An answer was given by Arrow and Debreu in 1954 [1] who showed that such an equilibrium would exist, under very mild conditions, if the utility functions were concave. Their proof was non-constructive and did not offer any algorithm to find such equilibrium prices.

Fisher was the first to consider an algorithm to compute equilibrium prices for a related and different model where agents are divided into two sets: producers and consumers; see Brainard and Scarf [2, 22]. Consumers spend money only to buy goods and maximize their individual utility functions of goods; producers sell their goods only for money. An equilibrium is an assignment of prices to goods so that when every consumer buys a maximal bundle of goods then the market clears, meaning that all the money is spent and all the goods are sold. Fisher's model is a special case of Walras' model when money is also considered a good so that Arrow and Debreu's result applies.

For certain utility functions, the equilibrium problem is actually a social utility maximization problem. For example, Eisenberg and Gale [12, 14] give a convex programming (or optimization) formulation whose solution yields equilibriums for the Fisher market with linear utility functions, and Eisenberg [13] extended this approach to derive a convex program for general concave and homogeneous functions of degree 1. Their program consists of maximizing an aggregate social utility function of all consumers over a convex polyhedron defined by supply-demand linear constraints. The Lagrange or dual multipliers of these constraints yield equilibrium prices. Thus, finding a Fisher equilibrium becomes solving a convex optimization problem, and it could be computed by the Ellipsoid method or by efficient interior-point methods in polynomial time. Here, polynomial time means that one can compute an ϵ approximate equilibrium in a number of arithmetic operations bounded by polynomial in n and $\log \frac{1}{\epsilon}$; or, if there is a rational equilibrium solution, one can compute an *exact* equilibrium in a number of arithmetic operations bounded by polynomial in n and L, where L is the bit-length of the input data, see, e.g., [16]. When the utility functions are linear, the current best arithmetic operations complexity bound is $O(\sqrt{mn}(m+n)^3L)$ given by [26]. Negative results also obtained for other utilities, see, e.g., Codenotti et al. [27, 8].

However, little is known on the computational complexity for competitive market equilibria with non-homogeneous utility functions ([6, 11, 5]), or utility functions that include goods purchased by other agents, which is the case in dynamic spectrum management. In the original paper of Arrow and Debreu [1], each user's utility function was described as a function of his or her own actions. Our paper is to study the existence and complexity of an equilibrium point that is characterized by the property that each individual is maximizing the pay-off to him or her by controlling his or her own actions, given the actions of the other agents, over the set of actions permitted him or her also in view of the other agents' actions.

We prove in this paper that

1. A competitive equilibrium always exists for the communication spectrum market with the Shannon utility for spectrum users and profit utility for the spectrum power provider.

2. Under an additional weak-interference and fixed supply condition, such equilibria form a convex or log-convex set and one can be computed in polynomial time.

3. Under the FMDA policy, the equilibrium is unique and can be computed in polynomial time.

2 Mathematical Notations

First, a few mathematical notations. Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space; $\mathbf{R}^n_+$ denote the subset of $\mathbf{R}^n$ where each coordinate is non-negative. $\mathbf{R}$ and $\mathbf{R}_+$ denote the set of real numbers and the set of non-negative real numbers, respectively.

Let $X \in \mathbf{R}^{mn}_+$ denote the set of ordered m-tuples $X = (\boldsymbol{x}_1, ..., \boldsymbol{x}_m)$ and let $\bar{X}_i \in \mathbf{R}^{(m-1)n}_+$ denote the set of ordered $(m-1)$-tuples $X = (\boldsymbol{x}_1, ..., \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, ..., \boldsymbol{x}_m)$, where $\boldsymbol{x}_i = (x_{i1}, ..., x_{in}) \in X_i \subset \mathbf{R}^n_+$ for $i = 1, ..., m$. For each i, suppose there is a real utility function u_i, defined over X. Let $A_i(\bar{\boldsymbol{x}}_i)$ be a subset of X_i defining for each point $\bar{\boldsymbol{x}}_i \in \bar{X}_i$, Then the sequence $[X_1, ..., X_m, u_1, ..., u_m, A_1(\bar{\boldsymbol{x}}_1), ..., A_m(\bar{\boldsymbol{x}}_m)]$ will be termed an abstract economy. Here $A_i(\bar{\boldsymbol{x}}_i)$ represent the feasible action set of agent i that is possibly restricted by the actions of others, such as the budget restraint that the cost of the goods chosen at current prices not exceed his income, and the prices and possibly some or all of the components of his income are determined by choices made by other agents. Similarly, utility function $u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i)$ for agent i depends on his or her actions $\boldsymbol{x}_i$, as well as actions $\bar{\boldsymbol{x}}_i$ made by all other agents. Also, denote $\boldsymbol{x}_j = (x_{1j}, ..., x_{mj}) \in \mathbf{R}^m$ for a given $\boldsymbol{x} \in X$.

A function $u : \mathbf{R}^n_+ \rightarrow \mathbf{R}_+$ is said to be *concave* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n_+$ and any $0 \leq \alpha \leq 1$, we have $u(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geq \alpha u(\boldsymbol{x}) + (1-\alpha)u(\boldsymbol{y})$; and it is strictly concave if $u(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) > \alpha u(\boldsymbol{x}) + (1-\alpha)u(\boldsymbol{y})$ for $0 < \alpha < 1$. It is *monotone increasing* if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n_+$, $\boldsymbol{x} \geq \boldsymbol{y}$ implies that $u(\boldsymbol{x}) \geq u(\boldsymbol{y})$. It is *homogeneous* of degree d if for any $\boldsymbol{x} \in \mathbf{R}^n_+$ and any $\alpha > 0$, $u(\alpha\boldsymbol{x}) = \alpha^d u(\boldsymbol{x})$.

3 Competitive Communication Spectrum Market

Let the multi-user communication system consist of m transmitter-receiver pairs sharing a common frequency band $f \in \Omega$. For simplicity, we will call each of such transmitter-receiver pair a "User". Upon normalization, we can assume Ω to be the unit interval $[0; 1]$. Each user i will be endowed a "monetary" budget $w_i > 0$ and use it to "purchase or exchange" for power spectra density, $x_i(f)$, across $f \in \Omega$, from an open market so as to maximize its own utility $u_i(x_i(f \in \Omega), \bar{x}_i(f \in \Omega))$, where $\bar{x}_i(f \in \Omega)$ represent power spectra densities obtained by all other users. w_i may not represent real money like a coupon. In some traffic flow applications, they represent "toll" budgets for users to pay toll routs. In other applications, w_i simply represents the "importance" weight of certain users; e.g., $w_i = 1$ for all i means that all users are treated uniformly important. One can also adjust w_i to maximize certain aggregate social utility.

There is a second-type agents, called power capacity “Producer or Provider”, who installs power capacity spectra density $s(f \in \Omega) \geq 0$ to the market from a convex and compact set S to maximize his or her utility.

The third agent, “Market”, sets power spectra unit “price” density $p(f \in \Omega) \geq 0$. $p(f)$ can be interpreted as a “preference or ranking” spectra density of f. For example, $p(f_1) = 1$ and $p(f_2) = 2$ simply mean that users can use one unit of $s(f_2)$ to trade for two units of $s(f_1)$. In traffic flow applications, $p(f)$ represents the toll fee for route f.

Then, User i's$(i = 1, ..., m)$ individual utility maximization problem is

$$\begin{array}{lll} \text{maximize}_{\; x_i(f\in\Omega)} & u_i(x_i(f \in \Omega), \bar{x}_i(f \in \Omega)) & \\ \text{subject to} & \int_\Omega p(f)x_i(f)df & \leq w_i \\ & x_i(f \in \Omega) & \geq 0; \end{array} \tag{1}$$

that is, the total payment of “purchased” power spectral density does not exceed his or her endowed budget w_i.

A commonly recognized utility for user i, $i = 1, ..., m$, in communication is the Shannon utility [9]:

$$u_i(x_i(f \in \Omega), \bar{x}_i(f \in \Omega)) = \int_\Omega \log\left(1 + \frac{x_i(f)}{\sigma_i(f) + \sum_{k\neq i} a_k^i(f)x_k(f)}\right) df \tag{2}$$

where $\sigma_i(f)$ denotes the normalized background noise power for user i at frequency f, and $a_k^i(f)$ is the normalized crosstalk ratio from user k to user i at frequency f. Due to normalization we have $a_i^i(f) = 1$ for all i and $f \in \Omega$. One may also subtract a physical cost of total “purchased” power spectra density from the Shannon utility:

$$u_i(x_i(f \in \Omega), \bar{x}_i(f \in \Omega)) = \int_\Omega \log\left(1 + \frac{x_i(f)}{\sigma_i(f) + \sum_{k\neq i} a_k^i(f)x_k(f)}\right) df - c_i \int_\Omega x_i(f)df,$$

where c_i is a constant cost rate for user i.

The power provider's individual utility maximization problem is

$$\begin{array}{ll} \text{maximize}_{\; s(f\in\Omega)} & u_s(s(f \in \Omega), p(f \in \Omega)) \\ \text{subject to} & s(f \in \Omega) \in S, \end{array} \tag{3}$$

where $u_s(s(f \in \Omega), p(f \in \Omega))$ represents the utility function of the power capacity provider who installs $s(f \in \Omega)$ and S is a physical feasible set (can be a fixed point). For example,

$$u_s(s(f \in \Omega), p(f \in \Omega)) = \int_\Omega p(f)s(f)df - c(s(f \in \Omega))$$

that is, the profit made by installing power spectra density $s(f)$ where $c(s(f \in \Omega))$ is a cost function.

A competitive market equilibrium is a density point $[x_1^*(f), ..., x_m^*(f),), s^*(f), p^*(f)],\ f \in \Omega$ such that

- (User optimality) $x_i^*(f \in \Omega)$ is a maximizer of (1) given $\bar{x}_i^*(f \in \Omega)$ and $p^*(f \in \Omega)$ for every i.
- (Producer optimality) $s^*(f \in \Omega)$ is a maximizer of (3) given $p^*(f \in \Omega)$.
- (Market efficiency) $p(f \in \Omega) \geq 0$, $\sum_{i=1}^m x_i^*(f) \leq s^*(f),\ p^*(f)(s^*(f) - \sum_{i=1}^m x_i^*(f)) = 0,\ \forall f \in \Omega$.

The last condition says that if power capacity $s^*(f)$ is greater than needed density, $\sum_{i=1}^m x_i^*(f))$, at frequency f, then its equilibrium price density $p^*(f) = 0$.

We remark that this is a rather *artificial* exchange or trading open market, where the price p and budget w do not necessarily have *physical interpretations*. The meaningful outputs may be only those power spectra assignments, and w and p are economic mechanisms mainly used to influence users' behavior over the communication channel selection to achieve a larger market utilization.

In reality, the frequency band range Ω is discretized by n tones: $\Omega = \{f_1, f_2, ..., f_n\}$, where the Shannon utility for user i, $i = 1, ..., m$, becomes

$$u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i) = \sum_{j=1}^n \log\left(1 + \frac{x_{ij}}{\sigma_{ij} + \sum_{k \neq i} a_{kj}^i x_{kj}}\right) \tag{4}$$

where variable $\boldsymbol{x}_i = (x_{i1}, ..., x_{in}) \in \mathbf{R}_+^n$ and x_{ij} is the power units purchased by User i for tone j, variables in $\bar{\boldsymbol{x}}_i \in \mathbf{R}_+^{(m-1)n}$ are power units purchased by all other users, parameter σ_{ij} denotes the normalized background noise power for user i at tone j, and parameter a_{kj}^i is the normalized crosstalk ratio from user k to user i at tone j. Due to normalization we have $a_{ij}^i = 1$ for all i, j. Clearly, $u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i)$ is a continuous concave and monotone increasing function in $\boldsymbol{x}_i \in \mathbf{R}_+^n$ for every $\bar{\boldsymbol{x}}_i \in \mathbf{R}_+^{(m-1)n}$. Again, one may also adjust the utility function to

$$u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i) = \sum_{j=1}^n \log\left(1 + \frac{x_{ij}}{\sigma_{ij} + \sum_{k \neq i} a_{kj}^i x_{kj}}\right) - c_i\left(\sum_{j=1}^n x_{ij}\right),$$

that is, subtracting a physical cost of total "purchased" tone powers from the Shannon utility.

Then, a competitive communication spectrum market equilibrium is a point of $[\boldsymbol{x}_1^*, ..., \boldsymbol{x}_m^*, \boldsymbol{s}^*, \boldsymbol{p}^*]$, where $\boldsymbol{s}^* = (s_1^*, ..., s_n^*) \in \mathbf{R}_+^n$ and s_j^* is the total power capacity units installed by the second power provider, and $\boldsymbol{p}^* = (p_1^*, ..., p_n^*) \in \mathbf{R}_+^n$ and p_j^* is the unit price for tone j set by Market; such that

1. (User optimality) $\boldsymbol{x}_i^*$, $i = 1, ..., m$, is a maximizer of

$$\begin{array}{lll} \text{maximize}_{\boldsymbol{x}_i} & u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i^*) & \\ \text{subject to} & \sum_{j=1}^n p_j^* x_{ij} & \leq w_i \\ & \boldsymbol{x}_i \in X_i \subset \mathbf{R}_+^n, & \end{array} \tag{5}$$

   where $\bar{\boldsymbol{x}}_i^* = [\boldsymbol{x}_1^*, ..., \boldsymbol{x}_{i-1}^*, \boldsymbol{x}_{i+1}^*, ..., \boldsymbol{x}_n^*]$ and X_i is a physical feasible set. For example,

$$X_i = \{\boldsymbol{x}_i \in \mathbf{R}_+^n : \sum_{j=1}^n x_{ij} \leq b_i\}$$

   where $b_i > 0$ is a physical battery power budget for user i.

2. (Producer optimality) $\boldsymbol{s}^*$ is a maximizer of

$$\begin{array}{ll} \text{maximize}_{\boldsymbol{s}} & u_s(\boldsymbol{s}, \boldsymbol{p}^*) \\ \text{subject to} & \boldsymbol{s} \in S \subset \mathbf{R}_+^n. \end{array} \tag{6}$$

3. (Market efficiency) $\boldsymbol{p}^* \geq \mathbf{0}$, $\sum_{i=1}^m x_{ij}^* \leq \boldsymbol{s}_j^*$, $p_j^*(s_j^* - \sum_{i=1}^m x_{ij}^*) = 0$ for all j.

The following theorem directly follows Arrow and Debreu [1].

Theorem 1. *Let $u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i)$ be a continuous and concave function in $\boldsymbol{x}_i \in \mathbf{R}_+^n$ for every $\bar{\boldsymbol{x}}_i \in \mathbf{R}_+^{(m-1)n}$, $u_s(\boldsymbol{s}, \boldsymbol{p})$ be a continuous and concave function in $\boldsymbol{s}$ for every $\boldsymbol{p}$, $X_i \subset \mathbf{R}_+^n$, $i = 1, ..., m$, and $S \subset \mathbf{R}_+^n$ be closed convex sets, and S be furthermore bounded. Then, the discretized communication spectrum market has a competitive equilibrium.*

The proof of the theorem is identical to the one given in [1] based on the Lemma of Abstract Economy developed by Debreu [10] and Nash [21], where money is a good whose price is normalized as 1. The only difference is that in [1], the utility function u_i is a function of $\boldsymbol{x}_i$ and nothing else. However, the Lemma of Abstract Economy actually allows agent i's utility u_i to be a function of $\boldsymbol{x}_i$, the action made by agent i, and $\bar{\boldsymbol{x}}_i$, the actions made by all other agents.

Furthermore, if $\boldsymbol{p}$ and $\boldsymbol{s}$ are fixed and only Users are the agents in the game, the equilibrium problem reduces to a Nash equilibrium problem. By allowing $\boldsymbol{p}$ and $\boldsymbol{s}$ vary in the game, we hope to potentially achieve a more efficient spectrum economy.

Since the Shannon utility function of (4) is continuous, concave and monotone increasing in $\boldsymbol{x}_i \in \mathbf{R}^n_+$ for for every $\bar{\boldsymbol{x}}_i \in \mathbf{R}^{(m-1)n}_+$, Our main result is the following corollary.

Corollary 1. *Let the power capacity provider utility $u_s(\boldsymbol{s}, \boldsymbol{p})$ be a continuous and concave function in $\boldsymbol{s}$ for every $\boldsymbol{p}$, and $X_i \subset \mathbf{R}^n_+$, $i = 1, ..., m$, and $S \subset \mathbf{R}^n_+$ be closed convex sets, and S be furthermore bounded. Then, the discretized communication spectrum market with the Shannon utility has a competitive equilibrium.*

4 Equilibrium Characterization

For simplicity, let the power capacity set S is represented by a polyhedron

$$S = \{\boldsymbol{s} : \ B\boldsymbol{s} \leq \boldsymbol{r}, \ \boldsymbol{s} \geq \boldsymbol{0}\}$$

for given constraint matrix B and resource vector $\boldsymbol{r}$, and let the power supply cost function be linear $c(\boldsymbol{s}) = \boldsymbol{c}^T\boldsymbol{s}$ for a given cost coefficient vector $\boldsymbol{c}$. Then, the optimality function of (6) is that, there is $\boldsymbol{y}^*$ such that

$$\begin{array}{rcl} B\boldsymbol{s}^* & \leq & \boldsymbol{r}, \\ \boldsymbol{p}^* - \boldsymbol{c} & \leq & B^T\boldsymbol{y}^*, \\ \boldsymbol{r}^T\boldsymbol{y}^* & = & (\boldsymbol{p}^* - \boldsymbol{c})^T\boldsymbol{s}^*, \\ \boldsymbol{s}^*, \boldsymbol{y}^* & \geq & \boldsymbol{0}, \end{array} \tag{7}$$

Now consider the optimality conditions of (5) where, for simplicity, let $X_i = \mathbf{R}^n_+$. They are

$$\begin{array}{rcl} w_i \cdot \nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*) & \leq & (\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*)^T\boldsymbol{x}_i^*) \cdot \boldsymbol{p}^*, \\ (\boldsymbol{p}^*)^T\boldsymbol{x}_i^* & = & w_i, \\ \boldsymbol{x}_i^* & \geq & \boldsymbol{0}, \end{array} \tag{8}$$

where $\nabla_{\boldsymbol{x}_i} u(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i) \in \mathbf{R}^n$ denotes any sub-gradient vector of $u(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i)$ with respect to $\boldsymbol{x}_i$.

The complete necessary and sufficient conditions for a competitive equilibrium can be summarized as:

$$\begin{array}{rcl} w_i \cdot \nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*) & \leq & (\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*)^T \boldsymbol{x}_i^*) \cdot \boldsymbol{p}^*, \ \forall i \\ B\boldsymbol{s}^* & \leq & \boldsymbol{r}, \\ \boldsymbol{p}^* - \boldsymbol{c} & \leq & B^T \boldsymbol{y}^*, \\ \sum_i \boldsymbol{x}_i^* & \leq & \boldsymbol{s}^*, \\ \boldsymbol{c}^T \boldsymbol{s}^* + \boldsymbol{r}^T \boldsymbol{y}^* & \leq & \sum_i w_i, \\ \boldsymbol{x}_i^*, \boldsymbol{p}^*, \boldsymbol{s}^*, \boldsymbol{y}^* & \geq & \mathbf{0}, \ \forall i. \end{array} \tag{9}$$

Note that the conditions $(\boldsymbol{p}^*)^T \boldsymbol{s}^* = \boldsymbol{c}^T \boldsymbol{s}^* + \boldsymbol{r}^T \boldsymbol{y}^*$ and $(\boldsymbol{p}^*)^T \boldsymbol{x}_i^* = w_i$ for all i are implied by the conditions in (9): multiplying $\boldsymbol{x}_i^* \geq \mathbf{0}$ to both sides of the first inequality, $\boldsymbol{y}^* \geq \mathbf{0}$ to both sides of the second inequality, $\boldsymbol{s}^* \geq \mathbf{0}$ to both sides of the third inequality in (9), we have $(\boldsymbol{p}^*)^T \boldsymbol{x}_i^* \geq w_i$ for all i, $\boldsymbol{r}^T \boldsymbol{y}^* \geq (\boldsymbol{s}^*)^T B^T \boldsymbol{y}^*$, and $(\boldsymbol{s}^*)^T B^T \boldsymbol{y}^* \geq (\boldsymbol{p}^* - \boldsymbol{c})^T \boldsymbol{s}^*$, which, together with other inequality conditions in (9), imply

$$\sum_i w_i \geq \boldsymbol{c}^T \boldsymbol{s}^* + \boldsymbol{r}^T \boldsymbol{y}^* \geq (\boldsymbol{p}^*)^T \boldsymbol{s}^* \geq (\boldsymbol{p}^*)^T \left(\sum_i \boldsymbol{x}_i^* \right) = \sum_i (\boldsymbol{p}^*)^T \boldsymbol{x}_i^* \geq \sum_i w_i,$$

that is, every inequality in the sequence must be tight, which implies $(\boldsymbol{p}^*)^T \boldsymbol{s}^* = \boldsymbol{c}^T \boldsymbol{s}^* + \boldsymbol{r}^T \boldsymbol{y}^*$, $(\boldsymbol{p}^*)^T (\sum_i \boldsymbol{x}_i^*) = (\boldsymbol{p}^*)^T \boldsymbol{s}^*$, and $(\boldsymbol{p}^*)^T \boldsymbol{x}_i^* = w_i$ for all i.

Thus, we have a characterization theorem of a competitive equilibrium.

Theorem 2. *Every equilibrium of the discretized communication spectrum market with the Shannon utility has the following properties*

1. $\boldsymbol{p}^* > \mathbf{0}$ *(every tone power has a price);*
2. $\sum_i \boldsymbol{x}_i^* = \boldsymbol{s}^*$ *(produce what is needed);*
3. $(\boldsymbol{p}^*)^T \boldsymbol{s}^* = \sum_i w_i$ *(all user budgets go to the provider);*
4. *If* $x_{ij}^* > 0$ *then* $(\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*)^T \boldsymbol{x}_i^*) \cdot p_j^* - w_i \cdot (\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*))_j = 0$ *for all* i, j *(every user only purchases most valuable tone power).*

Proof. Note that

$$(\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i))_j = \frac{1}{\sigma_{ij} + \sum_{k \neq i} a_{kj}^i x_{kj} + x_{ij}} > 0, \ \forall \boldsymbol{x} \geq \mathbf{0}.$$

Thus,

$$w_i \cdot \nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*) > \mathbf{0},$$

so that the first inequality of (9) implies that $\boldsymbol{p}^* > \mathbf{0}$.

The second property is from $(\boldsymbol{p}^*)^T(\sum_i \boldsymbol{x}_i^*) = (\boldsymbol{p}^*)^T \boldsymbol{s}^*$, $\sum_i \boldsymbol{x}_i^* \leq \boldsymbol{s}^*$ and $\boldsymbol{p}^* > \mathbf{0}$.

The third is from $(\boldsymbol{p}^*)^T \boldsymbol{x}_i^* = w_i$ for all i and $\sum_i \boldsymbol{x}_i^* = \boldsymbol{s}^*$.

The last one is from the complementarity condition of user optimality. □

5 Equilibrium for a Weak-Interference Market

The inequalities and equalities in (9) are all linear, except the first

$$w_i \cdot \nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*) \leq (\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*)^T \boldsymbol{x}_i^*) \cdot \boldsymbol{p}^*.$$

Now we consider a weak-interference communication channel where the Shannon utility function is approximated by

$$u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i) = \sum_{j=1}^{n} \log\left(1 + \frac{x_{ij}}{\sigma_{ij} + a_j^i\left(\sum_{k\neq i} x_{kj}\right)}\right) \tag{10}$$

where a_j^i represent the average of normalized crosstalk ratios for $k \neq i$. Furthermore, we assume $0 \leq a_j^i \leq 1$, that is, the average cross-interference ratio is not above 1 or it is less than the self-interference ratio (always normalized to 1); and $S = \{\boldsymbol{s}^*\}$ is a singleton, that is, the power supply $\boldsymbol{s}^*$ is fixed.

The partial derivative of $u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i)$ to x_{ij} is

$$(\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i))_j = \frac{1}{\sigma_{ij} + a_j^i\left(\sum_{k\neq i} x_{kj}\right) + x_{ij}}, \ \forall j$$

so that

$$\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i)^T \boldsymbol{x}_i = \sum_{j=1}^{n} \frac{x_{ij}}{\sigma_{ij} + a_j^i\left(\sum_{k\neq i} x_{kj}\right) + x_{ij}}.$$

At an equilbrium charaterized by (9),

$$\sum_{k\neq i} x_{kj}^* = s_j^* - x_{ij}^*, \ \forall j.$$

Thus,

$$(\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*))_j = \frac{1}{\sigma_{ij} + a_j^i s_j^* + (1 - a_j^i) x_{ij}^*}, \ \forall j$$

so that

$$\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*)^T \boldsymbol{x}_i = \sum_{j=1}^{n} \frac{x_{ij}^*}{\sigma_{ij} + a_j^i s_j^* + (1 - a_j^i) x_{ij}^*}.$$

Then, using the logarithmic transformation, one can rewrite the nonlinear inequality in (9) as

$$\log\left(\sigma_{ij} + a_j^i s_j^* + (1 - a_j^i) x_{ij}^*\right) + \log(p_j^*) + \log\left(\sum_{j=1}^{n} \frac{x_{ij}^*}{\sigma_{ij} + a_j^i s_j^* + (1 - a_j^i) x_{ij}^*}\right) \geq \log(w_i), \ \forall i, j. \quad (11)$$

This is actually a *convex* inequality, since the function on the left is a (strictly) concave function in x_{ij}^* and p_j^* for any constant $1 - a_j^i \geq 0$.

Therefore,

Theorem 3. *Under the weak-interference communication channel and fixed power supply condition, the competitive equilibrium set of the discretized communication spectrum market is convex.*

We remark that the cooperative utility maximization approach, with the Shannon utility given by (10) and fixed power supply, is still a non-convex optimization problem. However, its equilibrium set is convex!

It's known that the convex inequality (11) admits an efficient barrier function (see Deng et al. [11, 5]), so that

Corollary 2. *An equilibrium of the discretized communication spectrum market under the weak-interference communication channel and fixed power supply condition can be computed in polynomial time.*

Complementarity Property 4 of Theorem 2 implies

$$\log\left(\sigma_{ij} + a_j^i s_j^* + (1 - a_j^i) x_{ij}^*\right) + \log(p_j^*) + \log\left(\sum_{j=1}^{n} \frac{x_{ij}^*}{\sigma_{ij} + a_j^i s_j^* + (1 - a_j^i) x_{ij}^*}\right) = \log(w_i), \ \forall x_{ij}^* > 0.$$

Let $[(\boldsymbol{x}^*)^1, (\boldsymbol{p}^*)^1]$ and $[(\boldsymbol{x}^*)^2, (\boldsymbol{p}^*)^2]$ be two distinct equilibrium points, then Theorem 3 implies that $[0.5(\boldsymbol{x}^*)^1 + 0.5(\boldsymbol{x}^*)^2, \boldsymbol{s}^*, 0.5(\boldsymbol{p}^*)^1 + 0.5(\boldsymbol{p}^*)^2]$ is also an equilibrium, so that

$$\log\left(\sigma_{ij} + a_j^i s_j^* + (1 - a_j^i)(0.5(x^*)_{ij}^1 + 0.5(x^*)_{ij}^2)\right) + \log\left(0.5(p^*)_j^1 + 0.5(p^*)_j^2\right) +$$

$$\log\left(\sum_{j=1}^{n}\frac{0.5(x^*)^1_{ij}+0.5(x^*)^2_{ij}}{\sigma_{ij}+a^i_j s^*_j+(1-a^i_j)(0.5(x^*)^1_{ij}+0.5(x^*)^2_{ij})}\right)=\log(w_i),\ \forall\max\{(x^*)^1_{ij},(x^*)^2_{ij}\}>0.$$

However, the function on the left is strictly concave in $\boldsymbol{p}^*$, and in $\boldsymbol{x}^*$ if $a^i_j<1$, so that

$$\log\left(\sigma_{ij}+a^i_j s^*_j+(1-a^i_j)(0.5(x^*)^1_{ij}+0.5(x^*)^2_{ij})\right)+\log\left(0.5(p^*)^1_j+0.5(p^*)^2_j\right)+$$

$$\log\left(\sum_{j=1}^{n}\frac{0.5(x^*)^1_{ij}+0.5(x^*)^2_{ij}}{\sigma_{ij}+a^i_j s^*_j+(1-a^i_j)(0.5(x^*)^1_{ij}+0.5(x^*)^2_{ij})}\right)>$$

$$0.5\left(\log(\sigma_{ij}+a^i_j s^*_j+(1-a^i_j)(x^*)^1_{ij})+\log((p^*)^1_j)+\log\left(\sum_{j=1}^{n}\frac{(x^*)^1_{ij}}{\sigma_{ij}+a^i_j s^*_j+(1-a^i_j)(x^*)^1_{ij}})\right)\right)+$$

$$0.5\left(\log(\sigma_{ij}+a^i_j s^*_j+(1-a^i_j)(x^*)^2_{ij})+\log((p^*)^2_j)+\log\left(\sum_{j=1}^{n}\frac{(x^*)^2_{ij}}{\sigma_{ij}+a^i_j s^*_j+(1-a^i_j)(x^*)^2_{ij}})\right)\right)\geq$$

$$0.5\log(w_i)+0.5\log(w_i)=\log(w_i),\ \forall\max\{(x^*)^1_{ij},(x^*)^2_{ij}\}>0.$$

Thus, we must have $(\boldsymbol{p}^*)^1=(\boldsymbol{p}^*)^2>\boldsymbol{0}$, and $(x^*)^1_{ij}=(x^*)^2_{ij},\ \forall\max\{(x^*)^1_{ij},(x^*)^2_{ij}\}>0$ if $a^i_j<1$, which imply that the equilibrium point is unique.

Therefore,

Theorem 4. *Under the weak-interference communication channel and fixed power supply condition, the competitive price equilibrium of the discretized communication spectrum market is unique. Moreover, if the crosstalk ratio a^i_j is strictly less than 1, then the power allocation x^*_{ij} is also unique.*

6 The Spectrum Trading Market

Unlike the market described above where each user is equipped with an endowed "monetary" budget, in the spectrum trading market each user is endowed with a bundle of allocated spectrum power, say $\boldsymbol{w}_i\in\mathbf{R}^n$, where each entry represents the power allocation for user i on channel or tone j, $j=1,...,n$. Now they trade their spectrum power under a market price to maximize their individual utility function. Here, the total spectrum power capacities available on the market are

$$\bar{\boldsymbol{s}}=\sum_{i=1}^{m}\boldsymbol{w}_i,$$

and they are fixed.

A competitive communication spectrum market equilibrium is a point of $[\boldsymbol{x}_1^*, ..., \boldsymbol{x}_m^*, \boldsymbol{p}^*]$, where $\boldsymbol{p}^* = (p_1^*, ..., p_n^*) \in \mathbf{R}_+^n$ and p_j^* is the unit price on tone j; such that

1. (User optimality) $\boldsymbol{x}_i^*$, $i = 1, ..., m$, is a maximizer of

$$\begin{array}{ll} \text{maximize}_{\boldsymbol{x}_i} & u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i^*) \\ \text{subject to} & \sum_{j=1}^n p_j^* x_{ij} \quad \leq \boldsymbol{w}_i^T \boldsymbol{p}^* \\ & \boldsymbol{x}_i \geq \mathbf{0}, \end{array} \tag{12}$$

2. (Market efficiency) $\boldsymbol{p}^* \geq \mathbf{0}$, $\sum_{i=1}^m x_{ij}^* \leq \bar{s}_j$ and $p_j^*(\bar{s}_j - \sum_{i=1}^m x_{ij}^*) = 0$ for all j.

Note that (12) is homogeneous in $\boldsymbol{p}^*$, so that if $\boldsymbol{p}^*$ is an equilibrium price vector, so is $\alpha \cdot \boldsymbol{p}^*$ for any positive multiplier α.

Again, following the theorem of Arrow and Debreu [1], we have:

Corollary 3. *The trading spectrum market with the Shannon utility has a competitive equilibrium.*

From the optimality conditions of (12):

$$\begin{array}{rcl} (\boldsymbol{w}_i^T \boldsymbol{p}^*) \cdot \nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*) & \leq & (\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*)^T \boldsymbol{x}_i^*) \cdot \boldsymbol{p}^*, \\ (\boldsymbol{p}^*)^T \boldsymbol{x}_i^* & = & \boldsymbol{w}_i^T \boldsymbol{p}^*, \\ \boldsymbol{x}_i^* & \geq & \mathbf{0}, \end{array} \tag{13}$$

we can derive the complete necessary and sufficient conditions for a competitive equilibrium:

$$\begin{array}{rcl} (\boldsymbol{w}_i^T \boldsymbol{p}^*) \cdot \nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*) & \leq & (\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{x}}_i^*)^T \boldsymbol{x}_i^*) \cdot \boldsymbol{p}^*, \ \forall i \\ \sum_i \boldsymbol{x}_i^* & \leq & \bar{\boldsymbol{s}}, \\ \boldsymbol{x}_i^*, \boldsymbol{p}^* & \geq & \mathbf{0}, \ \forall i. \end{array} \tag{14}$$

Note that the condition $(\boldsymbol{p}^*)^T \boldsymbol{x}_i^* = \boldsymbol{w}_i^T \boldsymbol{p}^*$ for all i is implied by the conditions in (14): multiplying $\boldsymbol{x}_i^* \geq \mathbf{0}$ to both sides of the first inequality, we have $(\boldsymbol{p}^*)^T \boldsymbol{x}_i^* \geq \boldsymbol{w}_i^T \boldsymbol{p}^*$ for all i, which imply

$$\sum_i \boldsymbol{w}_i^T \boldsymbol{p}^* \leq \sum_i (\boldsymbol{p}^*)^T \boldsymbol{x}_i^* \leq (\boldsymbol{p}^*)^T \bar{\boldsymbol{s}} = \sum_i \boldsymbol{w}_i^T \boldsymbol{p}^*,$$

that is, every inequality in the sequence must be tight, which implies $(\boldsymbol{p}^*)^T \boldsymbol{x}_i^* = \boldsymbol{w}_i^T \boldsymbol{p}^*$ for all i and $\sum_i \boldsymbol{x}_i^* = \bar{\boldsymbol{s}}$.

Similarly, we have a characterization theorem of a competitive equilibrium.

Theorem 5. *Every equilibrium of the trading spectrum market with the Shannon utility has the following properties*

1. $\boldsymbol{p}^* > \mathbf{0}$ *(every tone power has a price);*

2. $\sum_i \boldsymbol{x}_i^* = \bar{\boldsymbol{s}}$ *(all power endowments are traded).*

3. *If the equilibrium allocation has a property that each tone is occupied by a single user, then the Shannon utility at the equilibrium is improved for every user (trading benefits every one).*

The proof of the last statement is based on the fact that, for every i,

$$u_i(\boldsymbol{x}_i^*, \mathbf{0}) \geq u_i(\boldsymbol{x}_i^*, \bar{\boldsymbol{w}}_i) \geq u_i(\boldsymbol{w}_i, \bar{\boldsymbol{w}}_i);$$

where the first inequality is from that there is no interference at the equilibrium and the second inequality holds since $\boldsymbol{w}_i$ is a feasible solution for (12) at the corresponding equilibrium price $\boldsymbol{p}^*$.

Under the same weak-interference communication channel condition and using the logarithmic transformation, one can rewrite the nonlinear inequality in (14) as

$$\log\left(\sigma_{ij} + a_j^i \bar{s}_j + (1 - a_j^i)x_{ij}^*\right) + \log(p_j^*) + \log\left(\sum_{j=1}^{n} \frac{x_{ij}^*}{\sigma_{ij} + a_j^i \bar{s}_j + (1 - a_j^i)x_{ij}^*}\right) \geq \log(\boldsymbol{w}_i^T \boldsymbol{p}^*),\ \forall i, j. \tag{15}$$

This is *not* a *convex* inequality, since the function on the right is a concave function in $\boldsymbol{p}^*$.

However, from Theorem 5 we can define $\log(p_j^*) = y_j^*$ for all j. Then, the above inequality becomes

$$\log\left(\sigma_{ij} + a_j^i \bar{s}_j + (1 - a_j^i)x_{ij}^*\right) + y_j^* + \log\left(\sum_{j=1}^{n} \frac{x_{ij}^*}{\sigma_{ij} + a_j^i \bar{s}_j + (1 - a_j^i)x_{ij}^*}\right) \geq \log(\sum_j w_{ij} e^{y_j^*}),\ \forall i, j.$$

This is now a *convex* inequality, since the function on the left is a (strictly) concave function in x_{ij}^* and y_j^* for any constant $1 - a_j^i \geq 0$, and the one on the right is a convex function in $\boldsymbol{y}^*$. Therefore,

Theorem 6. *Under the weak-interference communication channel condition, the competitive equilibrium set of the spectrum trading market is log convex (convex in allocation and log price), and an equilibrium can be computed in polynomial time.*

7 Equilibrium under the Frequency Division Multiple Access policy

For continuous communication spectrum management under the Frequency Division Multiple Access policy or even multiple users sharing a same tone but operating at different time point (that is, at any given time period, only one user utilizes the tone), each user's utility function is independent of other users, that is,

$$u_i(\boldsymbol{x}_i) = \sum_{j=1}^{n} \log\left(1 + \frac{x_{ij}}{\sigma_{ij}}\right) \tag{16}$$

Then, similar to the discussion in the previous section, one can rewrite the nonlinear inequality in (9) as

$$\log\left(\sigma_{ij} + x_{ij}^*\right) + \log(p_j^*) + \log\left(\sum_{j=1}^{n} \frac{x_{ij}^*}{\sigma_{ij} + x_{ij}^*}\right) \geq \log(w_i), \ \forall i, j; \tag{17}$$

or (14) as

$$\log\left(\sigma_{ij} + x_{ij}^*\right) + \log(p_j^*) + \log\left(\sum_{j=1}^{n} \frac{x_{ij}^*}{\sigma_{ij} + x_{ij}^*}\right) \geq \log(\boldsymbol{w}_i^T \boldsymbol{p}^*), \ \forall i, j. \tag{18}$$

Again, this is a *convex* or *log-convex* inequality. Therefore,

Theorem 7. *Under the FDMA or single-user-single-frequency at any time period policy, the competitive equilibrium of the discretized communication spectrum market or the spectrum trading market is unique, and it can be computed in polynomial time.*

For a single frequency j with multiple users $x_{ij}^* > 0$, one can implement an online or real time policy on tone j by a randomized algorithm by assigning user i with probability $\frac{x_{ij}^*}{\sum_k x_{kj}^*}$ at any time period or when multiple users request services on tone j.

8 An Illustration Example

Consider two channels f_1 and f_2 and two users $\boldsymbol{x}$ and $\boldsymbol{y}$; each of them have a physical power budget $b_1 = b_2 = 1$. Let the Shannon utility function for $\boldsymbol{x}$ be

$$\log(1 + \frac{x_1}{1 + y_1}) + \log(1 + \frac{x_2}{4 + y_2})$$

and one for user $\boldsymbol{y}$ be

$$\log(1+\frac{y_1}{2+x_1})+\log(1+\frac{y_2}{4+x_2});$$

and let the aggregate social utility be the sum of the two individual user utilities.

Clearly, the Nash equilibrium is $x_1 = y_1 = 1$ and $x_2 = y_2 = 0$, where the social utility has value 0.3010.

The cooperative social solution is $x_1 = y_2 = 1$ and $x_2 = y_1 = 0$, where the social utility has value 0.3979, but it is not individually optimal for user $\boldsymbol{y}$.

Now consider a competitive spectrum market with power supply for two channels $s_1 = s_2 = b_1 + b_2 = 2$ and the initial endowments for two users $w_x = w_y = 1$. Then the unique competitive equilibrium point is

$$p_1^* = 3/5 \quad \text{and} \quad p_2^* = 2/5,$$

$$x_1^* = 5/3 \quad \text{and} \quad x_2^* = 0,$$

$$y_1^* = 1/3 \quad \text{and} \quad y_2^* = 2,$$

with the social utility value 0.566. However, the social utility value of the Nash equilibrium with the same physical power budget $b_1 = 5/3$ and $b_2 = 7/3$ has value 0.4657.

Even we simply scale the spectrum allocation for the competitive market equilibrium to

$$x_1^* = 1,\ x_2^* = 0,\ y_1^* = 1/7,\ y_2^* = 6/7$$

Such that each user is allocated exactly 1 unit physical power. Then, this allocation still has a social utility value 0.3775 that is much higher than that at the Nash equilibrium.

Furthermore, if we adjust the initial endowment to $w_x = 6/5$ and $w_y = 4/5$, then the equilibrium price will remain the same and the unique equilibrium allocation will be

$$x_1^* = 2,\ x_2^* = 0,\ y_1^* = 0,\ y_2^* = 2.$$

Upon scaling, we obtain the socially optimal allocation. In other words, the exogenous factors $\boldsymbol{w}$ and $\boldsymbol{s}$ of the spectrum market can be further adjusted to achieve a better social solution while maintaining individual satisfaction under the open market equilibrium price.

Now, consider the spectrum trading market where the initial power endowment for users x and y are given as

$$\boldsymbol{w}_x = \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{w}_y = \frac{1}{3} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

respectively. Thus, we have total one unit power in each tone in the trading market. The unique equilibrium point is

$$p_1^* = 2 \quad \text{and} \quad p_2^* = 1,$$

$$x_1^* = 1 \quad \text{and} \quad x_2^* = 0,$$

$$y_1^* = 0 \quad \text{and} \quad y_2^* = 1.$$

Basically, user x have exchanged its 2/3 unit power in tone 2 for 1/3 unit power in tone 1 of user y. User x's utility value increased from 0.2382 to 0.3010, and user y's utility value increased from 0.0811 to 0.0969.

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