Finding Equitable Convex Partitions of Points in a Polygon Efficiently

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Previous work has developed algorithms for finding an equitable convex partition that partitions the plane into n convex pieces each containing an equal number of red and blue points. Motivated by a vehicle routing heuristic, we look at a related problem where each piece must contain one point and an equal fraction of the area of some convex polygon. We first show how algorithms for solving the older problem lead to approximate solutions for this new equitable convex partition problem. Then we demonstrate a new algorithm that finds an *exact* solution to our problem in $O(Nn \log N)$ time or operations, where n is the number of points, m the number of vertices or edges of the polygon, and $N := n + m$ the sum.

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1. INTRODUCTION

Practical Motivation Our problem can be motivated in the context of dynamic resource allocation or multiple depot vehicle routing. Suppose for example that there are n vehicles servicing customers in a convex region C. The vehicles start at the set of depots P. We want to route the vehicles such that all customers are visited while minimizing the time until the last vehicle returns to its depot [Carlsson et al. 2007].

The key step in dynamic resource allocation problems is to allocate resources to clients in a load-balanced manner. In our vehicle routing example, we allocate to each vehicle the customers it will service. Routing the vehicle from its depot, to its allocated customers, and then back to the depot in the shortest time is a traveling-salesman problem. Since we seek to minimize the maximum travel time, we expect

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the travel times of the vehicles to be roughly equal in an optimal solution.

One heuristic is to find an equitable partition of C and P (one which partitions C into n pieces of equal area each containing one point of P). Then we assign all customers in one part of C to the vehicle starting at the depot in that part. This heuristic is asymptotically optimal when we assume that there are many customers and that their locations are i.i.d. uniformly in C. This is because the following well-known theorem [Beardwood et al. 1959] shows that the length of each vehicle's route is asymptotically proportional to the square root of the area of the region it services and because the equitable partition splits C into regions of equal area.

THEOREM 1.1. *Suppose that p is a point in convex set R and that X_i's are random points i.i.d. uniformly in R. Then the length* $\mathrm{TSP}(p, X_1, \ldots, X_k)$ *of the optimal travelling salesman tour traversing points p and $X_1, \ldots, X_k$ satisfies*

$$k^{-1/2}\,\mathrm{TSP}(p, X_1, \ldots, X_k) \to \alpha\sqrt{\lambda(R)},\ a.s.$$

as $k \to \infty$, where α is some constant and $\lambda(R)$ denotes the area of region R.

Using a *convex* equitable-partition might have some further benefits. We expect that odd-shaped regions take longer to service than more compact regions with the same area. Convex regions also ensure that a vehicle's traveling-salesman tour remains in its service region and may be a robust way to handle some small uncertainty in the customers' exact locations.

Suppose that, instead of using traveling salesman tours to service the clients, we want to connect each of the clients using spanning (or Steiner) trees to some point in P so as to minimize the size of the largest tree. The theorem above holds not only for traveling salesman tours but also for minimum spanning trees and Steiner trees (but with different values of α) [Beardwood et al. 1959; Steele 1988]. Hence asymptotically in the case of many clients, the optimal solution partitions C into pieces of equal area each containing one point in P (as in the above case for vehicle routing).

Problem Definition We say $(R_1, \ldots, R_n)$ is a *convex partition* of some set R if the pieces form a partition of R (i.e., the pieces are disjoint and $R = R_1 \cup \cdots \cup R_n$) and for each piece $i = 1, \ldots, n$ the closure of piece i, $\overline{R_i}$, is convex. Consider two measures μ_1 and μ_2 on the plane, $\mathbb{R}^2$. An *equitable convex partition with respect to μ_1 and μ_2* is a convex partition of the plane into n pieces of equal measure with respect to both μ_1 and μ_2: the pieces (R_i) are disjoint, $\mathbb{R}^2 = R_1 \cup \cdots \cup R_n$, and for each piece $i = 1, \ldots, n$ (i) the closure $\overline{R_i}$ is convex, (ii) $\mu_1(R_i) = 1/n$, and (iii) $\mu_2(R_i) = 1/n$. When $n = 2$ this is the well-studied *ham sandwich problem* [Steinhaus et al. 1938].

Notation We use $|X|$ to denote the cardinality of a set X. For a set $X \subset \mathbb{R}^2$ in the plane, we denote its boundary by ∂X, its complement by X^c, its convex hull by $\mathrm{conv}(X)$, and its Lebesgue measure (i.e., its area) by $\lambda(X)$. We denote the symmetric difference of two sets by $X \triangle Y$. So if X is a half-plane, then ∂X is the line separating X and its complement X^c. For any three distinct points a, b, and c, $\overline{ab}$ is the line segment from a to b; $\overleftrightarrow{ab}$ is the line passing through a and b; $\overrightarrow{ab}$ is the ray originating from a and going through b; and $\triangle abc$ is the triangle with vertices a, b, and c.

Previous work has studied this problem for particular choices of measures μ_1 and μ_2.

Example 1 Theorem 12 of Bespamyatnikh et al. [2000] and theorem 1 of Sakai [2002] show that an equitable convex partition exists when μ_1 and μ_2 are both probability measures with densities:

THEOREM 1.2. *Let ρ_1 and ρ_2 be measurable functions $\mathbb{R}^2 \to [0, \infty)$ with $\int_{\mathbb{R}^2} \rho_i(x)\,\mathrm{d}x = 1$ for $i = 1, 2$. Then for any $n > 0$, there exists a partition of the plane into n convex pieces $R_1, \ldots, R_n$ such that $\int_{R_j} \rho_i(x)\,\mathrm{d}x = 1/n$ for $i = 1, 2$ and $j = 1, \ldots, n$.*

Example 2 Consider the case where $P_1, P_2 \subset \mathbb{R}^2$ are finite sets of points in general position, and μ_1 and μ_2 are defined as $\mu_i(S) := |S \cap P_i| \,/\, |P_i|$ for any set S. Here *general position* means no three points (of $P_1 \cup P_2$) are collinear. Then Ito et al. [1998], theorems 17 and 18 of Bespamyatnikh et al. [2000], and corollary 2 of Sakai [2002] show that an equitable convex partition exists when the number of points in both P_1 and P_2 are a multiple of n (Bespamyatnikh et al. also give an algorithm):

THEOREM 1.3. *Given ng red points and nh blue points in the plane in general position. A subdivision of the plane into n convex polygonal regions each of which contains g red and h blue points can be computed in $O(N^{4/3} \log^3 N \log n)$ time where $N = n(g + h)$.*

Related Problems Previous work has been done on problems closely related to equitable convex partitions. Bereg et al. [2006] and Bárány and Matoušek [2001] relax requirement (i) that the pieces of the partition are convex. Bárány and Matoušek [2001] examine partitions generated by fans (multiple rays emanating from a common point) while Bereg et al. [2006] solve a generalization of example 2: they assume the points are in some (not necessarily convex) polygon and find a partition into pieces that are *relatively*-convex. Kaneko and Kano [2001] go beyond equal partitions (requirements (ii) and (iii)) and ask whether for some given positive $\alpha_1 + \cdots + \alpha_n = 1$ there exists a convex partition such that $\mu_1(R_i) = \mu_2(R_i) = \alpha_i$ for each piece $i = 1, \ldots, n$ (an equitable convex partition sets $\alpha_i = 1/n$). They give some results for the atomic measures in example 2. Kaneko and Kano [2002] solve a variant of the equitable convex partition problem where μ_2 is not a standard measure on $\mathbb{R}^2$: $\mu_1(S)$ measures the area of a convex polygon C inside S and μ_2 measures the perimeter of C inside S. Koutsoupias et al. [1992] give a related result showing how to divide a convex polygon into two pieces with equal area while minimizing the total perimeter of the pieces. Bast and Hert [2000] provide an algorithm for the (less related) problem of partitioning a polygon (not necessarily convex) into connected components with given areas and hopefully small cut-lengths.

Our Problem We consider the equitable convex partition problem where μ_1 is an atomic measure as in example 2 and μ_2 is a probability measure with a density as in example 1. Specifically, consider a convex polygon C with m vertices and a set n of points $P \subset C$ in general position. Polygon C is specified by a list of its vertices in clockwise order. The goal of this paper is to find an equitable convex partition when we define $\mu_1(S) := |S \cap P| \,/\, |P|$ and $\mu_2(S) := \lambda(S \cap C)/\lambda(C)$ for any measurable set S.

Note that the Voronoi diagram of C for the n points satisfies conditions (i) and (ii) for an equitable convex partition (described previously in our Problem Definition paragraph) but not (iii). Hert and Lumelsky [1998], motivated by a terrain-covering problem in robotics, find solutions satisfying (ii) and (iii) but not (i).

Paper Overview In Section 2 we show how our problem can be approximated arbitrarily well using the discrete version of the region partition problem described in example 2. This generates an approximation algorithm for our problem and a theorem guaranteeing the existence of a solution. Section 3 directly constructs an exact region partition. With an approach similar to the one Ito et al. [1998] use for the discrete version of the problem (see example 2), we develop a divide-and-conquer strategy that reduces our problem into two or three smaller problems by partitioning C into two or three convex polygons, respectively.

Our algorithm first determines the approximate location of such a partition by performing some binary searches on the vertices of C or the points P (both of which are finite sets). Once we know how P is partitioned and which edges the partition of C intersects, we solve a one-variable quadratic or linear equation to determine the exact location of the partition.

The benefit of the direct algorithm described in Section 3 is that its only approximation is computing square roots of positive numbers to solve some quadratic equations. Because there exist algorithms that calculate square roots with quadratic convergence, one may say that the ϵ-dependency of our solution time is $\log\log\frac{1}{\epsilon}$ (which is typical for a floating point algorithm). This is much less than the $\epsilon^{-8/3}\log^3\frac{1}{\epsilon}$ dependence of the approximation algorithm described in Section 2.

Section 4 discusses how the algorithm from Section 3 can be extended to deal with the case where μ_2 is a probability measure with a density (instead of the specific case where $\mu_2(S) = \lambda(S \cap C)/\lambda(C)$ for some convex set C). In our vehicle routing example, customers might not be spread uniformly in some convex set but may occur with higher density in some locations than in others. Another example of this extension is the case where we drop the requirement that C be convex but we still define $\mu_2(S) := \lambda(S \cap C)/\lambda(C)$. As an additional example, suppose we seek an equitable convex partition in some higher dimension (i.e., μ_2 and P are defined in $\mathbb{R}^d$ for some $d > 2$). One approach is to seek an equitable convex partition into prisms (i.e., points in $\mathbb{R}^d$ are partitioned based on their first two coordinates). To find such a partition we first project P and μ_2 onto the first two coordinates (or some other plane); the projected measure is defined for any measurable set $S \subseteq \mathbb{R}^2$ as $\tilde{\mu}_2(S) := \mu_2(S \times \mathbb{R}^{d-2})$. Then we find an equitable convex partition for the two dimensional problem.

We conclude in Section 5 with a discussion of open problems.

2. APPROXIMATION AND EXISTENCE

Since the first measure, μ_1, is the same in our problem and in example 2, we can view our problem as a limiting case of example 2 where the second measure from example 2 (counting the fraction of points of P_2 in a given set) tends to the second measure in our problem (measuring the area-fraction of C in a a given set). Generalizing the measure used in example 2, we define for any finite set of points, Q, a measure $\xi(\cdot;Q)$ so that $\xi(S;Q) := |S \cap Q| \,/\, |Q|$ for any set S.

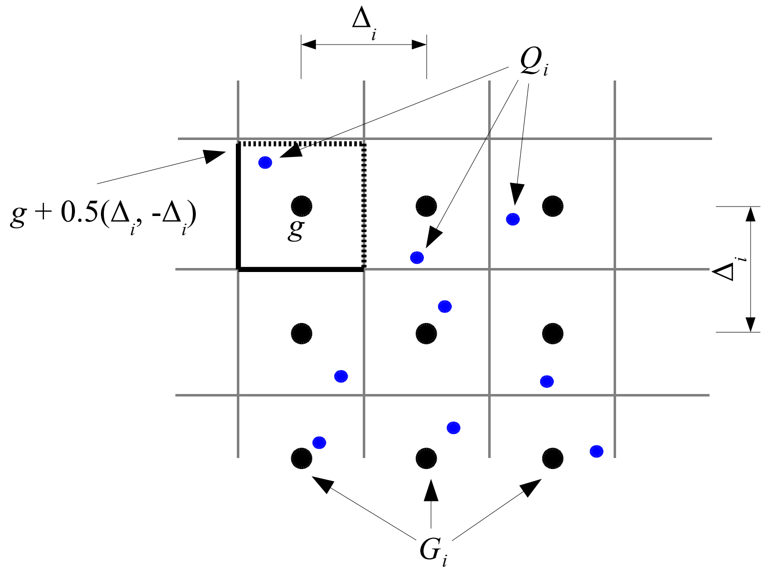


Fig. 1. Points Q_i and grid G_i.

Now consider a sequence (Q_i) of sets of more and more points, $|Q_i| \to \infty$, whose elements are spaced more or less uniformly in C. Assume specifically that there exists a sequence of grids, $(G_i) := (\Delta_i\mathbb{Z} \times \Delta_i\mathbb{Z})$ with decreasing grid sizes, $\Delta_i \searrow 0$, such that for any grid point in C, $g \in G_i \cap C$, there exists exactly one point of Q_i in $g + 0.5\Delta_i[-1, 1) \times 0.5\Delta_i[-1, 1)$ (i.e., closer to that grid point than any other). This is illustrated in Figure 1. Then the atomic measures, ξ, generated by the Q_i converge to the second measure in our problem (measuring the area-fraction).

Lemma 2.1. *For any (Lebesgue) measurable set S, $\xi(S; Q_i) \to \lambda(S \cap C)/\lambda(C)$ as $i \to \infty$. Furthermore if S is convex this convergence is uniform: for $|Q_i| \geq 156 + 437(\mathrm{diam}(C))^2/\lambda(C)$,*

$$|\xi(S; Q_i) - \lambda(S \cap C)/\lambda(C)| \leq 45\, |Q_i|^{-1/2}\, \mathrm{diam}(C)/\sqrt{\lambda(C)}$$

where $\mathrm{diam}(C)$ *is the diameter of set C.*

For the proof we will need the following lemma.

Lemma 2.2. *For $y \geq M_3 + M_4/\beta$, $0 \leq x$, $\beta \leq \pi/4$, and*

$$x \leq \frac{2}{y-4} + \sqrt{\left(\frac{2}{y-4}\right)^2 + \frac{\beta}{y-4}} \tag{1}$$

the following inequalities hold with $M_1 = 2$, $M_2 = 13$, $M_3 = 156$, and $M_4 = 437$

$$x \leq M_1\sqrt{\beta/y}, \quad x^2 \leq (\beta/y)(1 + M_2/\sqrt{\beta y}).$$

Proof of lemma 2.1. We first prove the second claim. Consider a convex set S and the grid G_i. The idea is to relate the area of $S \cap C$ to the tiles of G_i associated with the points $S \cap Q_i$. Suppose $S \cap Q_i$ has a height of a tiles and a width of b tiles. We can bound the area of S (measured in tiles) by expanding or shrinking the set of tiles associated with $S \cap Q_i$:

$$-2(a+b)+4 \leq \lambda(S \cap C)/\Delta_i^2 - |S \cap Q_i| \leq 2(a+b)+4, \tag{2}$$

$$\left|\lambda(S \cap C)/\lambda(C) - (|S \cap Q_i|+4)\Delta_i^2/\lambda(C)\right| \leq 2(a+b)\Delta_i^2/\lambda(C). \tag{3}$$

Since $a, b \leq \operatorname{diam}(C)/\Delta_i + 2$,

$$\left|\lambda(S \cap C)/\lambda(C) - (|S \cap Q_i|+4)\Delta_i^2/\lambda(C)\right| \leq (8\Delta_i^2 + 4\operatorname{diam}(C)\Delta_i)/\lambda(C), \tag{4}$$

$$\left|\lambda(S \cap C)/\lambda(C) - |S \cap Q_i|\,\Delta_i^2/\lambda(C)\right| \leq (12\Delta_i^2 + 4\operatorname{diam}(C)\Delta_i)/\lambda(C). \tag{5}$$

Applying (4) to C and noting that $Q_i \subset C$,

$$(|Q_i|+4)\Delta_i^2 - \lambda(C) \leq 8\Delta_i^2 + 4\operatorname{diam}(C)\Delta_i, \tag{6}$$

$$(|Q_i|-4)\Delta_i^2 - 4\operatorname{diam}(C)\Delta_i - \lambda(C) \leq 0. \tag{7}$$

Since $|Q_i| \geq 4$, it follows that

$$\frac{\Delta_i}{\operatorname{diam}(C)} \leq \frac{2}{|Q_i|-4} + \sqrt{\left(\frac{2}{|Q_i|-4}\right)^2 + \frac{\lambda(C)/(\operatorname{diam}(C))^2}{|Q_i|-4}} \tag{8}$$

and that $\Delta_i \leq \operatorname{diam}(C)$. Applying lemma 2.2 and noting that $\lambda(C)/(\operatorname{diam}(C))^2 \leq \pi/4$ we obtain the identities,

$$\Delta_i \leq M_1\sqrt{\frac{\lambda(C)}{|Q_i|}}, \quad \Delta_i^2 \leq \operatorname{diam}(C) M_1\sqrt{\frac{\lambda(C)}{|Q_i|}}, \quad \Delta_i^2 \leq \frac{\lambda(C)}{|Q_i|}\left(1 + \frac{M_2\operatorname{diam}(C)}{\sqrt{\lambda(C)\,|Q_i|}}\right).$$

Substituting them into (5),

$$\left|\lambda(S \cap C)/\lambda(C) - |S \cap Q_i|\,/\,|Q_i|\left(1 + \frac{M_2\operatorname{diam}(C)}{\sqrt{\lambda(C)\,|Q_i|}}\right)\right| \leq \frac{16M_1\operatorname{diam}(C)}{\sqrt{|Q_i|\,\lambda(C)}}. \tag{9}$$

Using the triangle inequality,

$$|\lambda(S \cap C)/\lambda(C) - |S \cap Q_i|\,/\,|Q_i|| \leq (16M_1 + M_2\,|S \cap Q_i|\,/\,|Q_i|)\frac{\operatorname{diam}(C)}{\sqrt{|Q_i|\,\lambda(C)}}. \tag{10}$$

Since $|S \cap Q_i|\,/\,|Q_i| \leq 1$,

$$|\lambda(S \cap C)/\lambda(C) - |S \cap Q_i|\,/\,|Q_i|| \leq (16M_1 + M_2)\operatorname{diam}(C)/\sqrt{|Q_i|\,\lambda(C)}. \tag{11}$$

Substituting the values for M_1 and M_2 from lemma 2.2 proves the second claim.

Having proved the second claim, we know that $\xi(S; Q_i) \to \lambda(S \cap C)/\lambda(C)$ for convex sets S. The first claim follows from the fact that the Lebesgue measurable sets are the completion of the sets generated by convex sets. □

PROOF OF LEMMA 2.2. We first show that the second identity implies the first. Taking the square root of both sides of the second identity gives $x \leq \sqrt{\beta/y}\sqrt{1 + M_2/\sqrt{\beta y}}$. Since $y \geq M_4/\beta$, $1/\sqrt{\beta y} \leq 1/\sqrt{M_4}$ and hence $x \leq \sqrt{\beta/y}\sqrt{1 + M_2/\sqrt{M_4}}$. Substituting our values for M_2 and M_4 proves the first identity.

To prove the second identity we first square both sides of (1),

$$x^2 \le \frac{8}{(y-4)^2} + \frac{\beta}{y-4} + \frac{4}{y-4}\sqrt{\left(\frac{2}{y-4}\right)^2 + \frac{\beta}{y-4}} \tag{12}$$

$$= \frac{1}{y-4}\left(\beta + \frac{8}{y-4} + 4\sqrt{\left(\frac{2}{y-4}\right)^2 + \frac{\beta}{y-4}}\right). \tag{13}$$

Since $\sqrt{\cdot}$ is subadditive, $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$. Hence

$$x^2 \le \frac{1}{y-4}\left(\beta + \frac{16}{y-4} + \frac{4\sqrt{\beta}}{\sqrt{y-4}}\right) = \frac{\beta}{y-4}\left(1 + \frac{4}{\sqrt{\beta(y-4)}}\left(1 + \frac{4}{\sqrt{\beta(y-4)}}\right)\right). \tag{14}$$

Since $M_3 \ge 4$, $y - 4 \ge M_4/\beta$ and hence $1/\sqrt{\beta(y-4)} \le 1/\sqrt{M_4}$. Therefore,

$$x^2 \le \frac{\beta}{y-4}\left(1 + \frac{4(1+4/\sqrt{M_4})}{\sqrt{\beta(y-4)}}\right) = \frac{\beta}{y}\left(1 + \frac{4}{y-4}\right)\left(1 + \frac{4(1+4/\sqrt{M_4})}{\sqrt{\beta(y-4)}}\right) \tag{15}$$

$$= \frac{\beta}{y}\left(1 + \frac{4(1+4/\sqrt{M_4})}{\sqrt{\beta(y-4)}}\left(1 + \frac{\beta}{(1+4/\sqrt{M_4})\sqrt{\beta(y-4)}} + \frac{4}{y-4}\right)\right). \tag{16}$$

Because $\beta \le \pi/4$ and $y - 4 \ge M_4/\beta$ it follows that $4/(y-4) \le \pi/M_4$ and $\beta/\sqrt{\beta(y-4)} \le \pi/(4\sqrt{M_4})$. This allows us to bound the term in the inner parentheses,

$$x^2 \le \frac{\beta}{y}\left(1 + \frac{4(1+4/\sqrt{M_4})}{\sqrt{\beta(y-4)}}\left(1 + \frac{\pi}{(1+4/\sqrt{M_4})4\sqrt{M_4}} + \frac{\pi}{M_4}\right)\right). \tag{17}$$

Since $y \ge M_3 \ge 8$, $1/(y-4) \le 2/y$. Therefore,

$$x^2 \le \frac{\beta}{y}\left(1 + \frac{8(1+4/\sqrt{M_4})}{\sqrt{\beta y}}\left(1 + \frac{\pi}{(1+4/\sqrt{M_4})4\sqrt{M_4}} + \frac{\pi}{M_4}\right)\right). \tag{18}$$

Substituting our value for M_4 proves the identity. □

An Approximation Algorithm We can find an approximate solution to our problem by applying the theorem from example 2, theorem 1.3, taking the first set of points (the red ones) as our points, $P_1 := P$, and the second set of points (the blue ones), P_2, as a set of hn points in general position that are spaced more or less uniformly in C (in the sense described above). Here h is some positive integer. Theorem 1.3 generates a partition of C into n convex pieces, $(C_1, \dots, C_n)$, each containing one point of P and h points of P_2. We then use lemma 2.1 to bound the area of each piece. For each piece i, $\xi(C_i; P_2) = 1/n$, and hence for large enough h, $|\lambda(C_i) - \lambda(C)/n| \le 45(hn)^{-1/2}\,\mathrm{diam}(C)\sqrt{\lambda(C)}$ or equivalently in terms of our second measure $|\mu_2(C_i) - 1/n| \le 45(hn)^{-1/2}\,\mathrm{diam}(C)/\sqrt{\lambda(C)}$.

Suppose we are seeking an ϵ-approximate equitable convex partition for our problem; that is, a partition of C into n convex pieces where the second measure of each piece i has a relative error of size ϵ: $|\mu_2(C_i) - 1/n| \le \epsilon/n$ or written in terms of areas

$|\lambda(C_i) - \lambda(C)/n| \leq \epsilon\lambda(C)/n$. We can obtain an ϵ-approximate solution if we implement the above approach taking $h \geq \epsilon^{-2} n 45^2 (\mathrm{diam}(C))^2/\lambda(C)$. The total number of points used in theorem 1.3 is $N = n + hn = n(1 + \epsilon^{-2} 45^2 (\mathrm{diam}(C))^2/\lambda(C))$. Substituting into the complexity bound from theorem 1.3 we conclude that our algorithm takes

$$\begin{aligned} &O(N^{4/3} \log^3 N \log n) \\ &= O(n^{4/3} \epsilon^{-8/3} (\mathrm{diam}(C))^{8/3} (\lambda(C))^{-4/3} \log^3 [n\epsilon^{-2} (\mathrm{diam}(C))^2/\lambda(C)] \log n) \end{aligned} \quad (19)$$

time to find an ϵ-approximate solution to our problem.

Existence The above approximation algorithm lets us find an ϵ-approximate solution for arbitrarily small ϵ. With a little more work, we can show that a sequence of ϵ-approximate solutions converges as $\epsilon \searrow 0$ and hence show the *existence* of an exact equitable convex partition for our problem.

Theorem 2.3. *There exists an exact solution for our problem.*

Proof Proof Sketch. Let V be the set of convex partitions of C with n pieces and $W := \{(R_1, \ldots, R_n) \in V : p_i \in R_i\}$ the set of convex partitions such that piece i contains point p_i. Now define a function $u : V \to \mathbb{R}$ on the set of convex partitions measuring the quality of approximate solutions, $u(R) := \max_{j=1}^n |\lambda(R_j)/\lambda(C) - 1/n|$. Then we can generate using the above approximation algorithm a sequence of approximate solutions, (C^i), in W of better and better quality, $u(C^i) \to 0$.

Each edge in a convex partition in V is shared by two pieces of the partition, but due to convexity each pair of pieces shares at most one edge. Hence each partition in V can be described as a collection of at most $n(n-1)/2$ edges. By explicitly parameterizing these edges we can construct for some d a function $h : \mathbb{R}^d \to V$ mapping some subset X of $\mathbb{R}^d$ to the set of convex partitions giving approximate solutions, $h(X) = W$. Hence there exists a sequence (x_i) in X such that $h(x_i) = C^i$. Then we will show that X is compact implying that the sequence (x_i) has a cluster point $x^* \in X$. Finally we will show that the function $\lambda \circ h_i$ taking the area of the ith piece of $h(x) := (h_1(x), \ldots, h_n(x))$ is continuous. Hence $u \circ h$ is continuous, implying that $u \circ h(x^*) = 0$ and that $C^* = h(x^*)$ is an exact solution to our problem. □

3. AN EXACT ALGORITHM

The $\epsilon^{-8/3}$ dependency of the approximation algorithm described in the previous section leads to the question: is there a polynomial-time exact algorithm solving our problem? We construct such an algorithm in this section.

Let $N := m + n$, and let $(v_1, \ldots, v_m)$ be the vertices of C listed in clockwise order. Our main claim is that

Theorem 3.1. *We can find an equitable convex partition of C and P in $O(nN \log N)$ time.*

To prove this claim we need to introduce the concept of an *equitable k-partition*. An equitable k-partition is a partition of C into k regions $C = C_1 \cup C_2 \cup \cdots \cup C_k$ so that for $i = 1, \ldots, k$, the closure $\overline{C_i}$ is a convex polygon specified by its vertices in clockwise order; and the area of each region is proportional to the number of points of P it contains, $\lambda(C_i)/\lambda(C) = |P \cap C_i| / |P|$. A *ham sandwich cut* is a line that bisects the area of C and has half the points of P on either side. So when n is even, it creates an equitable 2-partition. Let $q := \lfloor n/2 \rfloor$, and let $\mathcal{H}(\ell, x)$ denote the open half-plane containing point x and whose boundary is line ℓ.

LEMMA 3.2. *If n is even, we can find an equitable 2-partition of C and P (specifically a ham sandwich cut) in $O(N \log N)$ time.*

LEMMA 3.3. *If n is odd, we can find in $O(N \log N)$ time either an equitable 2-partition, or an equitable 3-partition where one polygon contains 1 point and the other two polygons contain q points.*

We are now ready to prove our main claim.

PROOF OF THEOREM 3.1. The idea is to apply lemmas 3.2 and 3.3 recursively. Applying either lemma to a polygon of k vertices specified in clockwise order creates two or three polygons each with at most $k + 2$ vertices specified in clockwise order. We need at most $n - 1$ applications of these lemmas. Hence the polygons so generated will have at most $m + 2n - 2$ vertices.

Note that both lemmas give $O(N \log N)$ complexity bounds when applied to C and P. So the complexity of any application of either lemma while we search for an equitable convex partition is at most $O((n+m+2n-2)\log(n+m+2n-2))$ or $KN \log N$ for some absolute constant K. Since there are at most $n-1$ applications of these lemmas, the total complexity for finding an equitable convex partition is $O(nN \log N)$. □

We state lemma 3.4 without proof. The claims in lemma 3.4 follow from the fact that C is convex and that its vertices are given in clockwise order.

LEMMA 3.4. (*1*) *In $O(m)$ time we can calculate the area of C.*

(*2*) *Given $\alpha \in (0, 1)$ and a point v on ∂C, we can find in $O(m)$ time the unique point $u \neq v$ on ∂C such that the area in C to the left of $\overrightarrow{vu}$ is $\alpha\lambda(C)$.*

(*3*) *Given half-planes H_1 and H_2 we can specify the vertices of the convex polygon $C \cap H_1 \cap H_2$ in $O(m)$ time.*

(*4*) *Given line ℓ we can compute its intersection with the boundary of C in $O(m)$ time.*

LEMMA 3.5. *Consider two half-planes, H_1 and H_2, whose boundaries go through $x \in C$ and whose intersections with C have areas α_1 and α_2, respectively: $\alpha_1 := \lambda(C \cap H_1)$ and $\alpha_2 := \lambda(C \cap H_2)$. Then for any $\alpha \in [\alpha_1, \alpha_2]$ we can find in $O(m)$ time a half-plane H^* between H_1 and H_2 (i.e., $\partial H^* \subset (H_1 \triangle H_2) \cup \{x\}$) cutting-off an area $\alpha = \lambda(C \cap H^*)$. If the boundaries of the half-planes intersect the same pair of edges, then we obtain a solution in $O(1)$ time.*

PROOF. Let $f := \partial H_1 \cap \partial C \cap H_2$ be the point where half-plane H_1 intersects C, and let $(v_1, \ldots, v_j)$ and $(w_1, \ldots, w_k)$ be lists in clockwise order of the vertices of C in $H_2 \setminus H_1$ and $H_1 \setminus H_2$ respectively. For each point v in either list, consider the half-plane $\mathcal{H}(\overleftrightarrow{xv}, f)$. Then sort these half-planes by angle into the list $(S_0 := H_1, S_1, \ldots, S_{j+k}, S_{j+k+1} := H_2)$. This can be done in $O(m)$ time because the lists

$(v_1, \dots, v_j)$ and $(w_1, \dots, w_k)$ are already sorted. Note that for $i = 0, \dots, j+k$, $C \cap S_i \setminus S_{i+1}$ and $C \cap S_{i+1} \setminus S_i$ are just two triangles, and hence their areas (A_i^- and A_i^+ respectively) can be computed in $O(1)$ time. Since $\lambda(C \cap S_0) = \alpha_1$ we now iteratively compute $B_i := \lambda(C \cap S_i)$ for all $i = 0, \dots, j+k+1$ in $O(m)$ time. Since $B_0 \le \alpha \le B_{j+k+1}$, there exists i^* such that $\lambda(C \cap S_{i^*}) = B_{i^*} \le \alpha \le B_{i^*+1} = \lambda(C \cap S_{i^*+1})$. Figure 2 illustrates this procedure.

If $B_{i^*} = \alpha$ then $H^* := S_{i^*}$ and we are done. In the remainder of the proof assume $B_{i^*} < \alpha$. The points of intersection $\partial S_{i^*} \cap \partial C$ and $\partial S_{i^*+1} \cap \partial C$ lie on the same pair of edges of C and form a convex quadrilateral Q whose diagonals intersect at x. Let $y_1 y_2 y_3 y_4$ be the vertices of Q listed clockwise such that $y_1 = \partial S_{i^*} \cap \partial C \cap S_{i^*+1}$. Our candidate solution is the half-plane $H^* := \mathcal{H}(\overleftrightarrow{rx}, f)$ where $r := (1-\beta_1)y_1 + \beta_1 y_2$, $\beta_1 \in (0,1]$ is a point on $\overline{y_1 y_2}$. The line $\overleftrightarrow{rx}$ intersects $\overline{y_3 y_4}$ at $(1-\beta_2)y_3 + \beta_2 y_4$ where $\beta_2 = \frac{(1+K)\beta_1}{K\beta_1+1}$ and $K > -1$ depends on the vertices of Q. The areas $\lambda(C \cap H^* \setminus S_{i^*}) = \beta_1 A_{i^*}^+$ and $\lambda(C \cap S_{i^*} \setminus H^*) = \beta_2 A_{i^*}^-$. Solving $\lambda(C \cap H^*) = \alpha$ involves finding β_1 solving

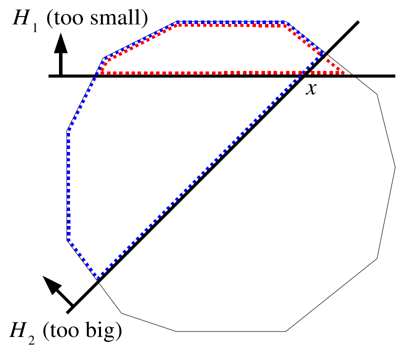
$$\alpha = B_{i^*} + \beta_1 A_{i^*}^+ - \frac{(1+K)\beta_1}{K\beta_1 + 1} A_{i^*}^-,$$
$$0 = f(\beta_1) := (\beta_1 A_{i^*}^+ - \alpha + B_{i^*})(K\beta_1 + 1) - (1+K)\beta_1 A_{i^*}^-. \qquad (20)$$

This equation has a unique solution in $(0,1]$ because $f(\cdot)$ is a quadratic equation with $f(0) = -\alpha + B_{i^*} < 0 \le (B_{i^*} + A_{i^*}^+ - A_{i^*}^- - \alpha)(1+K) = f(1)$. □

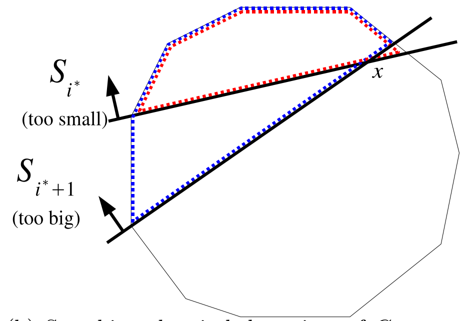
LEMMA 3.6. *We are given half-plane H containing $k \le n/2$ points, $k := |H \cap P|$, and an area too small, $\lambda(H \cap C) \le \lambda(C)k/n$. Then we can find in $O(N \log N)$ time an equitable 2-partition of C and P.*

PROOF. Find the intersections of ∂H with ∂C in $O(m)$ time. Label the points of intersection f and g such that H is to the left of $\overrightarrow{fg}$. We compute in $O(m)$ time the half-plane H_g whose boundary goes through g cutting-off an area of exactly $\lambda(H_g \cap C) = \lambda(C)k/n$ and with an orientation opposite of H (i.e., H_g does not contain point f). Since $k \le n/2$ and $\lambda(H \cap C) \le \lambda(C)k/n \le \lambda(C)/2$, it follows that $H \cap C \subset C \setminus H_g$.

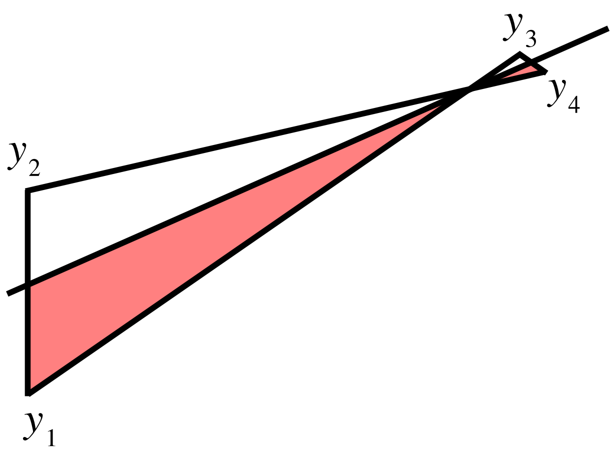
Case 1 H_g contains more than k points, $|H_g \cap P| > k$. Since $\left|H_g^c \cap P\right| / \lambda(H_g^c \cap C) \le n/\lambda(C) \le |H \cap P| / \lambda(H \cap C)$ we will attempt to find a half-plane between H and H_g^c creating an equitable 2-partition. For any point between the half-planes $p \in P \cap H_g^c \cap H^c$ calculate the angle $\angle fgp$. In $O(n \log n)$ time, sort these points by increasing angle into a list $(p_1, \dots, p_j)$. From this list we construct a sequence of half-planes, $J_0 := H$, $J_i := \mathcal{H}(\overleftrightarrow{gp_i}, f)$ for $i = 1, \dots, j$, and $J_{j+1} := H_g^c$. Note $J_i \cap C \subseteq J_{i'} \cap C$ for $i \le i'$. Now we perform a binary search to find i^* such that J_{i^*} cuts off too small an area and J_{i^*+1} cuts off too large an area, $\lambda(J_{i^*} \cap C)/(k+i^*) \le \lambda(C)/n \le \lambda(J_{i^*+1} \cap C)/(k+i^*+1)$. This takes $O(m \log n)$ time: $O(\log n)$ steps with each step taking $O(m)$ time to find the area of $J_i \cap C$. By construction there are no points of P between these two half-planes. We now in $O(m)$ time find through g a half-plane H^* between them that cuts off an area $\lambda(H^* \cap C) = \lambda(C)(k+i^*)/n$. Since half-plane H^* contains $k + i^*$ points, the polygons $C \cap H^*$ and $C \setminus H^*$ form an equitable 2-partition. We then specify the vertices of the two polygons in $O(m)$ time. Figure 3 illustrates this case.



(a) One half-plane cuts off an area that is too big while the other cuts off an area that is too small.



(b) Searching the circled vertices of C we reduce the problem to an hourglass figure.



(c) Solving a quadratic equation gives us the correct half-plane.

Fig. 2. Lemma 3.5.

Case 2 H_g contains at most k points, $|H_g \cap P| \leq k$. In this case, we compute in $O(m)$ time the half-plane H_f whose boundary goes through f that contains g and cuts off an area of exactly $\lambda(H_f \cap C) = \lambda(C)k/n$. Since $H \cap C \subseteq H_f$, we know that H_f contains at least k points. We assume that H_f contains more than k points, since otherwise we have found an equitable 2-partition.

For any point $v \in \partial C$ let H_v be the half-plane whose boundary goes through v that contains g and cuts off an area of exactly $\lambda(H_v \cap C) = \lambda(C)k/n$. Without loss of generality suppose $(v_1, \ldots, v_j)$ are the vertices of C between f and g traversed clockwise. Based on this list, we define a sequence of half-planes each cutting-off an area of $\lambda(C)k/n$, $J_0 := H_f$, $J_i := H_{v_i}$ for $i = 1, \ldots, j$, and $J_{j+1} := H_g$. Since H_f contains more than and H_g less than k points we can perform a binary search to determine in $O(\log m)$ steps i^* such that half-plane J_{i^*} contains at least k points and half-plane J_{i^*+1} contains at most k points. Since each step takes $O(n + m)$

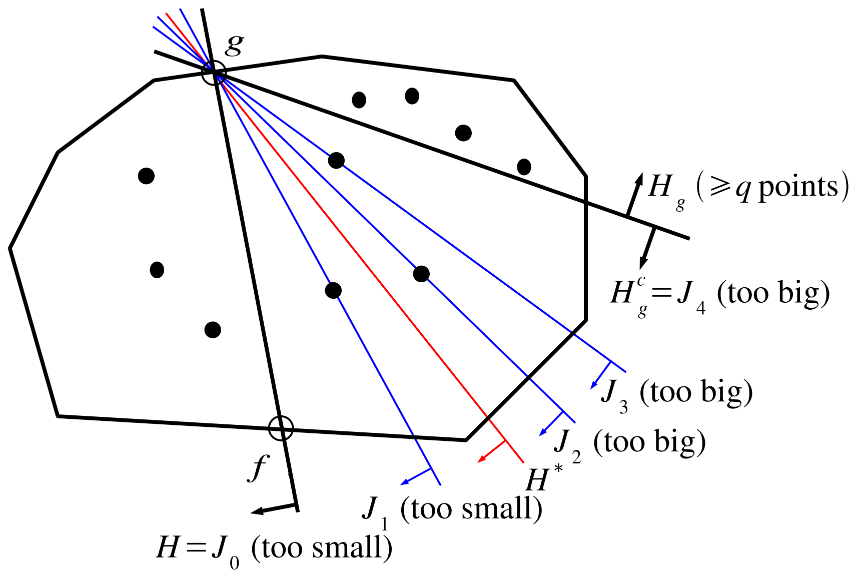
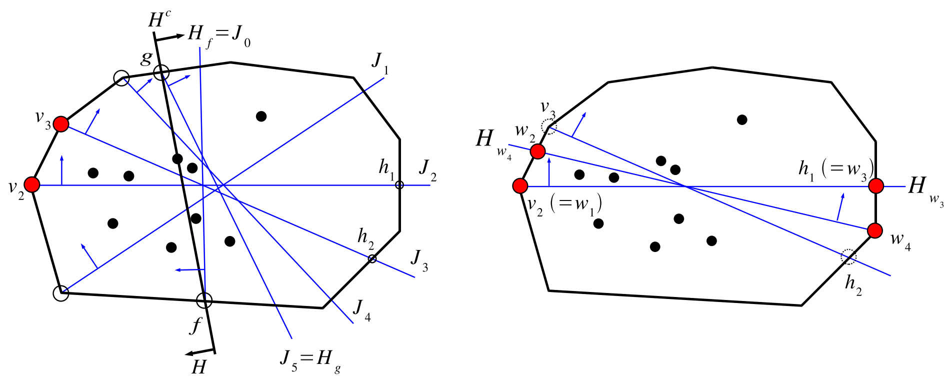


Fig. 3. In case 1 we search for a half-plane H^* through g creating an equitable 2-partition.



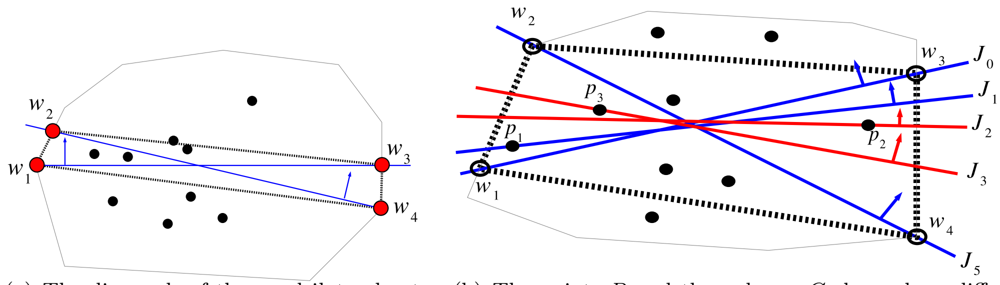
(a) Here $J_{i^*} = J_2$ cuts off at least and $J_{i^*+1} = J_3$ cuts off at most k points.

(b) We isolate w_3 and w_4 and label w_1 the dotted circle on ∂H_{w_4}.

Fig. 4. In case 2 we perform a binary search along the edges of C. In this example $i^* = 2$.

time ($O(m)$ time to construct J_i and $O(n)$ time to counts the number of points it contains) the binary search takes $O(N \log m)$ time. Figure 4(a) illustrates this binary search.

Let h_1 and h_2 be the intersections of ∂J_{i^*} and J_{i^*+1} respectively with the other side of ∂C. Now we apply a similar binary search to the vertices between h_1 and h_2 to isolate two points w_3 and w_4 sharing an edge (with w_3 closer to h_1 and w_4 closer to h_2). Finally, let H_{w_3} and H_{w_4} denote the half-planes whose boundaries go through w_3 and w_4 respectively that cut off an area of $\lambda(C)k/n$ and contain point g. This is illustrated by Figure 4(b).



(a) The diagonals of the quadrilateral cut off $\lambda(C)k/n$. Notice that H_{w_1} contains at least k while H_{w_2} contains at most k points.

(b) The points P and the polygon C shown here differ slightly from the previous figures. Notice that J_0 contains more than k points while J_5 contains fewer than k. Half-planes between J_2 and J_3 (shown in red) contain k points and solve our problem.

Fig. 5. The problem is reduced to a quadrilateral.

By construction, both ∂H_{w_3} and ∂H_{w_4} intersect edge $\overline{v_{i^*}v_{i^*+1}}$. Let w_1 and w_2 denote these two intersection points, respectively; so that $w_1w_2w_3w_4$ forms a quadrilateral Q whose diagonals w_1w_3 and w_2w_4 cut off an area $\lambda(C)k/n$ of C. Furthermore, H_{w_1} contains at least while H_{w_2} contains at most k points. See Figure 5(a) for a diagram.

We could continue the bisection procedure along the interval between w_1 and w_2 to find a point w_0 such that H_{w_0} contains k points and cuts off approximately an area $\lambda(C)k/n$ — essentially an equitable 2-partition. Instead we show how this can be done exactly.

Let p^* be a point such that $\mathcal{H}(\overleftrightarrow{p^*v}, g) \cap C$ has an area of $\lambda(C)k/n$ for any $v \in \overline{w_1w_2}$. The point p^* is unique (if it exists), because it must lie on the boundary of H_v for all $v \in \overline{w_1w_2}$ and these lines are all different.

Note that for $p \in Q$, $\mathcal{H}(\overleftrightarrow{pv}, g)$ with $v = (1-\alpha)w_1 + \alpha w_2$ and $0 \le \alpha \le 1$ is a half-plane cutting off an area $\lambda(C)k/n$ iff α solves the quadratic equation $a_2(Q,p)\alpha^2 + a_1(Q,p)\alpha + a_0(Q,p) = 0$ where the $a_i(\cdot)$ are functions of p and the vertices of Q that can be evaluated in $O(1)$ time. Let $P^* := P \cap Q \setminus \{p^*\}$, and for $p \in P^*$, let H_p and H'_p be those half-planes if they exist.

Let $\mathcal{V}$ be the union of H_{w_1}, H_{w_2}, and H_p and H'_p (when they exist) for $p \in P^*$. This set, $\mathcal{V}$ contains at most $2n+2$ half-planes and can be computed in $O(n)$ time. We then sort $\mathcal{V}$ in $O(n \log n)$ time by the size of their intersection with $\overline{w_1w_2}$ into a sequence of half-planes, $(J_1, \ldots, J_j)$. Since the points of P are in general position, we assume that the sets H_p and H'_p (if they exist) are distinct for different $p \in P^*$. Therefore, define p_i to be the unique point from P^* on the boundary of J_i. It turns out that for $i = 2, \ldots, j$, $|J_i \cap P| - |J_{i-1} \cap P|$ equals -1 if $p_{i-1} \notin J_i$ and equals $+1$ if $p_{i-1} \in \cap J_i$. Since $|J_0 \cap P| \ge k \ge |J_j \cap P|$ (noting that $J_0 = H_{w_1}$ and $J_j = H_{w_2}$) and the number of points in consecutive J_i differs by ± 1, we can just iterate through $1, \ldots, j$ in $O(n)$ time to find i^* such that J_{i^*} contains exactly k points. Figure 5(b) illustrate this procedure. The polygons $C \cap J_{i^*}$ and $C \setminus J_{i^*}$ form an equitable 2-partition whose vertices can be specified in $O(m)$ time.

Now we merely need to justify the claim that for $i = 2, \ldots, j$, $|J_i \cap P| - |J_{i-1} \cap P|$ equals -1 if $p_{i-1} \notin J_i$ and equals $+1$ if $p_{i-1} \in \cap J_i$. Equivalently we need to show that there are no points of P in the interior of the triangles $J_i \triangle J_{i-1}$. Clearly p^* is not in the interior of $J_i \triangle J_{i-1}$ because (if it exists) it is on the boundary of both J_i and J_{i-1}. We will later show that $(J_i \triangle J_{i-1}) \cap C \subseteq Q$. Suppose by contradiction that $p \in P^*$ is a point in the interior of $J_i \triangle J_{i-1}$. Note that $\lambda(\mathcal{H}(\overleftrightarrow{pu_i}, g) \cap C) \leq \lambda(C)k/n \leq \lambda(\mathcal{H}(\overleftrightarrow{pu_{i-1}}, g) \cap C)$. Then by lemma 3.5, H_p exists and should be between J_{i-1} and J_i in $\mathcal{V}^*$. This contradicts our sorting of $\mathcal{V}^*$.

Note that the boundaries of J_i and J_{i-1} intersect $\overline{w_1 w_2}$ by definition. Hence to show that $(J_i \triangle J_{i-1}) \cap C \subseteq Q$, it suffices to show that for any $v \in \overline{w_1 w_2}$, the boundary of H_v contains no points from $C \setminus Q$. Note that for any $v \in \overline{w_1 w_2}$, $\lambda(\mathcal{H}(\overleftrightarrow{vw_3}, g) \cap C) \leq \lambda(C)k/n \leq \lambda(\mathcal{H}(\overleftrightarrow{vw_4}, g) \cap C)$. Note that H_v is unique and has a boundary that intersects $\overline{w_3 w_4}$. This completes the proof. □

Proof of lemma 3.2. Let $(p_1, \ldots, p_n)$ be the points P sorted by their y-coordinates. Using a *selection algorithm* we can find in $O(n)$ time the two median points of the list, p_q and p_{q+1}. Then let H be a half-plane with a horizontal boundary that separates these two points: $|H \cap P| = q$ and $|H^c \cap P| = q$. We calculate in $O(m)$ time the area of the fraction of C cutoff by H, $\alpha := \lambda(H \cap C)/\lambda(C)$.

If this fraction is half, $\alpha = 1/2$, then the half-plane boundary ∂H is a ham sandwich cut. The convex polygons $C \cap H$ and $C \setminus H$ form an equitable 2-partition of C and P. We finish by specifying the vertices of these two polygons in $O(m)$ time.

If this is not the case then either H or H^c will cut off an area too small, $\lambda(H \cap C) < \lambda(C)/2$ or $\lambda(H^c \cap C) < \lambda(C)/2$. In that case we apply lemma 3.6 to obtain an equitable 2-partition. This equitable 2-partition is a ham sandwich cut because we end up in case 2 of the proof of that lemma. □

Proof of lemma 3.3. Pick a vertex u of conv(P) in $O(n)$ time (e.g., just take the point with the greatest y-coordinate). Next we use a *selection algorithm* to construct in $O(n)$ time a line ℓ through u such that q points lie to either side of ℓ.

Let ℓ_L be the half-plane to the left and ℓ_R the half-plane to the right of line ℓ. Label the resulting halves of C, $L := \ell_L \cap C$ on the left and $R := \ell_R \cap C$ on the right. Find the upper point of intersection of line ℓ and the boundary of C in $O(m)$ time and label the point of intersection b as shown in Figure 6. We can find the area of L and R in $O(m)$ time. We check that $\lambda(L)/\lambda(C), \lambda(R)/\lambda(C) \in [\frac{q}{n}, \frac{q+1}{n}]$.

If this is not the case, then, from $\lambda(L) + \lambda(R) = \lambda(C)$ we have $\min\{\lambda(L), \lambda(R)\} < \lambda(C)q/n$. Thus, using lemma 3.6 (with $k = q$) we can find in $O(N \log N)$ time an equitable 2-partition. We thus assume for the remainder of the proof that $\lambda(L)/\lambda(C), \lambda(R)/\lambda(C) \in [\frac{q}{n}, \frac{q+1}{n}]$.

In $O(n)$ time we find the vertices of conv(P) adjacent to u, $u_L \in L$ and $u_R \in R$. Define half-planes $H_L := \mathcal{H}(\overleftrightarrow{uu_L}, b)$ and $H_R := \mathcal{H}(\overleftrightarrow{uu_L}, b)$. This setup is illustrated again in Figure 6. Calculate in $O(m)$ time the area cutoff by these half-planes, $\lambda(H_L \cap C)$ and $\lambda(H_R \cap C)$. If either area is less than or equal to $\lambda(C)/n$, then we apply lemma 3.6 (with $k = 1$) to find an equitable 2-partition. For the remainder of this proof assume that both areas are strictly greater than $\lambda(C)/n$.

Calculate in $O(m)$ time the area of L and R excluding the two half-planes H_L and H_R: that is $\lambda(L \setminus H_L)$, $\lambda(L \setminus H_R)$, $\lambda(R \setminus H_L)$, and $\lambda(R \setminus H_R)$. We now examine

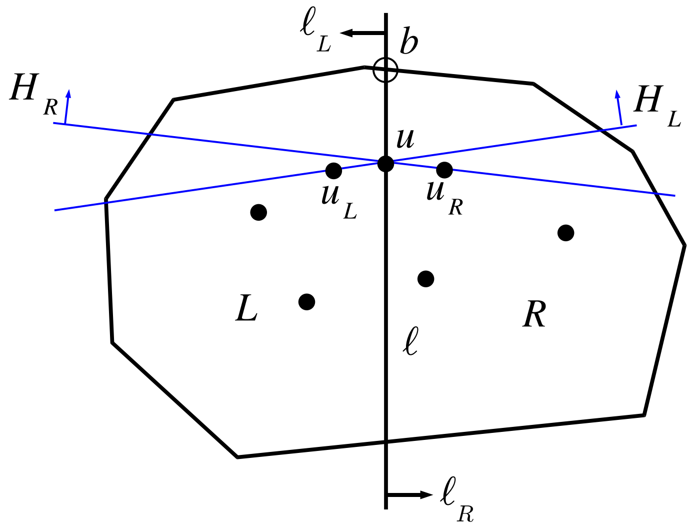


Fig. 6. There are q points on either side of ℓ and the areas cutoff by H_L and H_R are greater than $\lambda(C)/n$.

three possible cases as we compare these areas to $\lambda(C)q/n$.

Case 1 Suppose $\lambda(L \setminus H_L), \lambda(R \setminus H_L) < \lambda(C)q/n$ or $\lambda(L \setminus H_R), \lambda(R \setminus H_R) < \lambda(C)q/n$. Without loss of generality, assume that the first case is true. We are in a situation where H_L^c and ℓ_L are half-planes whose boundaries intersect u and the areas they cut off bracket $\lambda(C)q/n$: $\lambda(L \cap H_L^c) < \lambda(C)q/n \leq \lambda(L \cap \ell_L) = \lambda(L)$. We then find in $O(m)$ time a half-plane J_L through u cutting-off the desired area, $\lambda(L \cap J_L) = \lambda(C)q/n$. Since J_L is between H_L^c and ℓ_L, $J_L \cap L \supset H_L^c \cap L$ and hence $L \cap J_L$ contains q points (not counting u). This is illustrated by the left picture of Figure 7.

The situation with R is symmetric: the boundaries of half-planes H_L^c and ℓ_R intersect u and and $\lambda(R \cap H_L^c) < \lambda(C)q/n \leq \lambda(R \cap \ell_R) = \lambda(R)$. We then find in $O(m)$ time a half-plane J_R through u cutting-off the desired area, $\lambda(R \cap J_R) = \lambda(C)q/n$ and containing the necessary points, $|P \cap R \cap J_R| = q$ (not counting u). This is illustrated by the right picture of Figure 7. The convex polygons $L \cap J_L$ and $R \cap J_R$ containing q points each, and $C \setminus (J_L \cup J_R)$ containing point u form an equitable 3-partition of C. We can specify the vertices of these polygons in $O(m)$ time.

Case 2 Suppose $\lambda(R \setminus H_R) < \lambda(C)q/n \leq \lambda(R \setminus H_L)$ or $\lambda(L \setminus H_L) < \lambda(C)q/n \leq \lambda(L \setminus H_R)$. Without loss of generality assume the above holds for R. We now find in $O(m)$ time a half-plane S_R through u cutting-off the desired area, $\lambda(R \cap S_R) = \lambda(C)q/n$. Since S_R is between H_L and H_R it contains all the points, $|P \cap S_R| = n$. If $\lambda(C \setminus S_R) < \lambda(C)/n$, then apply lemma 3.6 (with $k = 1$) as above. For the remainder of this proof assume the opposite.

The first polygon of our equitable 3-partition will be $R \cap S_R$ and contain q points (not including u). This is shown in Figure 8(a). It follows that $\lambda(L \cap S_R) \leq \lambda(C)q/n$, since $C = (R \cap S_R) \cup (C \setminus S_R) \cup (L \cap S_R)$ and $\lambda(C \setminus S_R) \geq \lambda(C)/n$ and

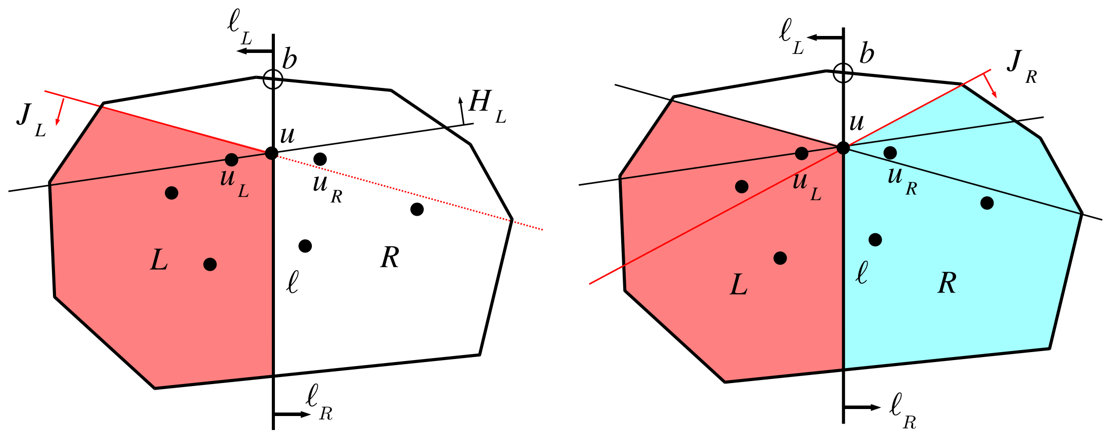
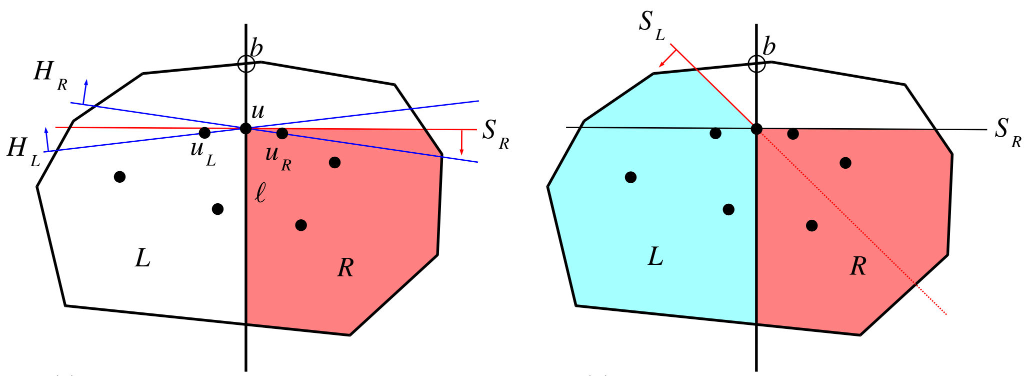


Fig. 7. Finding half-planes J_L and J_R in case 1 such that the areas of $J_L \cap L$ and $J_R \cap R$ are both $\lambda(C)q/n$.



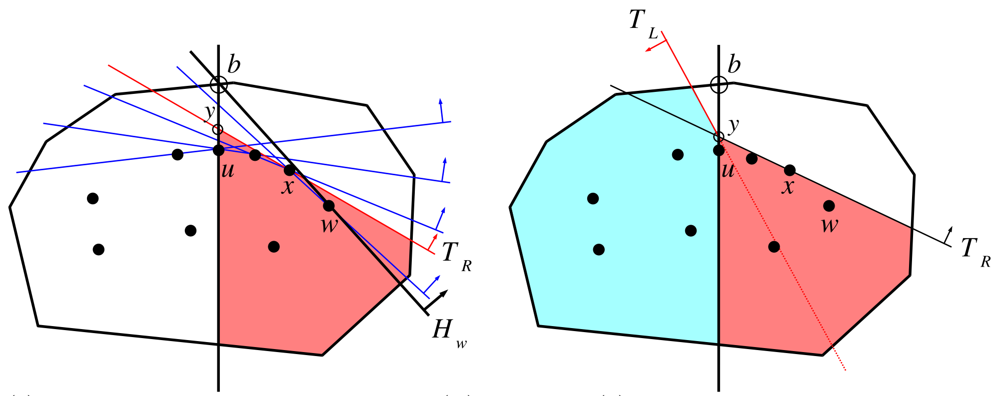
(a) First choose S_R between H_L^c and H_R^c. (b) Then choose S_L between S_R and ℓ_L.

Fig. 8. Choosing S_L and S_R in case 2 such that the areas of $S_L \cap L$ and $S_R \cap R$ are both $\lambda(C)q/n$.

$\lambda(R \cap S_R) = \lambda(C)q/n$. As above we are in a situation where $\lambda(L \cap S_R) \leq \lambda(C)q/n \leq \lambda(L \cap \ell_L)$. We find in $O(m)$ time a half-plane S_L through u cutting-off the desired area, $\lambda(L \cap S_L) = \lambda(C)q/n$ and containing all the points of L. This region, $L \cap S_L$, is the second polygon in the equitable 3-partition containing q points (not counting u). The third polygon is $C \setminus (S_L \cup S_R)$ and contains point u. We then specify the vertices of the polygons in $O(m)$ time. This is illustrated in Figure 8(b).

Case 3 In the remaining case suppose that either $\lambda(L \setminus H_L) \geq \lambda(C)q/n$ or $\lambda(R \setminus H_R) \geq \lambda(C)q/n$. Without loss of generality assume the above holds for R. For every point $p \in P \cap R$, determine in $O(n)$ time the angle of $\angle ubp$, and find the point w with maximum angle.

Let $H_w := \mathcal{H}(\overleftrightarrow{bw}, u)^c$ be the half-plane through b supporting $\text{conv}(P)$ at w. Calculate the area cutoff by the half-plane H_w check that it is too large, $\lambda(C \cap H_w) >$



(a) We find a half-plane T_R supporting conv(P) so that the remaining area of R is $\lambda(C)q/n$.

(b) Now we find half-plane T_L.

Fig. 9. Case 3.

$\lambda(C)/n$. If not then we find an equitable 2-partition using lemma 3.6 (with $k = 1$). Since $\lambda(R) \leq \lambda(C)(q+1)/n$ it follows that $\lambda(R \setminus H_w) < \lambda(C)q/n \leq \lambda(R \setminus H_R)$ where both H_R and H_w are half-planes supporting conv($P \cap R$). Hence there is some half-plane T_R supporting conv(P) between u and w that cuts off the right area, $\lambda(R \setminus T_R) = \lambda(C)q/n$. To find it we first determine in $O(n \log n)$ time all the edges of conv(P) between u and w. We then perform a binary search among these edges to find in $O(n \log n)$ time vertex x of conv(P) such that one of the half-planes supporting conv(P) along an edge of conv(P) adjacent to x cuts off more than $\lambda(R) - \lambda(C)q/n$ and the half-plane supporting conv(P) along the other edge adjacent to x cuts off less than $\lambda(R) - \lambda(C)q/n$. We now apply lemma 3.5 to find a half-plane T_R supporting conv(P) at x with the desired area, $\lambda(R \setminus T_R) = \lambda(C)q/n$. This is illustrated in Figure 9(a). Again we check that $\lambda(C \cap T_R) > \lambda(C)/n$, otherwise we find an equitable 2-partition using lemma 3.6 (with $k = 1$). The first polygon of our equitable 3-partition is $R \setminus T_R$ and containing q points (including u but not including x).

Since now $\lambda(C \cap T_R) > \lambda(C)/n$, $\lambda(R \setminus T_R) = \lambda(C)q/n$, and $C = (C \cap T_R) \cup (R \setminus T_R) \cup (L \setminus T_R)$ we have $\lambda(L \setminus T_R) < \lambda(C)q/n$. Let y be the intersection of the boundary of T_R and ℓ (note that y is between u and b). Because $\lambda(L \cap T_R^c) < \lambda(C)q/n \leq \lambda(L \cap \ell_L)$ we can find in $O(m)$ time a half-plane T_L such that $\lambda(L \cap T_L) = \lambda(C)q/n$. This is illustrated in Figure 9(b). Since T_L is between T_R^c and ℓ_L, $L \cap T_L$ contains q points (not counting u). This gives us a three partition, $L \cap T_L$ containing q points (including u), $C \cap T_R$ containing x, and $R \setminus T_R$ containing q points (including u but not x). To conclude our proof we note that the vertices of the 3-partition can be found in $O(m)$ time. □

4. EXTENSION TO NONUNIFORM DENSITIES

Here we note that the algorithm from the previous section extends easily to the case where μ_2 is merely a probability measure with a density (provided we can

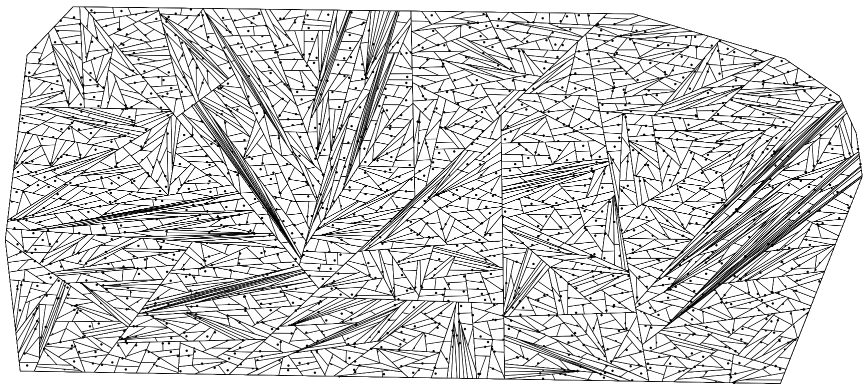


Fig. 10. An equitable convex partition of a region with 2048 points.

evaluate μ_2 efficiently). This is because the algorithm is based on the ability to find a solution to the problem posed in lemma 3.5 (i.e., given half-planes H_1 and H_2 whose boundaries go through some point with $\mu_2(H_1) \leq \alpha \leq \mu_2(H_2)$, can we find an intermediate half-plane with measure α). In this extension, we may not be able to find an exact solution analytically, but we can still find an ϵ-approximation (i.e., a half-plane H^* such that $|\mu_2(H^*) - \alpha| \leq \epsilon$) using the bisection method in $O(\log \frac{1}{\epsilon})$ steps. The algorithm from the previous section extended in this way takes $O(n^2 \log \frac{n}{\epsilon})$ steps and queries to μ_2 to find an ϵ-approximate equitable convex partition (where, similar to Section 2, the second measure of each piece is within a factor ϵ of $1/n$). For the case considered in the previous section, theorem 3.1 gives a tighter analysis by exploiting the fact that for a convex polygon S with k vertices we can calculate $\lambda(S)$ in $O(k)$ time.

5. OPEN PROBLEMS

Extending our algorithm for convex equitable partitions to higher dimensions is an open problem. Based on the ham sandwich theorem we believe a convex equitable partition of $\mathbb{R}^d$ with respect to d measures, $\mu_1, \ldots, \mu_d$, exists; we hope to find an algorithm that partitions $\mathbb{R}^d$ into n convex pieces $(R_1, \ldots, R_n)$ such that $\mu_i(R_j) = 1/n$ for all i and j. Consider, for example, a three-dimensional convex polytope C containing ng blue points and nh red points. We believe there is a convex partition of the polytope C where each piece contains g blue points, h red points, and the same volume.

When we implement our algorithm we find that some of the pieces of the partition are long and skinny as in Figure 10. For many applications one would like the pieces to be not only convex but also "fat" (i.e., to have small diameters). It would be interesting to explore how fat the individual pieces can be made while still having an equitable convex partition or how fat the pieces can be made if we allow their areas to be slightly unequal. For a related problem, Carmi and Katz [2005] try to minimize the average distance between any point in C and its depot.

We plan to work on problems found in applications that are similar to equitable convex partitions. These applications include redrawing congressional districts and the efficient surveillance of an area with multiple vehicles. The robotics community has already developed some heuristics for the latter problem [Jäger and Nebel 2002].

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