Dynamic Spectrum Management with the Competitive Market Model

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Abstract

[1, 2] have shown for the dynamic spectrum allocation problem that a competitive market model (which sets a price for transmission power on each channel) leads to a greater social utility (by reducing cross talk) than the Nash equilibrium. We show that the market equilibrium is the solution of a linear complementarily problem, and hence the market model possesses no additional computational complexity beyond that of the Nash equilibrium model and can be calculated efficiently. We also show that under reasonable conditions, any tâtonnement process for adjusting the prices will converge to the equilibrium prices. The conditions are that users of a channel experience the same noise levels and that the cross-talk effects between users are low-rank and weak.

1 Introduction

Dynamic spectrum management (DSM) is a technology to efficiently share the frequency spectrum among users in a communication system. This technology can be used in digital subscriber line (DSL) systems to reduce cross-talk interference and improve total system throughput (see [3, 4],

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and [5] for a survey). DSM is also a promising candidate for multiple access in overlay cognitive radio technology [6, 7].

In DSM, multiple users coexist in a channel and this causes the co-channel interferences. This interference (an externality) distinguishes DSM from other network-flow control problems [8, 9]. The goal of DSM is to manage the power allocations in all the channels in order to maximize the sum of the data rates of all the users [3]. Unfortunately, the problem of maximizing the total rate subject to power constraints is non-convex and cannot be solved efficiently in polynomial time.

Recently, the game-theoretic formulation of DSM has attracted interest in a variety of contexts: for DSL [10, 3, 4], for frequency-selective Gaussian interference channels [11], for CDMA uplink [12], for fading wireless channels [13], and for the general problem of power control (Chapter 6 of [14]). In the game-theoretic formulation, each user maximizes her data rate, the Shannon utility function, with the knowledge of other users' current power allocations. This is a competitive game whose pure Nash equilibrium exists and is unique under certain conditions. One merit of the game-theoretic formulation is that the user's problem is convex and can be solved efficiently when other users' power allocations are fixed. Various iterative water filling algorithms have been proposed to solve the Nash equilibrium (e.g., [10]).

However, the power allocation in a Nash equilibrium may not be socially optimal. Because of the non-cooperative nature of the Nash equilibrium, users tend to compete for "good" channels regardless of the interferences caused to others to the detriment of overall system efficiency, when they may all be better off by avoiding interference through the use of different channels. This is an example of the well-known "tragedy of the commons" in economics [15]. A simple example of this problem in DSM with two users and two channels was presented in [1].

Therefore we turn to the competitive market model for the DSM problem in [1], where in each channel a fictitious price is imposed on power allocations. In the competitive market model, each user maximizes her total data rate by purchasing some power in each channel, given her budget constraint. The price for power in each channel is determined by some central office to balance the channel load and to make the demand meet the supply. The existence of a market equilibrium was proven in [1]. Also, it was shown in [2] that the market equilibrium achieves higher social utility (total rate) than the Nash equilibrium, if we properly assign budgets to guarantee fairness among all users. However, the challenge is to find the market equilibrium efficiently, especially the equilibrium prices. Traditionally, the price is determined by an auction type algorithm called the tâtonnement process [15]. However, it is not known whether this process converges with Shannon utility functions.

In this paper, we focus on solving this competitive market model for DSM and make three contributions. We first show that the competitive market model can be formulated as a linear complementarity problem (LCP) [16], as the Nash equilibrium model could [4], even though the original problem is nonlinear. The key lies in a change of variables from {price, power allocation} to {price, revenue}. Thus, the competitive market model possesses no additional computational complexity beyond that of the Nash equilibrium model. Secondly, we show that when the interference coefficients are user symmetric, then the problem is equivalent to finding KKT points of a QP, for which an FPTAS (fully polynomial-time approximation scheme) exists [17]. Lastly, we prove that under certain low-rank conditions, a simple distributed price-adjust tâtonnement process converges to the equilibrium price, which is the first convergence result in spectrum management with the Shannon utility functions.

The paper is organized as follows. The next section presents the problem formulation. Section 3 presents the LCP formulation and the FPTAS result, and section 4 is about the decentralized price-adjustment tâtonnment process. We conclude in section 5 and present some of the technical proofs in the appendices.

2 Problem Formulation

The notation in this paper is conventional. We use lower case, bold letters for vectors and capital, bold letters for matrices. $\boldsymbol{X} \geq 0$ and $\boldsymbol{x} \geq 0$ are elementwise inequalities while $\boldsymbol{X} \succeq 0$ and $\boldsymbol{X} \succ 0$ indicate that $\boldsymbol{X}$ is semi-positive definite and positive definite, respectively. In addition, $\boldsymbol{I}$ is the identity matrix; $\rho(\boldsymbol{X})$ is the spectral radius of $\boldsymbol{X}$; $\boldsymbol{X}^\dagger$ is the Moore-Penrose pseudoinverse of $\boldsymbol{X}$; and $(x)^+ := \max\{x, 0\}$.

Consider a communication system consists of n users and m channels. The users coexist in each channel and may transmit at the same time, causing interference to each other. Suppose the power allocated by user i to channel j is $x_{ij} \geq 0$. The total demand in channel j, the total power allocated by all the users in that channel, is $\sum_{i=1}^n x_{ij}$. This may not exceed the channel limit c_j.

(Regulatory reasons may limit the total transmission power in a channel.)

For overlay cognitive radio [6], a natural extension would be to limit the weighted sum, $\sum_{k=1}^{n} a_{0k}^{j} x_{kj} \leq c_j$, of the transmission powers of the secondary users in order to limit the interference experienced by the channel's primary user. Here a_{0k}^{j} is the interference coefficient on channel j of user k with the primary user. This extension would not substantially change our results.

An efficient allocation of spectrum requires that the total power in each channel is at its limit, $\sum_{i=1}^{n} x_{ij} = c_j$ for all j. To achieve this, our competitive market model associates a price $p_j > 0$ with each channel j. Alternatingly, users adjust their power allocations based on these prices and the spectrum manager adjusts the prices so that eventually the market clears: the demand in each channel equals the supply. For a given vector of prices, $\boldsymbol{p} := [p_1, \ldots, p_m]^\top$, each user i chooses the power allocation $\boldsymbol{x}_i := [x_{i1}, \ldots, x_{im}]^\top$ that maximizes her utility function subject to her budget w_i[1].

User i uses the Shannon utility [18]

$$u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i) := \sum_{j=1}^{m} \log \left(1 + \frac{x_{ij}}{\sigma_{ij} + \sum_{k \neq i} a_{ik}^{j} x_{kj}} \right). \tag{1}$$

because it represents the user's total data rate across all channels. Here $\bar{\boldsymbol{x}}_i := [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_n]^\top$ is the power allocation of the other $n-1$ users; $\sigma_{ij} > 0$ is the noise level for user i on channel j; and $a_{ik}^{j} \geq 0$ is the cross-talk coefficient for interference on channel j from user $k \neq i$ on user i. We do not normalize the channel capacities (by scaling the allocations $\{x_{ij}\}$ and noise levels $\{\sigma_{ij}\}$) in order to show explicitly their influence on our results. The optimal power allocation $\boldsymbol{x}_i^*(\boldsymbol{p}, \bar{\boldsymbol{x}}_i)$ of user i, when she faces prices $\boldsymbol{p}$ and power allocations $\bar{\boldsymbol{x}}_i$ of the other users, is determined by the following convex optimization problem

$$\begin{aligned} \boldsymbol{x}_i^*(\boldsymbol{p}, \bar{\boldsymbol{x}}_i) = \arg\max_{\boldsymbol{x}_i}\ & u_i(\boldsymbol{x}_i, \bar{\boldsymbol{x}}_i) \\ \text{subject to } & \boldsymbol{p}^\top \boldsymbol{x}_i \leq w_i, \\ & \boldsymbol{x}_i \geq 0. \end{aligned} \tag{2}$$

This optimization problem has a unique solution because it is strictly convex. Fig. 1 illustrates this

[1] [2] discusses how to choose the users' budgets.

competitive market model for spectrum allocation.

Each users' power-allocation problem has a water filling solution (Appendix A gives the derivation)

$$x_{ij}^* = \left(\frac{\nu_i}{p_j} - \sigma_{ij} - \sum_{k \neq i} a_{ik}^j x_{kj} \right)^+ \tag{3}$$

where the dual variable ν_i is determined by the budget constraint

$$\boldsymbol{p}^\top \boldsymbol{x}_i = w_i, \tag{4}$$

which is tight. Despite the use of prices, our derivation in Appendix A is analogous to the one in [10].

The competitive equilibrium (CE) [19] of this model is the vector of prices $\boldsymbol{p}^*$ and the corresponding user-optimal power allocations $\{x_{ij}^*\}$ so that the market clears. [1] proved that in this model a CE exists. In the next section we describe the CE as an LCP and then describe a decentralized price-adjustment process in section 4.

3 CE as LCP

Given a vector of prices $\boldsymbol{p}$, the simultaneous solution of all users' power-allocation problems is a Nash equilibrium. By applying the fact that $x = y^+$ is equivalent to $x \geq y \wedge x(x-y) = 0 \wedge x \geq 0$ to (3) and (4), we obtain the following LCP for the Nash equilibrium [4]:

$$\begin{aligned} x_{ij} \geq \frac{\nu_i}{p_j} - \sigma_{ij} - \sum_{k \neq i} a_{ik}^j x_{kj} \quad & \forall ij, \\ x_{ij} \left(x_{ij} - \frac{\nu_i}{p_j} + \sigma_{ij} + \sum_{k \neq i} a_{ik}^j x_{kj} \right) = 0 \quad & \forall ij, \\ x_{ij} \geq 0 \quad & \forall ij, \\ \boldsymbol{p}^\top \boldsymbol{x}_i = w_i \quad & \forall i. \end{aligned} \tag{5}$$

In the CE, prices $\boldsymbol{p}$ are also equilibrium variables. Thus simply adding the market clearing condition to (5) results in a nonlinear complementarity problem for the CE. We now reformulate the CE as an LCP. Let the revenue of user i on channel j be $r_{ij} := x_{ij} p_j$. Define the vectors

$\boldsymbol{r}_j := [r_{1j}, \ldots, r_{nj}]^\top$, $\boldsymbol{\sigma}_j := [\sigma_{1j}, \ldots, \sigma_{nj}]^\top$, $\boldsymbol{w} := [w_1, \ldots, w_n]^\top$, $\boldsymbol{\nu} := [\nu_1, \ldots, \nu_n]^\top$, and the slack vectors $\boldsymbol{s}_j$. Also define the interference coefficient matrix $\boldsymbol{A}_j$ with ones on the diagonal, $[\boldsymbol{A}_j]_{ii} = 1$, and $[\boldsymbol{A}_j]_{ik} = a^j_{ik}$ for $k \neq i$. Then (5) is equivalent to

$$
\begin{aligned}
\boldsymbol{A}_j \boldsymbol{r}_j + \boldsymbol{\sigma}_j p_j - \boldsymbol{\nu} - \boldsymbol{s}_j = 0 \quad \forall j,\\
r_{ij} s_{ij} = 0 \quad \forall ij,\\
\boldsymbol{r}_j, \boldsymbol{s}_j \geq \mathbf{0} \quad \forall j,\\
\sum_j \boldsymbol{r}_j = \boldsymbol{w}.
\end{aligned} \tag{6}
$$

A competitive equilibrium is a set of prices and allocations that satisfy the above equilibrium conditions (6) and the market clearing condition: $\sum_i x_{ij} = c_j$ or equivalently $\mathbf{1}^\top \boldsymbol{r}_j = c_j p_j$ for each channel j.

One can use this condition to eliminate prices from (6) and obtain the following LCP for the CE:

$$
\begin{aligned}
\left(\boldsymbol{A}_j + \frac{1}{c_j}\boldsymbol{\sigma}_j \mathbf{1}^\top\right) \boldsymbol{r}_j - \boldsymbol{\nu} - \boldsymbol{s}_j = 0 \quad \forall j,\\
r_{ij} s_{ij} = 0 \quad \forall ij,\\
\boldsymbol{r}_j, \boldsymbol{s}_j \geq \mathbf{0} \quad \forall j,\\
\sum_j \boldsymbol{r}_j = \boldsymbol{w}.
\end{aligned} \tag{7}
$$

To see the structure of this LCP, consider the case of two channels, $m = 2$, and let $\boldsymbol{M}_j := \boldsymbol{A}_j + \frac{1}{c_j}\boldsymbol{\sigma}_j \mathbf{1}^\top$. Then, (7) becomes

$$
\begin{pmatrix} \boldsymbol{M}_1 & \mathbf{0} & -I \\ \mathbf{0} & \boldsymbol{M}_2 & -I \\ I & I & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{\nu} \end{pmatrix} = \begin{pmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \\ \boldsymbol{w} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{pmatrix} \geq \mathbf{0}, \quad \text{and} \quad \begin{pmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \end{pmatrix} \geq \mathbf{0}, \tag{8}
$$

where we look for a complementarity solution $\boldsymbol{r}_1^\top \boldsymbol{s}_1 + \boldsymbol{r}_2^\top \boldsymbol{s}_2 = 0$. If both $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are monotone matrices, that is, $\boldsymbol{M}_1 + \boldsymbol{M}_1^\top$ and $\boldsymbol{M}_2 + \boldsymbol{M}_2^\top$ are positive semidefinite, then the LCP matrix

$$\begin{pmatrix} \boldsymbol{M}_1 & \boldsymbol{0} & -I \\ \boldsymbol{0} & \boldsymbol{M}_2 & -I \\ I & I & \boldsymbol{0} \end{pmatrix} \tag{9}$$

is also monotone.

Note that the solution of an LCP is determined by a system of linear equations. Furthermore, a solution of an LCP can be computed in polynomial time if the LCP matrix is monotone [16], and a KKT point of a quadratic optimization problem can be computed by a fully polynomial-time approximation scheme (FPTAS) [17]. Therefore, formulation (7) leads to our first few main results.

Theorem 1. *Consider the competitive market model for spectrum management.*

(i) Let the parameters w_i, c_j, σ_{ij}, and a_{ik}^j be rational. Then, there exists a CE with rational entries, that is, the entries of the equilibrium point are rational values.

(ii) If the matrix $\boldsymbol{A}_j + \frac{1}{c_j}\boldsymbol{\sigma}_j\boldsymbol{1}^\top$ is monotone for all j, then a CE can be computed in polynomial time.

(iii) If the matrix $\boldsymbol{A}_j + \frac{1}{c_j}\boldsymbol{\sigma}_j\boldsymbol{1}^\top$ is symmetric (in particular, if $\boldsymbol{A}_j$ is symmetric and $\sigma_{1j} = \sigma_{ij}$ for all i) for all j, then the competitive equilibria are the KKT points of the following QP

$$\begin{aligned} \underset{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_m}{\text{minimize}} \quad & \sum_j \frac{1}{2}\boldsymbol{r}_j^\top \left(\boldsymbol{A}_j + \frac{1}{c_j}\boldsymbol{\sigma}_j\boldsymbol{1}^\top\right)\boldsymbol{r}_j \\ \textit{subject to} \quad & \sum_j \boldsymbol{r}_j = \boldsymbol{w} \quad \textit{(with } \boldsymbol{\nu} \textit{ as the Lagrange multiplier)} \\ & \boldsymbol{r}_j \geq \boldsymbol{0} \quad \forall j \quad \textit{(with } \boldsymbol{s}_j \textit{ as the Lagrange multiplier).} \end{aligned} \tag{10}$$

(iv) There is a FPTAS to compute a CE equilibrium if the matrix $\boldsymbol{A}_j + \frac{1}{c_j}\boldsymbol{\sigma}_j\boldsymbol{1}^\top$ is symmetric for all j.

Assuming strict monotonicity (replacing "positive semidefinite" with "positive definite" in the definition of monotonicity) ensures that the CE is unique.

Corollary 2. *There is a unique CE if the matrix $\boldsymbol{A}_j + \frac{1}{c_j}\boldsymbol{\sigma}_j\boldsymbol{1}^\top$ is strictly monotone for all j.*

For example, a symmetric and weak-interference condition, that is, for all j, $\sum_{k \neq i} a_{ik}^j < 1$ for all i and $\sum_{i \neq k} a_{ik}^j < 1$ for all k, will ensure that $\boldsymbol{A}_j$ is strictly monotone for all j. In addition, if we have equal noise: $\sigma_{1j} = \sigma_{ij}$, $\forall i, j$, then $\boldsymbol{A}_j + \frac{1}{c_j} \boldsymbol{\sigma}_j \mathbf{1}^\top$ will be strictly monotone for all j. The equal noise condition is that all users of a channel experience the same background noise level.

4 Tâtonnement Process for Spectrum Management

In a centralized approach, the manager gathers all the parameters (budgets, noise levels, and interference coefficients); solves (2) or (7); and then recovers and publishes the optimal power allocations and prices. We now describe a decentralized spectrum management approach similar to an open-outcry auction for a divisible good (or one for multiple copies of the same item). At such an auction, the auction manager announces a price, notes the bids, and then repeats the process, adjusting the price (up or down), until the quantity demanded equals the quantity for sale.

Unlike the centralized approach where the manager knows all the parameters and does all the computation, this approach reduces the communication between users and distributes the computation so that each user or channel determines its own power allocations using the water-filling solution, a simple computation requiring only knowledge of prices and the local interference level. In the decentralized approach each user must only send its power allocations and receive the channel prices from the spectrum manager (unlike the centralized approach where each user must transmit the noise levels and interference coefficients to the manager). Such low communication overhead is important for system such as distributed sensor networks where energy for sending messages is limited. In fact, the spectrum manager need not exist if each user can observe the total demand on each channel and calculate the price updates themselves. In this case the process would be fully distributed and require no communication between users.

In our approach, the manager publish an initial vector of prices, $\boldsymbol{p}$. Users then arrive at power allocations forming a Nash equilibrium for these prices, that is, they arrive at $\boldsymbol{r}_j$ and $\boldsymbol{\nu}$ for fixed $\boldsymbol{p}$

that satisfy the conditions in (6):

$$\begin{aligned} A_j \boldsymbol{r}_j - \boldsymbol{\nu} - \boldsymbol{s}_j = -\boldsymbol{\sigma}_j p_j \quad \forall j, \\ r_{ij} s_{ij} = 0 \quad \forall ij, \\ \boldsymbol{r}_j, \boldsymbol{s}_j \geq \mathbf{0} \quad \forall j, \\ \sum_j \boldsymbol{r}_j = \boldsymbol{w}. \end{aligned} \tag{11}$$

Then, the auction manager checks the excess demand $\mathbf{1}^\top \boldsymbol{x}_j - c_j$ or $\mathbf{1}^\top \boldsymbol{r}_j - c_j p_j$ and makes a price adjustment for each channel j. This repeats, with the spectrum manager adjusting the prices each iteration, until the market clears, i.e., until the demand in each channel equals the supply:

$$\mathbf{1}^\top \boldsymbol{x}_j = \mathbf{1}^\top \boldsymbol{r}_j / p_j = c_j, \quad \forall j. \tag{12}$$

Analogous to the CE LCP in section 3, if the matrix $\boldsymbol{A}_j$ is monotone for all j, then a Nash equilibrium of the LCP (11) can be computed in polynomial time for a given vector of prices $\boldsymbol{p}$. Moreover, the Nash equilibrium is unique if the matrix $\boldsymbol{A}_j$ is strictly monotone for all j. Proposition 2 of [4] gives a similar condition for uniqueness. (The model in [4] does not have prices, but these can easily be incorporated without affecting the results, by scaling the power allocations and the noise levels.) On the other hand, if the matrix $\boldsymbol{A}_j$ is symmetric for all j, then the Nash equilibria of (11) are the KKT points of the following QP

$$\begin{aligned} \underset{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_m}{\text{minimize}} \quad & \sum_j \left(\frac{1}{2} \boldsymbol{r}_j^\top \boldsymbol{A}_j \boldsymbol{r}_j + p_j \boldsymbol{\sigma}_j^\top \boldsymbol{r}_j \right) \\ \text{subject to} \quad & \sum_j \boldsymbol{r}_j = \boldsymbol{w} \quad (\text{with } \boldsymbol{\nu} \text{ as the Lagrange multiplier}) \\ & \boldsymbol{r}_j \geq \mathbf{0} \quad \forall j \quad (\text{with } \boldsymbol{s}_j \text{ as the Lagrange multiplier}). \end{aligned} \tag{13}$$

Thus, similar to the way we constructed an FPTAS for the CE in section 3, there is an FPTAS to compute a Nash equilibrium of (11) after the prices $\boldsymbol{p}$ are posted. However, these methods are still centralized.

4.1 Decentralized Computation of a Nash Equilibrium

We now describe decentralized methods for solving (11) for a given vector of prices $\boldsymbol{p}$. Our first method works for the case where the matrices $\boldsymbol{A}_j$ are symmetric and positive semidefinite. Since (13) is convex and Slater's condition holds, it is equivalent to its Lagrange dual

$$\begin{aligned} \max_{\boldsymbol{\nu}} \min_{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_m} \; & \sum_j \left(\frac{1}{2} \boldsymbol{r}_j^\top \boldsymbol{A}_j \boldsymbol{r}_j + p_j \boldsymbol{\sigma}_j^\top \boldsymbol{r}_j + \boldsymbol{\nu}^\top \boldsymbol{r} \right) - \boldsymbol{\nu}^\top \boldsymbol{w} \\ \text{subject to} \quad & \boldsymbol{r}_j \geq \boldsymbol{0} \quad \forall j. \end{aligned} \tag{14}$$

Note that the inner minimization is separable across channels,

$$\max_{\boldsymbol{\nu}} -\boldsymbol{\nu}^\top \boldsymbol{w} + \sum_j \left\{ \begin{aligned} & \min_{\boldsymbol{r}_j} \frac{1}{2} \boldsymbol{r}_j^\top \boldsymbol{A}_j \boldsymbol{r}_j + (p_j \boldsymbol{\sigma}_j + \boldsymbol{\nu})^\top \boldsymbol{r}_j \\ & \text{subject to} \quad \boldsymbol{r}_j \geq \boldsymbol{0}. \end{aligned} \right\} \tag{15}$$

Since the outer optimization problem is convex, almost any hill-climbing method for adjusting $\boldsymbol{\nu}$ will converge. We could for example increase ν_i for each user i where $\sum_j r_{ij} < w_i$ (and otherwise decrease ν_i). In a similar fashion, the CE problem (10) can be decomposed across channels.

An iterative water-filling algorithm (IWFA) is a second decentralized method for finding a Nash equilibrium (11) after prices are posted. It is a simple round-robin approach where users take turns updating their power allocations. Corollary 1 in [4] shows that the IWFA converges linearly if the matrix $\boldsymbol{A}_j$ is symmetric for all j. (As noted above, the model in [4] does not have prices, but these can easily be incorporated.) The difference between our decetralized approach and the IWFA is that we decompose the problem across channels while the IWFA decomposes the problem across users.

4.2 Price Adjustment Based on Excess Demand

We return to the key question: how to adjust the prices and ensure that the process quickly converges to a CE. Tâtonnement processes [15] are simple approaches that adjust the price based on the excess demand: if the supply of power for a channel, c_j, exceeds the total demand, $\sum_i x_{ij}$, then the price of power on that channel, p_j, increases (and decreases if demand falls short of supply). This is a very broad class of price-update rules, that can be applied in continuous time with prices and allocations continuously adjusting or it can be applied iteratively (i.e., in discrete time), alternating between updating prices and power allocations. The condition for the convergence of a tâtonnement process is weak gross substitutability (WGS). Theorem 4.1 of [20] (also found in the classic [15]) shows that WGS is a sufficient condition for continuous-time tâtonnement processes while [21] shows that it is a sufficient condition for discrete-time tâtonnement processes.

Theorem 3. *(i) Suppose that prices for each product j are adjusted continuously by*

$$\frac{dp_j(t)}{dt} = f_j(y_j(\boldsymbol{p}(t))), \tag{16}$$

where $f_j(\cdot)$ is a sign preserving function (i.e., $\operatorname{sign} f_j(y) = \operatorname{sign} y$) and y_j is a measure of the excess of product j. Then $\boldsymbol{y} \to \boldsymbol{0}$ if weak gross substitutability holds, that is, $\partial_l y_j(\boldsymbol{p}) \geq 0$ for all $l \neq j$.

(ii) Suppose that prices for each product j are adjusted discretely by

$$p_j^{t+1} = p_j^t + f_j(y_j(\boldsymbol{p}^t)), \tag{17}$$

where $f_j(\cdot)$ is a sign preserving function (i.e., $\operatorname{sign} f_j(y) = \operatorname{sign} y$) and y_j is a measure of the excess of product j. Then $\boldsymbol{y}^t \to \boldsymbol{0}$ if weak gross substitutability holds, that is, $\partial_l y_j(\boldsymbol{p}) \geq 0$ for all $l \neq j$.

With some conditions, we can prove WGS for our spectrum management problem. For algebraic simplicity we use excess revenue instead of excess demand (this is without loss of generality since for each j the factor p_j could easily be incorporated into $f_j(y)$).

Theorem 4. *For each channel j define $y_j(\boldsymbol{p}) = p_j(\sum_i x_{ij}^* - c_j)$. Assume the following conditions*

(i) symmetric, weak-interference condition: $\sum_{k\neq i} a_{ik}^j < 1$ and $\sum_{k \neq i} a_{ki}^j < 1$, $\forall j$;

(ii) low-rank condition: the interference coefficient matrices $\boldsymbol{A}_j$ can be written as $\boldsymbol{A}_j = \boldsymbol{D}_j + \boldsymbol{a}_j \boldsymbol{b}_j^\top$, $\forall j$ where $\boldsymbol{D}_j$ diagonal, $\boldsymbol{D}_j, \boldsymbol{a}_j, \boldsymbol{b}_j \geq 0$, and $\boldsymbol{a}_j, \boldsymbol{b}_j$ in the range of $\boldsymbol{D}_j$; and

(iii) equal noise condition: $\sigma_{ij} = \sigma_j$, $\forall i, j$.

Then our spectrum model satisfies WGS, i.e., $\partial_l y_j(\boldsymbol{p}) \geq 0$ *for all* $l \neq j$*, so that both continuous and discrete tâtonnement price-adjustment processes converge.*

Proof. See Appendix B. □

We remark that condition (ii) in theorem 4 was used in [1, 2] to show the convexity of the equilibrium set, where they assumed that $a_{ik}^j = a_i^j \leq 1$ for all i, j, k. Thus, for all j, we can write $\boldsymbol{A}_j = \boldsymbol{D}_j + \boldsymbol{a}_j \boldsymbol{1}^\top$.

To the best of our knowledge, theorem 4 is the first convergence result for a tâtonnement price-adjustment process applied to a spectrum management problem with the Shannon utility functions. The conditions in theorem 4, excluding condition (ii), are similar to those in (2) for the CE to have a unique solution. We conjecture that condition (ii) is not necessary and that WGS will hold when only conditions (i) and (iii) are satisfied. For two channels, $m = 2$, convergence of the tâtonnement process can be proven with weaker conditions.

Theorem 5. *If* $m = 2$ *and the weak-interference condition holds for* $\boldsymbol{A}_1^\top$ *and* $\boldsymbol{A}_2^\top$*, that is,* $\sum_{k \neq i} a_{ki}^j < 1$ *for all* ij*, then weak gross substitutability holds and tâtonnement price-adjustment processes converge.*

Proof. See Appendix B. □

5 Conclusions

We considered the competitive market model for dynamic spectrum management of communication systems. We showed that the problem of finding the market equilibrium can be formulated as a linear complementarity problem (LCP) and solved efficiently. Besides the centralized LCP approach, we also studied decentralized tâtonnement processes. We proved conditions for these simple price-adjustment processes to converge.

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A Derivation of the Water Filling Solution

The Lagrangian of user i's power allocation problem (1) and (2) is

$$-\sum_j \log\left(1+\frac{x_{ij}}{\sigma_{ij}+\sum_{k\neq i} a_{ik}^j x_{kj}}\right)+\tilde{\nu}_i\left(\sum_j p_j x_{ij}-w_i\right)-\sum_j \lambda_{ij}x_{ij}, \tag{18}$$

where $\lambda_{ij}\geq 0$ and $\tilde{\nu}_i\geq 0$ are the associated Lagrange multipliers. Setting the gradient of the Lagrangian with respect to x_{ij} to zero, we have

$$p_j\tilde{\nu}_i-\frac{1}{\sigma_{ij}+x_{ij}+\sum_{k\neq i} a_{ik}^j x_{kj}}=\lambda_{ij}\geq 0. \tag{19}$$

Since p_j and σ_{ij} are positive and the a_{ik}^j are nonnegative, it follows that $\tilde{\nu}_i>0$. Thus, the budget constraint, $\sum_j p_j x_{ij}\leq w_i$, is tight, and we can define $\nu_i:=1/\tilde{\nu}_i$. Using (19), the complementary slackness condition $\lambda_{ij}x_{ij}=0$, implies that if $x_{ij}>0$, then $\lambda_{ij}=0$, or equivalently

$$x_{ij}=\frac{\nu_i}{p_j}-\sigma_{ij}-\sum_{k\neq i} a_{kj}^i x_{kj}. \tag{20}$$

Similarly, $x_{ij}=0$ implies $\lambda_{ij}\geq 0$, or equivalently turns the equality in (20) into $\geq$. This leads to the water filling solution for the power allocation in (3) and (4).

B Proof of WGS

Proof of theorem 4. Let $[\boldsymbol{r}_1^*(\boldsymbol{p}),\ldots,\boldsymbol{r}_m^*(\boldsymbol{p})]$ be the solution to (6). We rewrite $y_j(\boldsymbol{p})=\mathbf{1}^\top\boldsymbol{r}_j^*(\boldsymbol{p})-p_j$. For $j\neq l$, we will show that both the left and right hand limits are $\partial_l y_j(\boldsymbol{p})=\mathbf{1}^\top\partial_l\boldsymbol{r}_j^*(\boldsymbol{p})\geq 0$. Let us look at the left hand limit (the right hand limit will be similar). Then there is a small open interval $(t-\epsilon,t)$ in which the active set of the LCP is constant. Let the set S_j be the active set of channel j, $S_j:=\{i:s_{ij}=0\}$ and $\boldsymbol{I}_j$ the $n\times n$ matrix so that $[\boldsymbol{I}_j]_{il}:=1$ if $i=l\in S_j$ and 0 otherwise. Note that $\boldsymbol{I}_j\boldsymbol{s}_j=\mathbf{0}$ and $r_{ij}=0$ for $i\notin S_j$, thus $\boldsymbol{I}_j\boldsymbol{r}_j=\boldsymbol{r}_j$. Thus the first equation in (6) becomes

$$\boldsymbol{I}_j\boldsymbol{A}_j\boldsymbol{I}_j\boldsymbol{r}_j=\boldsymbol{I}_j\boldsymbol{\nu}-p_j\boldsymbol{I}_j\boldsymbol{\sigma}_j. \tag{21}$$

Defining $\bar{\boldsymbol{A}}_j := \boldsymbol{I}_j \boldsymbol{A}_j \boldsymbol{I}_j$ it follows that $\bar{\boldsymbol{A}}_j{}^\dagger \boldsymbol{I}_j = \bar{\boldsymbol{A}}_j{}^\dagger$ and that one solution is

$$\boldsymbol{r}_j = \bar{\boldsymbol{A}}_j{}^\dagger \boldsymbol{\nu} - p_j \bar{\boldsymbol{A}}_j{}^\dagger \boldsymbol{\sigma}_j. \tag{22}$$

Then the budget constraint (the last equation in (6)) gives us

$$\sum_k \bar{\boldsymbol{A}}_k^\dagger \boldsymbol{\nu} - p_k \bar{\boldsymbol{A}}_k^\dagger \boldsymbol{\sigma}_k = \boldsymbol{w}. \tag{23}$$

Thus one solution for $\boldsymbol{\nu}$ is

$$\boldsymbol{\nu} = \left(\sum_k \bar{\boldsymbol{A}}_k^\dagger \right)^\dagger \left(\boldsymbol{w} + \sum_k p_k \bar{\boldsymbol{A}}_k^\dagger \boldsymbol{\sigma}_k \right). \tag{24}$$

Thus for $j \neq l$,

$$\frac{\partial y_j}{\partial p_l} = \frac{\partial}{\partial p_l} \boldsymbol{1}^\top \boldsymbol{r}_j = \boldsymbol{1}^\top \bar{\boldsymbol{A}}_j^\dagger \frac{\partial \boldsymbol{\nu}}{\partial p_l}, \tag{25}$$

$$\frac{\partial y_j}{\partial p_l} = \boldsymbol{1}^\top \bar{\boldsymbol{A}}_j^\dagger \left(\sum_k \bar{\boldsymbol{A}}_k^\dagger \right)^\dagger \bar{\boldsymbol{A}}_l^\dagger \boldsymbol{\sigma}_l. \tag{26}$$

The equal noise condition and lemma 9 then prove the claim. □

Proof of theorem 5. Following the proof of theorem 4 we need to show that $\frac{\partial y_2}{\partial p_1}$ given by (26) is nonnegative:

$$\frac{\partial y_2}{\partial p_1} = \boldsymbol{1}^\top \bar{\boldsymbol{A}}_2^\dagger \left(\bar{\boldsymbol{A}}_1^\dagger + \bar{\boldsymbol{A}}_2^\dagger \right)^\dagger \bar{\boldsymbol{A}}_1^\dagger \boldsymbol{\sigma}_1 = \boldsymbol{1}^\top \left(\bar{\boldsymbol{A}}_2 + \bar{\boldsymbol{A}}_1 \right)^\dagger \boldsymbol{\sigma}_1. \tag{27}$$

Since $0.5(\bar{\boldsymbol{A}}_2 + \bar{\boldsymbol{A}}_1)^\top$ is a channel matrix obeying weak interference we can apply lemma 8 to show that $\boldsymbol{1}^\top \left(\bar{\boldsymbol{A}}_2 + \bar{\boldsymbol{A}}_1 \right)^\dagger$ is a nonnegative vector. The fact that $\boldsymbol{\sigma}_1 \geq 0$ completes the proof. □

Lemma 6 (Sherman-Morrison Formula)**.** *Provided that $\boldsymbol{C}^{-1}$ exists and $1 + \boldsymbol{b}^\top \boldsymbol{C}^{-1} \boldsymbol{a} \neq 0$, then*

$$(\boldsymbol{C} + \boldsymbol{a}\boldsymbol{b}^\top)^{-1} = \boldsymbol{C}^{-1} - \frac{(\boldsymbol{C}^{-1}\boldsymbol{a})(\boldsymbol{b}^\top \boldsymbol{C}^{-1})}{1 + \boldsymbol{b}^\top \boldsymbol{C}^{-1} \boldsymbol{a}}. \tag{28}$$

Lemma 7. *For $i = 1, \ldots, m$, let $\boldsymbol{A}_j = \boldsymbol{D}_j + \boldsymbol{a}_j \boldsymbol{b}_j^\top$ where $\boldsymbol{D}_j$ diagonal, $\boldsymbol{D}_j, \boldsymbol{a}_j, \boldsymbol{b}_j \geq 0$, and $\boldsymbol{a}_j, \boldsymbol{b}_j$ in the range of $\boldsymbol{D}_j$. If for each i there exists j such that $[\boldsymbol{D}_j]_{ii} > 0$, then, $\left(\sum_j \boldsymbol{A}_j^\dagger \right)^{-1}$ exists and is* *nonnegative.*

Proof. Applying lemma 6 to the range of $\boldsymbol{A}_j$ we obtain $\boldsymbol{A}_j^\dagger = \boldsymbol{D}_j^\dagger - \boldsymbol{B}_j$, where $\boldsymbol{B}_j := \frac{\boldsymbol{D}_j^\dagger \boldsymbol{a}_j \boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger}{1+\boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j}$. Since $\boldsymbol{D}_j \geq 0$, $\boldsymbol{D}_j^\dagger \geq 0$. Therefore, $\boldsymbol{a}_j, \boldsymbol{b}_j \geq 0$ implies $\boldsymbol{B}_j \geq 0$. Thus $\sum_j \boldsymbol{A}_j^\dagger$ can be written as

$$\sum_j \boldsymbol{A}_j^\dagger = \boldsymbol{D} - \boldsymbol{B}, \tag{29}$$

where $\boldsymbol{D} := \sum_j \boldsymbol{D}_j^\dagger$ and $\boldsymbol{B} := \sum_j \boldsymbol{B}_j$. Since $\boldsymbol{D} \succ 0$, $\boldsymbol{D}^{-1}$ exists and we may define $\boldsymbol{C} := \boldsymbol{D}^{-1/2}\boldsymbol{B}\boldsymbol{D}^{-1/2}$. Note that $\boldsymbol{D} \geq 0$, $\boldsymbol{B} \geq 0$, and $\boldsymbol{D}$ diagonal. Thus $\boldsymbol{D}^{-1/2} \geq 0$ and $\boldsymbol{C} \geq 0$. Note that for any $\boldsymbol{x} \neq \boldsymbol{0}$,

$$\left|\boldsymbol{x}^\top \boldsymbol{C}\boldsymbol{x}\right| = \left|\boldsymbol{x}\boldsymbol{D}^{-1/2}\boldsymbol{B}\boldsymbol{D}^{-1/2}\boldsymbol{x}\right| \leq \sum_j \left|\boldsymbol{x}\boldsymbol{D}^{-1/2}\boldsymbol{B}_j\boldsymbol{D}^{-1/2}\boldsymbol{x}\right| \tag{30}$$

$$= \sum_j \left|\frac{\boldsymbol{x}\boldsymbol{D}^{-1/2}\boldsymbol{D}_j^\dagger \boldsymbol{a}_j \boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{D}^{-1/2}\boldsymbol{x}}{1+\boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j}\right| \leq \sum_j \frac{\rho((\boldsymbol{D}_j^\dagger)^{1/2}\boldsymbol{a}_j\boldsymbol{b}_j^\top (\boldsymbol{D}_j^\dagger)^{1/2}) \left\|(\boldsymbol{D}_j^\dagger)^{1/2}\boldsymbol{D}^{-1/2}\boldsymbol{x}\right\|_2^2}{1+\boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j} \tag{31}$$

$$= \sum_j \frac{(\boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j)(\boldsymbol{x}^\top \boldsymbol{D}^{-1/2}\boldsymbol{D}_j^\dagger \boldsymbol{D}^{-1/2}\boldsymbol{x})}{1+\boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j} \leq \lambda \sum_j \boldsymbol{x}^\top \boldsymbol{D}^{-1/2}\boldsymbol{D}_j^\dagger \boldsymbol{D}^{-1/2}\boldsymbol{x} \tag{32}$$

where $\lambda = \max_j \frac{\boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j}{1+\boldsymbol{b}_j^\top \boldsymbol{D}_j^\dagger \boldsymbol{a}_j}$. Since $\boldsymbol{D}_j, \boldsymbol{a}, \boldsymbol{b} \geq 0$, $\lambda \geq 0$ and since $\boldsymbol{a}$ and $\boldsymbol{b}$ are in the range of $\boldsymbol{D}_j$, $\lambda < 1$. Therefore, for any $\boldsymbol{x} \neq \boldsymbol{0}$,

$$\left|\boldsymbol{x}^\top \boldsymbol{C}\boldsymbol{x}\right| < \sum_j \boldsymbol{x}^\top \boldsymbol{D}^{-1/2}\boldsymbol{D}_j^\dagger \boldsymbol{D}^{-1/2}\boldsymbol{x} = \boldsymbol{x}^\top \boldsymbol{D}^{-1/2}\boldsymbol{D}\boldsymbol{D}^{-1/2}\boldsymbol{x} = \boldsymbol{x}^\top \boldsymbol{x}. \tag{33}$$

Hence, $\rho(\boldsymbol{C}) < 1$ and thus $(\boldsymbol{I} - \boldsymbol{C})^{-1} = \sum_{k=0}^\infty \boldsymbol{C}^k \geq 0$. Therefore, $(\sum_j \boldsymbol{A}_j^\dagger)^{-1} = (\boldsymbol{D} - \boldsymbol{B})^{-1} = \boldsymbol{D}^{-1/2}(\boldsymbol{I} - \boldsymbol{C})^{-1}\boldsymbol{D}^{-1/2} \geq 0$. ☐

Lemma 8. *If $\boldsymbol{A}$ is a channel matrix satisfying the weak-interference assumption, that is, $\sum_{k\neq i} a_{ik}^j < 1$ for all i, then $\boldsymbol{A}^{-1}\boldsymbol{1} \geq 0$.*

Proof. Since $\boldsymbol{A}$ is a channel matrix we can write $\boldsymbol{A} = \boldsymbol{I} + \boldsymbol{B}$ for some $\boldsymbol{B} \geq 0$. Hence

$$\boldsymbol{A}^{-1}\boldsymbol{1} = (\boldsymbol{I}+\boldsymbol{B})^{-1}\boldsymbol{1} = (\boldsymbol{I}+\boldsymbol{B})^{-1}(\boldsymbol{I}-\boldsymbol{B})^{-1}(\boldsymbol{I}-\boldsymbol{B})\boldsymbol{1} = \left(\boldsymbol{I}-\boldsymbol{B}^2\right)^{-1}(\boldsymbol{I}-\boldsymbol{B})\boldsymbol{1}. \tag{34}$$

The weak interference assumption implies that $\rho(\boldsymbol{B}) < 1$. Hence $\left(\boldsymbol{I} - \boldsymbol{B}^2\right)^{-1}$ exists and equals $\sum_{k=0}^{\infty} \boldsymbol{B}^{2k} \geq 0$. In addition, $(\boldsymbol{I} - \boldsymbol{B})\mathbf{1} > 0$, due to the weak interference assumption. Thus $A^{-1}\mathbf{1} \geq 0$. □

However, $\boldsymbol{A}^{-1}\boldsymbol{\sigma}$ may be negative for asymmetric noise levels.

Lemma 9. *Assume conditions (i)–(iii) of theorem 4 hold. For each j consider a set S_j and construct $\bar{\boldsymbol{A}}_j$ so that $[\bar{\boldsymbol{A}}_j]_{il} := [\boldsymbol{A}_j]_{il}$ if $i, l \in S_j$ and 0 otherwise. Then*

$$\mathbf{1}^\top \bar{\boldsymbol{A}}_j^\dagger \left(\sum_k \bar{\boldsymbol{A}}_k^\dagger \right)^\dagger \bar{\boldsymbol{A}}_l{}^\dagger \mathbf{1} \geq 0 \quad \forall j, l. \tag{35}$$

Proof. Applying lemma 8 to the range of $\bar{\boldsymbol{A}}_l$ implies that $\bar{\boldsymbol{A}}_l^\dagger \mathbf{1} \geq 0$. Similarly for $\bar{\boldsymbol{A}}_k$. Applying lemma 7 to the union of the ranges of $\boldsymbol{D}_j$ shows that $\left(\sum_k \bar{\boldsymbol{A}}_k^\dagger\right)^\dagger \geq 0$. This proves the claim because the product of nonnegative vectors and a nonnegative matrix is nonnegative. □

References

[1] Y. Ye, “Competitive communication spectrum economy and equilibrium,” *Working Paper*, Oct. 2007.

[2] M. Ling, J. Tsai, and Y. Ye, “Budget allocation in a competitive communication spectrum economy,” *submitted to EURASIP Journal on Advances in Signal Processing*, Sept. 2008.

[3] N. Yamashita and Z.-Q. Luo, “A nonlinear complementatity approach to multiuser power control for digital subscriber lines,” *Optimization Methods and Software*, no. 19, 2004.

[4] Z.-Q. Luo and J.-S. Pang, “Analysis of iterative waterfilling algorithm for multiuser power control in digital subscriber lines,” *EURASIP Journal on Applied Signal Processing*, Article ID: 24012 2007.

[5] Z.-Q. Luo and S. Zhang, “Dynamic spectrum management: Complexity and duality,” *IEEE Journal of Selected Topics in Signal Processing*, vol. 2, pp. 57–73, Feburary 2008.

[6] A. Goldsmith, S. A. Jafar, I. Maric, and S. Srinivasa, "Breaking spectrum gridlock with cognitive radios: An information theoretic perspective," *Proceedings of the IEEE*, to appear 2008.

[7] D. Niyato and E. Hossain, *Microeconomic Models for Dynamic Spectrum Management in Cognitive Radio Networks*, ch. 14. Cognitive Wireless Communication Networks, US: Springer, 2008.

[8] F. P. Kelly, A. K. Maulloo, and D. K. H. Tan, "Rate control for communication networks: shadow prices, proportional fairness and stability," *Journal of the Operational Research Society*, vol. 49, 1998.

[9] E. Anderson, F. Kelly, and R. Steinberg, "A contract and balancing mechanism for sharing capacity in a communication network," *Management Science*, vol. 52, January 2006.

[10] W. Yu, G. Ginis, and J. Cioffi, "Distributed multiuser power control for digital subscriber lines," *IEEE Journal on Selected Areas in Communications*, vol. 20, 2002.

[11] S. T. Chung, S. J. Kim, J. Lee, and J. M. Cioffi, "A game-theoretic approach to power allocation in frequency-selective gaussian interference channels," in *Information Theory, 2003. Proceedings. IEEE International Symposium on*, p. 316, 2003.

[12] T. Alpcan, T. Başar, R. Srikant, and E. Altman, "CDMA uplink power control as a noncooperative game," *Wireless Networks*, vol. 8, pp. 659–670, November 2002.

[13] L. Lai and H. E. Gamal, "The water-filling game in fading multiple-access channels," *IEEE Transactions on Information Theory*, vol. 54, May 2008.

[14] M. Chiang, P. Hande, T. Lan, and C. Tan, *Power control in wireless cellular networks.* Foundation and Trends in Networking Sample, now Publishers Inc., 2008.

[15] P. Samuelson, *Foundations of economic analysis.* Harvard University Press, 1983.

[16] R. W. Cottle, J. S. Pang, and R. E. Stone, *The linear complementarity problem.* Computer Science and Scientific Computing, Academic Press, February 1992.

[17] Z. Zhu, C. Dang, and Y. Ye, "A FPTAS for computing a symmetric leontief competitive economy equilibrium," *Proceedings of 4th International Workshop on Internet and Network Economics (WINE)*, December 2008.

[18] J. G. Proakis, *Digital Communications.* NY, USA: McGraw-Hill, 3 ed., 1995.

[19] A. Mas-Colell, M. D. Whinston, and J. R. Green, *Microeconomic theory.* Oxford University Press, 1995.

[20] K. J. Arrow, H. D. Block, and L. Hurwicz, "On the stability of the competitive equilibrium, ii," *Econometrica*, vol. 27, no. 1, pp. 82–109, 1959.

[21] B. Codenotti, B. McCune, and K. Varadarajan, "Market equilibrium via the excess demand function," in *STOC '05: Proceedings of the thirty-seventh annual ACM symposium on Theory of computing*, (New York, NY, USA), pp. 74–83, ACM, 2005.

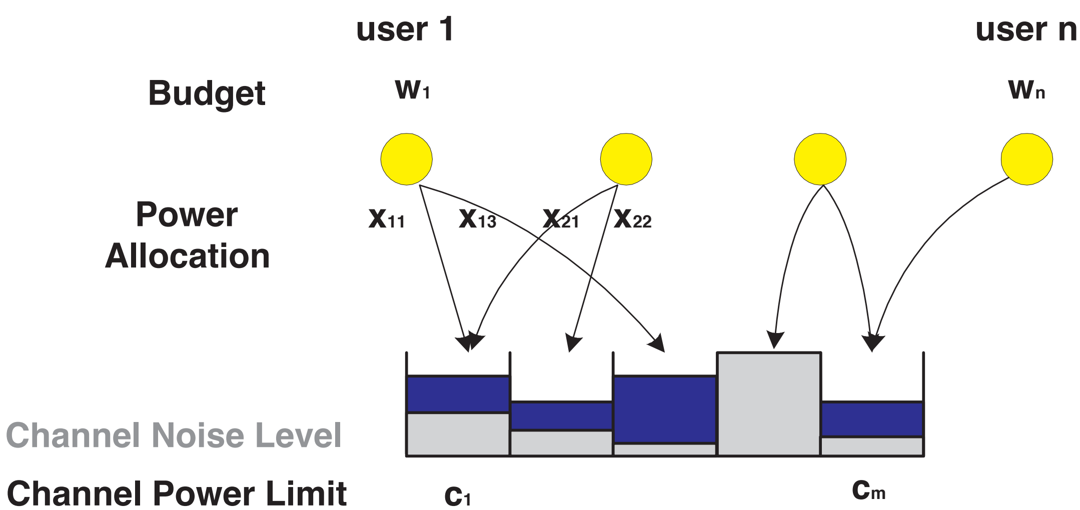

Figure 1: Competitive spectrum market model.