A Path to the Arrow-Debreu Competitive Market Equilibrium

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Abstract: We present polynomial-time interior-point algorithms for solving the Fisher and Arrow-Debreu competitive market equilibrium problems with linear utilities and n players. Both of them have the arithmetic operation complexity bound of $O(n^4 \log(1/\epsilon))$ for computing an ϵ-equilibrium solution. If the problem data are rational numbers and their bit-length is L, then the bound to generate an exact solution is $O(n^4 L)$ which is in line with the best complexity bound for linear programming of the same dimension and size. This is a significant improvement over the previously best bound $O(n^8 \log(1/\epsilon))$ for approximating the two problems using other methods. The key ingredient to derive these results is to show that these problems admit convex optimization formulations and efficient barrier functions. We also present a continuous path leading to the set of the Arrow-Debreu equilibrium, similar to the central path developed for linear programming interior-point methods. This path is derived from the weighted logarithmic utility and barrier functions and the Brouwer fixed-point theorem. The defining equations are bilinear and possess some primal-dual structure for the application of Newton's path-following method.

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1 Introduction

We consider the Arrow-Debreu competitive market equilibrium problem which was first formulated by Leon Walras in 1874 [33]. In this problem everyone in a population of n players has an initial endowment of a divisible good and a utility function for consuming all goods—their own and others. Every player sells the entire initial endowment and then uses the revenue to buy a bundle of goods such that his or her utility function is maximized. Walras asked whether prices could be set for everyone's good such that this is possible. An answer was given by Arrow and Debreu in 1954 [1] who showed that such equilibrium would exist if the utility functions were concave. Their proof was non-constructive and did not offer any algorithm to find such equilibrium prices.

Fisher was the first to consider algorithm to compute equilibrium prices for a related and different model where players are divided into two sets: producer and consumer; see Brainard and Scarf [3, 31]. Consumers have money to buy goods and maximize their individual utility functions; producers sell their goods for money. The price equilibrium is an assignment of prices to goods so that when every consumer buys a maximal bundle of goods then the market clears, meaning that all the money is spent and all the goods are sold. Fisher's model is a special case of Walras' model when money is also considered a commodity so that Arrow and Debreu's result applies.

Eisenberg and Gale [11, 15] gave a convex optimization setting to formulate Fisher's model with linear utility functions. They constructed a concave objective function that is maximized at the equilibrium. Thus, finding an equilibrium became solving a convex optimization problem, and it could be obtained by using the Ellipsoid method in polynomial time. Here, polynomial time means that one can compute an ϵ approximate equilibrium in a number of arithmetic operations bounded by polynomial in n and $\log \frac{1}{\epsilon}$. Devanur et al. [9] recently developed a "combinatorial" algorithm for solving Fisher's model with linear utility functions too. Both the ellipsoid method and the combinatorial algorithm have running times of the order of $O(n^8 \log(1/\epsilon))$. Neither approach, Eisenberg-Gale or Devanur et al., applied to the more general Walras model. The ϵ based complexity result seems more appropriate for analyzing these problems because solutions may be irrational, when the economy model or utility function is more general, even all input data are rational.

Solving the Arrow-Debreu problem proved to be more difficult. Eaves [12] showed that the problem with linear utility can be formulated as a linear complementarity problem (e.g. Cottle et al. [6]) so that Lemke's algorithm could compute the equilibrium, if it existed, in a finite time. It was also proved there that there is an equilibrium solution whose entries were rational as a solution to an n^2-dimension system of linear equations of the original rational inputs. In a later paper [13], Eaves also proved that the problem with Cobb-Douglas utility could be solved in strongly polynomial time of $O(n^3)$. Other effective algorithms to solve the problem include Primak [29], Dirkse and Ferris [10], and Rutherford [30]; see the excellent survey by Ferris and Pang [14]. None of these are proved to be polynomial-time algorithms.

More recently, however, Jain [17] has showed that Walras's model can be also formulated as a convex optimization, more precisely, a convex inequality problem, so that the ellipsoid method again can be used in solving it. Remarkably, it turned out that the very same formulation was developed by Nenakhov and Primak [27] more than twenty years earlier. They found out a clean set of posinomial inequalities to describe the problem which are necessary and sufficient. This set of inequalities can be logarithmically transformed into a set of convex inequalities, a technique which was used in the early '60s for geometric programming.

The goal of this paper is threefold. First, we develop a polynomial-time interior-point algorithm to solve Fisher's model with linear utility. The complexity bound, $O(n^4 \log \frac{1}{\epsilon})$, of this algorithm is significantly lower than that of either the ellipsoid or the "combinatorial" algorithm mentioned above. Secondly, we present an interior-point algorithm, which is not primal-dual, for solving the Arrow-Debreu pure exchange market equilibrium problem with linear utility. The algorithm has an efficient barrier function for every convex inequality where the self-concordant coefficient is at most 2. Thus, the number of arithmetic operations of the algorithm is again bounded by $O(n^4 \log \frac{1}{\epsilon})$, which is substantially lower than the one obtained by the ellipsoid method. If the input data are rational, then an exact solution can be obtained by solving the identified system of linear equations and inequalities, such as in Eaves' model, when $\epsilon < 2^{-L}$, where L is the bit length of the input data. Thus, the arithmetic operation bound becomes $O(n^4 L)$, which is in line with the best complexity bound for linear programming of the same dimension and size.

Finally, we develop a convex optimization setting for Walras' model, and present a continuous path leading to the set of Arrow-Debreu equilibria, similar to the central path developed for linear programming interior-point methods (see, e.g., Megiddo [22]). The path is derived from the weighted logarithmic utility and barrier functions and the Brouwer fixed-point theorem. The defining equations are bilinear and possess some primal-dual structure for the application of Newton's method. We also discuss some extensions of our results at the end of the paper.

2 An Interior-Point Algorithm for Solving the Fisher Equilibrium Problem

In Fisher's model the players are divided into two sets: producer and consumer. Consumer i, $i \in C$, has given money endowment w_i to spend and buy goods to maximize their individual utility functions; producer j, $j \in P$, sells their goods for money. The price equilibrium is an assignment of prices to goods so that when every consumer buys a maximal bundle of goods then the market clears, meaning that all the money is spent and all the goods are sold. Eisenberg and Gale [11] gave a convex optimization formulation, where, without losing generality, each producer has one unit of his or her good.

$$\begin{array}{ll} \text{maximize} & \sum_{i\in C} w_i \log\left(\sum_{j\in P} u_{ij}x_{ij}\right) \\ \text{subject to} & \sum_{i\in C} x_{ij} = 1, \quad \forall j \in P \\ & x_{ij} \geq 0, \quad \forall i, j. \end{array}$$

Here, player i, $i \in C$, has a linear utility function

$$u_i(x_{i\cdot}) = u(x_{i1}, ..., x_{in}) = \sum_j u_{ij}x_{ij},$$

where $u_{ij} \geq 0$ is the given utility coefficient of player i for producer j's good and x_{ij} represents the amount of good bought from producer j by consumer i. They proved that the optimal Largrange multipliers of this convex problem is the market clearing prices.

Through out this paper, we make the following assumptions:

Assumption 1. *Every consumer's initial money endowment $w_i > 0$, at least one $u_{ij} > 0$ for every $i \in C$, and at least one $u_{ij} > 0$ for every $j \in P$.*

This is to say that every consumer in the market has money to spend and he or she likes at least one good; and every good is valued by at least one consumer. We will see that, with these assumptions, each good can have a positive equilibrium price. If a consumer has zero budget or his or her utility has zero value for every good, then buying nothing is an optimal solution for him or her so that he or she can be removed from the market; if a good has zero value to every consumer, then it is a "free" good with zero price in a price equilibrium and can be arbitrarily distributed among the consumers so that it can be removed from the market too.

2.1 The weighted analytic center

The Eisenberg-Gale model can be rewritten as

$$\begin{array}{rc} \text{maximize} & \sum_{i \in C} w_i \log(u_i) \\ \text{subject to} & \sum_{i \in C} x_{ij} = 1, \quad \forall j \in P \\ & u_i - \sum_{j \in P} u_{ij} x_{ij} = 0, \quad \forall i \in C \\ & u_i,\ x_{ij} \geq 0, \quad \forall i, j. \end{array} \tag{1}$$

Consider a more general problem

$$\begin{array}{rc} \text{maximize} & \sum_{j=1}^{n} w_j \log(x_j) \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \tag{2}$$

where the given A is an $m \times n$-dimensional matrix with full row rank, b is an m-dimensional vector, and w_j is the nonnegative weight on the jth variable. Any x who satisfies the constraints is called a primal feasible solution, while any optimal solution to the problem is called a weighted analytic center.

If the weighted analytic center problem has an optimal solution, the optimality conditions are

$$\begin{aligned} Sx &= w, \\ Ax &= b, \ x \geq 0, \\ -A^T y + s &= 0, \ s \geq 0, \end{aligned} \tag{3}$$

where y and s are the Largrange or KKT multipliers (also dual variable and slacks of the dual linear program: min $b^T y$ subject to $s = A^T y \geq 0$), and S is the diagonal matrix with slack vector s on its diagonals. Let the feasible set of (2) be bounded and has a (relative) interior, i.e., has a strictly feasible point $x > 0$ with $Ax = b$ (clearly holds for problem (1)). Then, there is a strictly feasible dual solution $s > 0$ with $s = A^T y$ for some y. Moreover, from the literature of interior-point algorithms (e.g., Megiddo and Kojima et al. [22, 21] and Güler [16]) here are what we know about the problem:

- The mapping $u(x, s) = Sx$ maps $F_{++} := \{(x, s) > 0 : \ Ax = b, \ s = A^T y\}$ onto $R^n_{++} := \{u > 0 \in R^n\}$ diffeomorphically, or $u(\cdot, \cdot)$ is continuous, differentiable and one-to-one, i.e., for any $w \in R^n_{++}$, system (3) has a unique solution.

- The inverse mapping maps $R^n_+ := \{u \geq 0 \in R^n\}$ to $F_+ := \{(x, s) \geq 0 : \ Ax = b, \ s = A^T y\}$ upper semi-continuously. In particular, let $w := \mu \bar{w}$, where vectors $\bar{w} > 0$ is fixed, and consider the solution of (3) parameterized by scalar $\mu > 0$. Then, the path of the solution is a one-dimensional smooth curve and it converges as μ tends to 0 from above.

When $w_j > 0$ for all j and integral for all j, a weight-scaling interior-point algorithm was developed by Atkinson and Vaidya [2] where the arithmetic operation complexity bound is $O(n^3 \log(\frac{\max(w)}{\min(w)}))$ to compute a solution such that

$$\begin{aligned} \|Sx - w\| &\leq O(\min(w)), \\ Ax &= b, \; x \geq 0, \\ -A^T y + s &= 0, \; s \geq 0. \end{aligned}$$

They start with an approximate analytic center where all weights equal $\min(w)$, and then scale them up to w iteratively. It is not clear how their algorithm can be adapted or analyzed when some of w_j are zeros, which is the case of Fisher's model (1).

2.2 A modified primal-dual path-following algorithm

In this subsection, we modify the standard primal-dual path-following algorithm (e.g., Kojima et al. [20], Monteiro and Adler [25] and Mizuno et al. [24]) for solving problems (2) and (1) and analyze their complexity to computing an ϵ-solution for any $\epsilon > 0$:

$$\begin{aligned} \|Sx - w\| &\leq \epsilon, \\ Ax &= b, \; x \geq 0, \\ -A^T y + s &= 0, \; s \geq 0. \end{aligned} \tag{4}$$

Let $x > 0$ with $Ax = b$ and $(y, s > 0)$ with $s = A^T y$ be a primal-dual interior-point pair such that

$$\|Sx - \hat{w}\| \leq \eta\mu, \tag{5}$$

where $\mu \geq 0$ represents an error measure (similar to the complementarity gap in classical interior-point algorithms for linear programming), η is a positive constant less than 1, and

$$\hat{w}_j = \max\{\mu, \; w_j\}. \tag{6}$$

Such a point pair is called an approximate central-path point pair of the primal-dual feasible set F_{++}.

Now we solve a prima-dual system of linear equations for d_x, d_y and d_s:

$$\begin{aligned} Sd_x + Xd_s &= \hat{w}^+ - Xs, \\ Ad_x &= 0, \\ -A^T d_y + d_s &= 0, \end{aligned} \tag{7}$$

where

$$\hat{w}_j^+ = \max\{(1 - \frac{\eta}{\sqrt{n}})\mu, w_j\}. \tag{8}$$

Note that $d_x^T d_s = d_x^T A^T d_y = 0$ here. The work involved in solving the system is to form the normal matrix ADA^T, where D is a diagonal matrix whose diagonal entries are strictly positive, and factorize it. More precisely, pre-multiplying S^{-1} to both sides of the first equation of (7) we have

$$d_x + S^{-1}Xd_s = S^{-1}(\hat{w}^+ - Xs);$$

pre-multiplying A and noting $Ad_x = 0$ we have

$$AS^{-1}Xd_s = AS^{-1}(\hat{w}^+ - Xs);$$

substituting $d_s = A^T d_y$ we have

$$AS^{-1}XA^T d_y = AS^{-1}(\hat{w}^+ - Xs),$$

where $AS^{-1}XA^T$ is the normal matrix with $D = S^{-1}X$.

After obtaining (d_x, d_y, d_s) let

$$\begin{aligned} x^+ &:= x + d_x, \\ y^+ &:= y + d_y, \\ s^+ &:= s + d_s. \end{aligned} \tag{9}$$

Then, we prove that x^+ and (y^+, s^+) are an interior-point feasible pair, and

$$\|(S^+)x^+ - \hat{w}^+\| \leq \eta\mu^+ \tag{10}$$

where

$$\mu^+ = (1 - \frac{\eta}{\sqrt{n}})\mu,$$

so that the computation can repeat.

First, it is helpful to re-express d_x and d_s. Let

$$\begin{aligned} p &:= X^{-.5}S^{.5}d_x, \\ q &:= X^{.5}S^{-.5}d_s, \\ r &:= (XS)^{-.5}(\hat{w}^+ - Xs), \end{aligned} \tag{11}$$

Note that

$$p + q = r \quad \text{and} \quad p^T q = 0$$

so that p and q represent an orthogonal decomposition of r.

Secondly, from (5), (6), and (8), we have

$$x_j s_j \geq \hat{w}_j - \eta\mu \geq (1-\eta)\mu$$

and

$$\|\hat{w}^+ - Xs\| = \|\hat{w}^+ - \hat{w} + \hat{w} - Xs\| \leq \|\hat{w}^+ - \hat{w}\| + \|\hat{w} - Xs\| \leq \eta\mu + \eta\mu = 2\eta\mu,$$

which implies that

$$\|r\| \leq \|(XS)^{-.5}\|\|\hat{w}^+ - Xs\| \leq \frac{2\eta\sqrt{\mu}}{\sqrt{1-\eta}}.$$

Moreover, it is also proved in Mizuno et al. [24] that

$$\|p\|^2 + \|q\|^2 = \|r\|^2 \quad \text{and} \quad \|Pq\| \leq \frac{\sqrt{2}}{4}\|r\|^2.$$

Thus,

$$\begin{aligned} \|(S^+)x^+ - \hat{w}^+\|^2 &= \|(S + D_s)(x + d_x) - \hat{w}^+\|^2 \\ &= \|Sx + Sd_x + Xd_s - \hat{w}^+ + D_s d_x\|^2 \\ &= \|D_s d_x\|^2 \\ &= \|Pq\|^2 \\ &\leq \frac{\sqrt{2}}{4}\|r\|^2 \\ &\leq \frac{\sqrt{2}\eta^2}{1-\eta}\mu \\ &\leq \frac{\sqrt{2}\eta^2}{(1-\eta)^2}\mu^+. \end{aligned}$$

Thus, if we choose constant η such that

$$\frac{\sqrt{2}\eta^2}{(1-\eta)^2} \le \eta$$

(for example, $\eta = 1/4$), then condition (10) holds. Moreover,

$$\begin{aligned}
\|X^{-1}(x^+ - x)\| &= \|X^{-1}d_x\| \\
&= \|(XS)^{-.5}p\| \\
&\le \|(XS)^{-.5}\|\|p\| \\
&\le \frac{\|p\|}{\sqrt{(1-\eta)\mu}} \\
&\le \frac{\|r\|}{\sqrt{(1-\eta)\mu}} \\
&\le \frac{2\eta}{1-\eta} < 1,
\end{aligned}$$

which implies that $x^+ > 0$. Similarly, we have $s^+ > 0$. That is, (x^+, y^+, s^+) is a feasible interior-point pair.

We can generate an initial point pair $x^0 > 0$ and $s^0 > 0$ such that

$$\|S^0x^0 - \mu^0 e\| \le \eta\mu^0$$

where $\mu^0 = \max(w)$ and e is the vector of all ones. Such a point pair corresponds to an approximate analytic center of the bounded primal feasible and dual objective-level set. In problem (1), the primal feasible set has a relative interior and it is bounded, which implies that the dual feasible set has a relative interior and its objective-level set is bounded. The complexity to generate such an initial point pair is $O(n^3 \log n)$ arithmetic operations which will be seen in the next section. Since the dual feasible set is homogeneous, we can always scale (y, s) so that $\mu^0 = \max(w)$.

Note that μ is decreased at a geometric rate $(1 - \eta/\sqrt{n})$ and it starts at $\max(w)$. Also, if $w_j = 0$ for some j, then

$$s_jx_j \le \frac{\epsilon}{\sqrt{n}}$$

from

$$|s_jx_j - \mu| \le \eta\mu$$

as soon as $\mu \le \frac{\epsilon}{\sqrt{n}(1+\eta)}$. Thus, we have

Theorem 1. *The primal-dual path-following algorithm solves the partial weight analytic center problem (2) in $O(\sqrt{n}\log(n\max(w)/\epsilon))$ iterations and each iteration solves a system of linear equations in $O(nm^2+m^3)$ arithmetic operations. If Karmarkar's rank-one update technique is used, the average arithmetic operations per iteration can be reduced to $O(n^{1.5}m)$.*

If the predictor and corrector algorithm of Mizuno et al. [24] is used, the quadratic convergence result of [35] (also see [26]) applies to solving problem (2). We have

Corollary 1. *The primal-dual predictor-corrector algorithm solves the partial weight analytic center problem (2) in $O(\sqrt{n}(\log(n\max(w)C(A,b)) + \log\log(1/\epsilon))$ iterations and each iteration solves a system of linear equations in $O(nm^2+m^3)$ arithmetic operations. Here, $C(A,b)$ is a positive fixed number depending on the data A and b, and if the entries of A and b are rational numbers then $C(A,b) \le 2^{O(L(A,b))}$ where $L(A,b)$ is the bit-length of A and b.*

These results indicate that the complexity of the weighted analytic center problem is in line with linear programming of the same dimension and size.

2.3 Complexity analysis of solving the Fisher equilibrium

In solving Fisher's problem with n producers and n consumers formulated by Eisenberg and Gale in (1), the number of variables becomes n^2+n and the number of equalities is $2n$. We can assign the initial x^0 such that

$$x^0_{ij} = \frac{1}{n}, \quad \forall i,j$$

so that

$$u^0_i = \frac{1}{n}\sum_{j\in P} u_{ij}, \quad \forall i.$$

Let the dual vector $y=(p;\pi)$ and set the dual variable with equality constraint $j \in P$

$$p^0_j = 2n\beta$$

and dual variable with equality constraint $i \in C$ be

$$\pi_i^0 = \frac{\beta}{u_i^0}.$$

Then, we have slack variable $s_i^0 = \pi_i^0$ and u_i^0

$$\pi_i^0 u_i^0 = \beta, \quad \forall i$$

and slack variable s_{ij}^0 and x_{ij}^0

$$s_{ij}^0 x_{ij}^0 = (p_j^0 - \pi_i^0 u_{ij})/n = 2\beta - \frac{u_{ij}\beta}{\sum_{k\in P} u_{ik}}, \quad \forall i, j$$

which is between β and 2β. Using at most $O(\log(n))$ interior-point iterations, we will have an interior-point pair satisfying condition (5) (e.g., see [36]).

Moreover, matrix A of (1) is sparse and each of its columns has at most two nonzeros. Thus, ADA^T can be formed in at most $O(n^2)$ operations, and it can be factorized in $O(n^3)$ arithmetic operations. Thus, we have

Theorem 2. *The modified primal-dual path-following algorithm solves the Fisher equilibrium problem (1) with n producers and n consumers in at most $O(n \log(n \max(w)/\epsilon))$ iterations and each iteration solves a system of linear equations in $O(n^3)$ arithmetic operations.*

This results a significant improvement over the $O(n^8 \log(n/\epsilon))$ arithmetic operation bound of either the ellipsoid method and the combinatorial algorithm mentioned earlier.

In addition to the feasibility conditions, the optimality conditions of the Eisenberg-Gale formulation can be written as

$$\begin{array}{rcl} p_j \cdot \sum_{k\in P} u_{ik} x_{ik} & \geq & w_i u_{ij}, \quad \forall i, j \\ x_{ij} p_j \cdot \sum_{k\in P} u_{ik} x_{ik} & = & x_{ij} w_i u_{ij}, \quad \forall i, j. \end{array}$$

One can see that Assumption 1 on w_i and u_{ij} implies $p > 0$. Moreover, an optimal solution x_{ij} and p of the Eisenberg-Gale formulation is a solution of the system equations and inequalities:

$$\begin{array}{llll} p_j = \frac{w_i u_{ij}}{\sum_{k\in P} u_{ik} x_{ik}}, & x_{ij} > 0, & \forall (i,j) \in B^* \\ p_j = \frac{w_i u_{ij}}{\sum_{k\in P} u_{ik} x_{ik}}, & x_{ij} = 0, & \forall (i,j) \in Z^* \\ p_j > \frac{w_i u_{ij}}{\sum_{k\in P} u_{ik} x_{ik}}, & x_{ij} = 0, & \forall (i,j) \in N^* \\ \sum_{i\in C} x_{ij} = 1, & p_j > 0, & \forall j \end{array}$$

where B^* is the set of the optimal super-basic variables x_{ij} which can be positive at an optimal primal solution, N^* is the set of optimal dual slacks

$$s_{ij} = p_j - \frac{w_i u_{ij}}{\sum_{k \in P} u_{ik} x_{ik}}$$

which can be positive at an optimal dual solution, and Z^* contains the rest. Since the optimal solution set of the Eisenberg-Gale formulation is convex, (B^*, Z^*, N^*) is a unique partition of all variables, and an optimal solution pair with $x_{ij} > 0$ for all $(i,j) \in B^*$ and $s_{ij} > 0$ for all $(i,j) \in N^*$ is called a (relative) interior-point or maximal-cardinality solution pair. A rounding procedure for interior-point algorithms was developed to identify the partition and to round an approximate solution to an exact (relative) interior-point solution for solving a range of convex optimization problems; see, e.g., [23, 36].

Note that for any given $(i,j) \in B^*$ we have $u_{ij} > 0$ and if $(i,k) \in B^*$

$$\frac{u_{ij}}{p_j} = \frac{u_{ik}}{p_k};$$

and if $(i,k) \notin B^*$

$$\frac{u_{ij}}{p_j} \geq \frac{u_{ik}}{p_k}.$$

For any i, let

$$\lambda_i = \frac{p_k}{u_{ik}}, \ \forall (i,k) \in B^*.$$

Then, for any i, we have

$$\begin{aligned}
\sum_{k \in P} u_{ik} x_{ik} &= \sum_{k \in P} \frac{u_{ik}}{p_k} p_k x_{ik} \\
&= \sum_{k:(i,k) \in B^*} \frac{u_{ik}}{p_k} p_k x_{ik} \\
&= \sum_{k:(i,k) \in B^*} \frac{1}{\lambda_i} p_k x_{ik} \\
&= \frac{1}{\lambda_i} \sum_{k:(i,k) \in B^*} p_k x_{ik} \\
&= \frac{1}{\lambda_i} \sum_{k \in P} p_k x_{ik}.
\end{aligned}$$

Therefore, if we view products $p_j x_{ij}$ as new variables y_{ij}, then the above system becomes a system of *linear equations and inequalities*:

$$\begin{array}{lll} u_{ij}\lambda_i = p_j, & y_{ij} > 0, & \forall (i,j) \in B^* \\ u_{ij}\lambda_i = p_j, & y_{ij} = 0, & \forall (i,j) \in Z^* \\ u_{ij}\lambda_i < p_j, & y_{ij} = 0, & \forall (i,j) \in N^* \\ & \sum_{j\in P} y_{ij} = w_i, & \forall i \\ & \sum_{i\in C} y_{ij} = p_j, & \forall j. \end{array}$$

(Note the network-flow structure of the system which was explored by Devanur et al. [9].) Hence, there exist a solution where entries of y^*_{ij}, p^*_j and λ^*_i must be rational numbers and their size is bounded by the bit-length L of all input data u_{ij} and w_i. Moreover, there is a relative interior-point solution to the system such that

$$y^*_{ij} \geq 2^{-L}, \ \forall (i,j) \in B^*$$

and

$$2^L \geq p^*_j \geq u_{ij}\lambda^*_i + 2^{-L}, \ \forall (i,j) \in N^*.$$

These bounds are transformed back on the corresponding solution to the original system

$$\begin{array}{c} x^*_{ij} \geq 2^{-2L}, \ \forall (i,j) \in B^* \\ s^*_{ij} = p^*_j - \frac{w_i u_{ij}}{\sum_{k\in P} u_{ik} x^*_{ik}} \geq 2^{-2L}, \ \forall (i,j) \in N^*. \end{array} \tag{12}$$

Thus, the interior-point algorithm rounding technique (e.g., [23, 36]) can be applied to identify the partition and to compute an exact solution of the above system in $O(nL)$ interior-point algorithm iterations. I now give a complete proof below.

Consider the more general problem (2) and let $W = \{j : \ w_j > 0, \ j = 1, ..., n\}$. Then, the pair (x^*_j, s^*_j) must satisfy $x^*_j s^*_j = w_j$ for $j \in W$ and $x^*_j s^*_j = 0$ for $j \notin W$ in any optimal solution pair (x^*, s^*) of (2). Let (x, s) be any feasible solution pair (x, s) who satisfies the centering condition (5) and (6) for $\mu \leq \min\{w_j : \ j \in W\}$. Then,

$$w_j - \eta\mu \leq x_j s_j \leq w_j + \eta\mu, \ \forall j \in W$$

and

$$(1-\eta)\mu \le x_j s_j \le (1+\eta)\mu, \ \forall j \notin W.$$

For simplicity, let $\{1\} \notin W$ and $x_1^* > 0$ ($s_1^* = 0$) in a relative interior-point solution pair (x^*, s^*) of (2), i.e., $\{1\} \in B^*$. Since

$$(x - x^*)^T (s - s^*) = 0$$

we have

$$s^T x^* + x^T s^* = x^T s + (x^*)^T s^* \le \sum_{j \in W} (2w_j + \eta\mu) + \sum_{j \notin W} (1+\eta)\mu$$

or

$$\sum_{j \in W} (x_j^* s_j + s_j^* x_j) + \sum_{j \notin W} (x_j^* s_j + s_j^* x_j) \le \sum_{j \in W} (2w_j + \eta\mu) + \sum_{j \notin W} (1+\eta)\mu. \tag{13}$$

For every $j \in W$, we have

$$x_j^* s_j + s_j^* x_j \ge 2\sqrt{(x_j s_j)(x_j^* s_j^*)} = 2\sqrt{(x_j s_j) w_j} \ge 2w_j \sqrt{1 - \frac{\eta\mu}{w_j}} \ge 2w_j (1 - \frac{\eta\mu}{w_j}) = 2w_j - 2\eta\mu,$$

which, together with (13) and $\eta = 1/4$, imply

$$\begin{aligned} \sum_{j \notin W} (x_j^* s_j + s_j^* x_j) &\le \sum_{j \in W} (2w_j + \eta\mu) + \sum_{j \notin W} (1+\eta)\mu - \sum_{j \in W} (2w_j - 2\eta\mu) \\ &= \sum_{j \in W} (3\eta\mu) + \sum_{j \notin W} (1+\eta)\mu \le n(1+\eta)\mu. \end{aligned}$$

Therefore, in particular, we have

$$s_1 x_1^* = s_1 x_1^* + x_1 s_1^* \le n(1+\eta)\mu,$$

so that

$$x_1 n(1+\eta)\mu \ge x_1 s_1 x_1^* \ge (1-\eta)\mu x_1^*$$

which implies that

$$x_1 \ge \frac{1-\eta}{n(1+\eta)} x_1^* \quad \text{and} \quad s_1 \le n(1+\eta) \frac{\mu}{x_1^*}.$$

Similarly, if $s_1^* > 0$ ($x_1^* = 0$) in the pair (x^*, s^*), i.e., $\{1\} \in N^*$, we have

$$s_1 \geq \frac{1-\eta}{n(1+\eta)} s_1^* \quad \text{and} \quad x_1 \leq n(1+\eta)\frac{\mu}{s_1^*}.$$

Now we define the set

$$P^k = \{j : \ x_j^k \geq s_j^k\}$$

where $\{x^k, s^k\}$ is the solution sequence generated by the interior-point algorithm proposed earlier, and have

$$B^* \subset P^k \quad \text{and} \quad N^* \not\subset P^k,$$

as soon as

$$\mu^k < \frac{1-\eta}{n^2(1+\eta)^2} \min\{(x_j^* + s_j^*)^2 : \ j \in B^* \cup N^*\}.$$

We have just shown in (12) that for solving the Fisher problem

$$\min\{(x_j^* + s_j^*)^2 : \ j \in B^* \cup N^*\} \geq 2^{-2L}.$$

Thus, after $O(nL)$ interior-point algorithm iterations (we omit $\log n$), the algorithm guarantees $B^* \subset P^k$ and $N^* \not\subset P^k$. Then we formulate the system of *linear equations and inequalities*:

$$\begin{array}{lll} u_{ij}\lambda_i = p_j, & y_{ij} \geq 0, & \forall (i,j) \in P^k \\ u_{ij}\lambda_i \leq p_j, & y_{ij} = 0, & \forall (i,j) \notin P^k \\ & \sum_{j \in P} y_{ij} = w_i, & \forall i \\ & \sum_{i \in C} y_{ij} = p_j, & \forall j, \end{array}$$

where the system is for sure to have a feasible solution and any solution is a Fisher price equilibrium. One can simply apply an interior-point *linear programming* algorithm to computing an exact feasible solution in no more than $O(n^4L)$ arithmetic operations. To summarize, we have

Theorem 3. *The modified primal-dual path-following algorithm, coupled with the rounding procedure, solves the Fisher equilibrium problem (1) with n producers and n consumers exactly in at most $O(n^4L)$ arithmetic operations, where L is the bit-length of the input data u_{ij} and w_i.*

This theorem indicates, for the first time, that the complexity of the Fisher equilibrium problem is completely in line with linear programming of the same dimension and size.

3 An Interior-Point Algorithm for Solving the Arrow-Debreu Equilibrium Problem

Here, without loss of generality, let each of the n players have exactly one unit of divisible good for trade (we will relax this assumption later), and let player i, $i = 1, ..., n$, have the linear utility function

$$u_i(x_{i1}, ..., x_{in}) = \sum_j u_{ij} x_{ij},$$

where u_{ij} is the given utility coefficient of player i for player j's good and x_{ij} represents the amount of good bought from player j by player i. Again, assume that at least one $u_{ij} > 0$ for every i, and at least one $u_{ij} > 0$ for every j; that is, every player in the market likes at least one good; and every good is valued by at least one player. We will see that, with these assumptions, each good can have a positive equilibrium price.

The main difference between Fisher's and Walras' models is that, in the latter, each player is both producer and consumer and the initial endowment of player i is *not* given and will be the price assigned to his or her good. Nevertheless, we can still write a (parametric) convex optimization model:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n w_i \log\left(\sum_{j=1}^n u_{ij} x_{ij}\right) \\ \text{subject to} & \sum_{i=1}^n x_{ij} = 1, \quad \forall j \\ & x_{ij} \geq 0, \quad \forall i, j, \end{array}$$

or

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n w_i \log(u_i) \\ \text{subject to} & \sum_{i=1}^n x_{ij} = 1, \quad \forall j \\ & u_i - \sum_{j=1}^n u_{ij} x_{ij} = 0, \quad \forall i \\ & u_i,\ x_{ij} \geq 0, \quad \forall i, j, \end{array} \tag{14}$$

where we wish to select weights w_i's such that the optimal dual prices equal these weights respectively.

For given w's, the necessary and sufficient optimality conditions of the model are:

$$\begin{array}{rcl} u_i \pi_i &=& w_i, \ \forall i \\ x_{ij}(p_j - u_{ij}\pi_i) &=& 0, \ \forall i, j \\ p_j - u_{ij}\pi_i &\geq& 0, \ \forall i, j \\ \sum_{i=1}^n x_{ij} &=& 1, \ \forall j \\ u_i - \sum_{j=1}^n u_{ij} x_{ij} &=& 0, \ \forall i \\ u_i, \ \pi_i, \ x_{ij} &\geq& 0, \ \forall i, j, \end{array}$$

where p is the n-dimensional optimal dual price vector of the first n equality constraints and π is the n-dimensional optimal dual price vector of the second n equality constraints in (14). We call the first set of equations the weighted centering condition, the second set of equations the complementarity condition, the third set of inequalities the dual feasibility condition, and the fourth and fifth set the primal feasibility conditions.

Later, we will prove that there is indeed a $w \geq 0$ such that $p_i = w_i$ in these conditions, that is, there are (u, x) and (p, π) such that

$$\begin{array}{rcl} u_i \pi_i &=& p_i, \ \forall i \\ x_{ij}(p_j - u_{ij}\pi_i) &=& 0, \ \forall i, j \\ p_j - u_{ij}\pi_i &\geq& 0, \ \forall i, j \\ \sum_{i=1}^n x_{ij} &=& 1, \ \forall j \\ u_i - \sum_{j=1}^n u_{ij} x_{ij} &=& 0, \ \forall i \\ u_i, \ \pi_i, \ x_{ij} &\geq& 0, \ \forall i, j. \end{array} \tag{15}$$

3.1 A self-dual weighted analytic center

Consider a more general problem

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^l w_j \log(x_j) \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \tag{16}$$

where given A is an $m \times n$ matrix with full row rank,

$$b = \begin{pmatrix} e \\ 0 \end{pmatrix} \in R^m,$$

and e is the $l(\leq m)$-dimension vector of all ones.

We prove the following theorem:

Theorem 4. *Assume that the feasible set of (16) is bounded and it has a nonempty relative interior, and the dual feasibility $A^T y \geq 0$ implies $y_1, ..., y_l \geq 0$. Then, there exist $w_1, ..., w_l \geq 0$ such that the entries of an optimal dual vector, corresponding to the first l equality constraints of (16), equal $w_1, ..., w_l$, respectively. When w_j's satisfy this property, we call a solution of (16) a self-dual weighted analytic center of the feasible set.*

Proof. For any given $w_1, ..., w_l \geq 0$, and, without loss of generality, let $e^T w = \sum_{j=1}^{l} w_j = 1$, the optimality conditions of (16) are

$$\begin{array}{rcl} s_j x_j &=& w_j, \ j = 1, ..., l \\ s_j x_j &=& 0, \ j = l+1, ..., n \\ s - A^T y &=& 0, \\ Ax &=& b, \\ x, \ s &\geq& 0. \end{array} \tag{17}$$

These conditions are necessary and sufficient since the feasible set of (16) is bounded and has a nonempty relative interior.

Summing up the top n equalities, we have

$$\sum_{j=1}^{n} s_j x_j = \sum_{j=1}^{l} w_j = 1.$$

But from the rest conditions

$$1 = \sum_{j=1}^{n} s_j x_j = x^T s = x^T (Ay) = (Ax)^T y = b^T y = \sum_{i=1}^{l} y_i.$$

Let $x^0 > 0$ such that $Ax^0 = b$. Then, for any solution (x, y, s) of (17) we have

$$s^T (x^0 - x) = 0$$

or

$$s^T x^0 = s^T x = 1$$

which implies that s is bounded, and so is y since A has full row rank. That is, the dual objective level set

$$\{y: \ A^T y \geq 0, \ b^T y = 1\}$$

is bounded.

From the assumption, $y_j \geq 0$ for $j = 1, ..., l$ as long as $A^T y \geq 0$. Thus, $y(w) := (y_1, ..., y_l)$ is a map, called the Fisher map, from $w = (w_1, ..., w_l)$ in the simplex $S = \{w_j \geq 0 : \sum_{j=1}^{l} w_j = 1\}$ to itself. In general, this mapping may not be one-to-one. But we show that it is upper semi-continuous on S. Our proof is a simplified version of Güler [16]) for proving a more general problem. Let $w^k \in S$, $w^k \to w^* \in S$, and (x^k, y^k, s^k) be any convergent solution sequence of (17) with $w = w^k$, $k = 1, 2, ...$ We show that the limit point (y^∞, s^∞) of the sequence (y^k, s^k), which is bounded in the dual objective level set, is a dual solution of (17) with $w = w^*$.

Let (x^*, y^*, s^*) be a solution of (17) with $w = w^*$. Let $w_j^* > 0$ for $j = 1, ..., l'(\leq l)$ and the rest of them equal zeros. Then, we have $x_j^* s_j^* = w_j^* > 0$ for $j = 1, ..., l'$ and at least one of x_j^* and s_j^* equals 0 for $j = l' + 1, ..., n$. Similarly, both $x_j^\infty s_j^\infty = w_j^*$ for $j = 1, ..., l'$; and at least one of x_j^∞ and s_j^∞ equals 0 for $j = l' + 1, ..., n$, since, otherwise, $x_j^k s_j^k = w_j^k \not\to 0$. Suppose that there is one $\bar{j} \in \{1, ..., n\}$ such that

$$(x_{\bar{j}}^* - x_{\bar{j}}^\infty)(s_{\bar{j}}^* - s_{\bar{j}}^\infty) \neq 0.$$

Note that

$$(x^* - x^\infty)^T (s^* - s^\infty) = \sum_{j=1}^{n} (x_j^* - x_j^\infty)(s_j^* - s_j^\infty) = 0.$$

Then, we must have at least one $\bar{j} \in \{1, ..., n\}$ such that

$$(x_{\bar{j}}^* - x_{\bar{j}}^\infty)(s_{\bar{j}}^* - s_{\bar{j}}^\infty) > 0.$$

Without loss of generality assume $x^*_{\bar{j}} > x^\infty_{\bar{j}} \geq 0$. Then, if $\bar{j} \leq l'$, we must have $s^*_{\bar{j}} \geq s^\infty_{\bar{j}} > 0$ and $w^*_{\bar{j}} = x^*_{\bar{j}} s^*_{\bar{j}} > x^\infty_{\bar{j}} s^\infty_{\bar{j}} = w^*_{\bar{j}}$, which is a contradiction; if $\bar{j} > l'$, we must have $s^*_{\bar{j}} \geq s^\infty_{\bar{j}} \geq 0$ which, from $x^*_{\bar{j}} s^*_{\bar{j}} = 0$, implies that $0 = s^*_{\bar{j}} = s^\infty_{\bar{j}}$, which is also a contradiction. Therefore, we must have $(x^*_j - x^\infty_j)(s^*_j - s^\infty_j) = 0$ for all j, which implies that

$$x^*_j = x^\infty_j \quad \text{and} \quad s^*_j = s^\infty_j, \quad \forall j = 1, ..., l',$$

and

$$\text{either} \quad x^*_j = x^\infty_j = 0 \quad \text{or} \quad s^*_j = s^\infty_j = 0, \quad \forall j = l' + 1, ..., n.$$

Thus, $(x^*, y^\infty, s^\infty)$ satisfy all conditions of (17) with $w = w^*$, so that (y^∞, s^∞) is a dual solution of (17) with $w = w^*$.

Since the mapping $y(w)$ is upper semi-continuous on W, the result follows from the Kakutani fixed-point theorem (see, e.g., [31, 32, 34]). ∎

Todd has suggested a simpler proof of the upper semi-continuousness by considering the graph of the map: (x, y, s, w) who satisfies

$$\begin{aligned} s_j x_j &= w_j, \ j = 1, ..., l \\ s_j x_j &= 0, \ j = l + 1, ..., n \\ s - A^T y &= 0, \\ Ax &= b, \\ e^T w &= 1, \\ x, \ s, \ w &\geq 0. \end{aligned}$$

This is a closed set, so the correspondence is closed. Moreover, the correspondence is bounded from the proof shown above. Thus, the result follows from [32]. However, using the proof of Theorem 4, we can develop a stronger corollary for the Fisher equilibrium formulated in (1).

Corollary 2. *The Fisher price equilibrium is unique under Assumption 1, that is, the Fisher map is one-to-one.*

Proof. From $w_i > 0$ for all i and the proof of Theorem 4, $u^*_i > 0$ is unique for all i in any optimal solution of (1). From the constraint structure of (1), at least one $x^*_{ij} > 0$ or at least one pair $(i, j) \in B^*$ for every j, so that $p_j^* = \frac{w_i}{u_i^* u_{ij}}$ for every j. Thus, p_j^* is also unique for every j. ∎

This corollary is also true even if some $w_i = 0$ since we can remove consumer i from the market, or all $u_{ij} = 0$ for good j since we can set it as a "free" good with price 0. Thus, the Fisher map with linear utilities is a one-to-one map, and the proof of Theorem 4 implies that it is also continuous.

Theorem 4 establishes an alternative to Arrow-Debreu's general proof of equilibria restricted to the case of linear utility. There may be academic advantage of the constructed proof, however. First this proof can be seen as an extension of Eisenberg-Gale proof. Second, this proof reduces the Walras model (by Arrow-Debreu's setting) to the Fisher Model. This justifies an approximation algorithm of Jain et al. [18] to compute an approximate equilibrium. Their approximation algorithm reduces the Walras setting to the Fisher map, and it can be simply stated as

1. Starts with arbitrary w_i's.
2. Compute the $p_i(w)$'s.
3. Replace the w_i's with $p_i(w)$'s plus a "residual", and repeat the loop until the p_i's computed are almost equal to the w_i's used in the loop. (It is proved that the "residual" keeps going down linearly in the process.)

They have proved that this simple and elegant algorithm converges in a time bounded by $\frac{1}{\epsilon}$; see [18]. Note that, in general, this "budget (welfare) adjustment" scheme does not work. Consider an example of two consumers where $w_1 = 2$, $w_2 = 1$, $u_{11} = u_{22} = 1$ and $u_{12} = u_{21} = 2$. The Fisher prices of the problem are $p_1 = 1$ and $p_2 = 2$ so that the adjusted budgets will be $w_1 = 1$ and $w_2 = 2$—a complete reverse of the initial budget allocation. This implies that simply using $p(w)$ to replace w cycles and does not terminate.

Overall, the conditions for a self-dual weighted analytic center of the feasible set of (16) can be written as

$$\begin{aligned} s_j x_j &= y_j, \ j = 1, ..., l \\ s_j x_j &= 0, \ j = l+1, ..., n \\ s - A^T y &= 0, \\ Ax &= b, \\ x, \ s &\geq 0. \end{aligned}$$

Note that the system is homogeneous in (y, s) so that we may add a normalizing constraint

$$\sum_{j=1}^{l} y_j = 1$$

to the conditions.

However, from the remaining conditions but excluding the second one, we have

$$\sum_{j=l+1}^{n} s_j x_j = s^T x - \sum_{j=1}^{l} s_j x_j = s^T x - \sum_{j=1}^{l} y_j = b^T y - \sum_{j=1}^{l} y_j = 0,$$

that is, the second condition is implied by the rest of them. This fact was first proved in [27] for the Arrow-Debreu equilibrium problem, which is a special case of problem (16).

Corollary 3. *Assume that the feasible set of (16) is bounded and it has a nonempty interior, and the dual feasibility $A^T y \geq 0$ implies $y_1, ..., y_l \geq 0$. Then, a self-dual weighted analytic center of the feasible set of (16) satisfies the following necessary and sufficient conditions:*

$$\begin{aligned} s_j x_j &= y_j, \ j = 1, ..., l \\ s - A^T y &= 0, \\ Ax &= b, \\ x, \ s &\geq 0. \end{aligned} \tag{18}$$

3.2 A convex minimization formulation

Nenakhov and Primak [27] (and Jain [17]) have also shown that $p_i > 0$ for all i under our assumption on u_{ij} in problem (15). Thus, by deleting the complementarity condition and substituting u_i and π_i from the equalities, the Arrow-Debreu equilibrium is a point (x, p) that satisfies

$$\begin{array}{rcl} \sum_j u_{ij} x_{ij} & \geq & u_{ij} \frac{p_i}{p_j}, \ \forall i, j \\ \sum_i x_{ij} & = & 1, \ \forall j \\ p_i & > & 0, \ \forall i \\ x_{ij} & \geq & 0, \ \forall i, j. \end{array} \tag{19}$$

Then, the problem of finding such (x, p) can be formulated as the following Phase I optimization problem:

$$\begin{array}{lc} \text{minimize} & \theta \\ \text{subject to} & \sum_i x_{ij} = 1 + \theta \quad \forall j \\ & \sum_j u_{ij} x_{ij} \geq u_{ij} \frac{p_i}{p_j} \quad \forall i, j: \ u_{ij} \neq 0 \\ & x_{ij} \geq 0, \ p_i > 0 \quad \forall i, j. \end{array} \tag{20}$$

Here θ can be viewed as an inflated units of each player's good, i.e., initially every player pretends to have $1 + \theta$ units of good. Then θ is gradually moved down to 0. One can easily see that the problem is strictly feasible with a suitably large θ. Furthermore,

Lemma 1. *For any feasible solution of Problem (20), we must have* $\theta \geq 0$.

Proof. For all i, j, we have

$$x_{ij} p_j \sum_j u_{ij} x_{ij} \geq p_i u_{ij} x_{ij}.$$

Summing these inequalities over j, we have

$$\left(\sum_j x_{ij} p_j \right) \left(\sum_j u_{ij} x_{ij} \right) \geq p_i \left(\sum_j u_{ij} x_{ij} \right).$$

Thus,

$$\sum_j x_{ij} p_j \geq p_i.$$

Summing these inequalities over i, we have

$$\sum_i \sum_j x_{ij} p_j \geq \sum_i p_i,$$

or

$$(1+\theta) \sum_j p_j \geq \sum_i p_i$$

which implies $\theta \geq 0$. ∎

According to Arrow and Debreu [1], we must also have

Lemma 2. *The minimal value of Problem (20) is $\theta = 0$.*

3.3 The logarithmic transformation and efficient barrier functions

Let $y_j = \log p_j, \ \forall j$. Then problem (20) becomes

$$\begin{array}{rc} \text{minimize} & \theta \\ \text{subject to} & \sum_i x_{ij} - \theta = 1 \quad \forall j \\ & \sum_j u_{ij} x_{ij} \geq u_{ij} e^{y_i - y_j} \quad \forall i, j: \ u_{ij} \neq 0 \\ & x_{ij} \geq 0 \quad \forall i, j. \end{array} \tag{21}$$

Note that the new problem is a convex optimization problem since $e^{y_i - y_j}$ is a convex function in y. This type of transformation has been used in geometric programming.

The question arises: is there an efficient barrier function for the inequality

$$\sum_j u_{ij} x_{ij} \geq u_{ij} e^{y_i - y_j}, \ u_{ij} \neq 0?$$

The answer is "yes", and the barrier function is

$$-\log\left(\sum_j u_{ij} x_{ij}\right) - \log\left(\log\left(\sum_j u_{ij} x_{ij}\right) - \log u_{ij} - y_i + y_j\right)$$

with self-concordant parameter 2; see Proposition 5.3.3 of Nesterov and Nemirovskii [28]. One may also construct the dual, the Legendre transformation, of the barrier function.

Let $\bar{u}_{ij} = \log u_{ij}$ and, for simplicity, $u_{ij} > 0$ for all i, j in the following. Then, we can formulate the problem as a barrier optimization problem:

$$\begin{array}{ll} \text{minimize} & \theta - \mu \sum_{i,j} \left(\log x_{ij} + \log \left(\sum_j u_{ij} x_{ij} \right) + \log \left(\log \left(\sum_j u_{ij} x_{ij} \right) - \bar{u}_{ij} - y_i + y_j \right) \right) \\ \text{subject to} & \sum_i x_{ij} - \theta = 1 \quad \forall j, \end{array} \tag{22}$$

where the barrier parameter $\mu > 0$. Similar to what we did in the Eisenberg-Gale model (1), rewrite the problem as

$$\begin{array}{ll} \text{minimize} & \theta - \mu \sum_{i,j} \left(\log x_{ij} + \log (u_i) + \log (\log (u_i) - \bar{u}_{ij} - y_i + y_j) \right) \\ \text{subject to} & \sum_i x_{ij} - \theta = 1 \quad \forall j, \\ & u_i - \sum_j u_{ij} x_{ij} = 0 \quad \forall j. \end{array}$$

There are $n^2 + 2n + 1$ variables in this formulation. The Hessian matrix H of the barrier objective function has a block diagonal structure: the diagonal block with respect to x_{ij} is a $n^2 \times n^2$ positive diagonal matrix and the other diagonal block with respect to the remaining variables is a $(2n + 1) \times (2n + 1)$ positive definite matrix. Thus, the numerical construction and factorization of H needs $O(n^3)$ arithmetic operations. Then the computation and factorization of $A^T H^{-1} A$ is also bounded by $O(n^3)$ arithmetic operations, since the constraint matrix A is sparse and each of its columns has at most two nonzeros. Therefore, one can develop an interior-point path-following or potential reduction algorithm to compute an ϵ-optimal solution, i.e., $\theta < \epsilon$. Since the total self-concordant coefficient of the barrier function is $O(n^2)$, and each iteration uses at most $O(n^3)$ arithmetic operations, we have

Theorem 5. *There is an interior-point algorithm to generate a solution to problem (20) with $\theta < \epsilon$ in $O(n \log \frac{1}{\epsilon})$ iterations and each iteration uses $O(n^3)$ arithmetic operations.*

Note that this worst-case complexity bound is significantly lower than that using the Ellipsoid method by Nenakhov and Primak [27] and Jain [17].

3.4 Alternative optimization setting

An alternative Phase I problem is

$$
\begin{array}{lc}
\text{minimize} & \theta \\
\text{subject to} & \sum_i x_{ij} = 1 \quad \forall j \\
& \theta \cdot \sum_j u_{ij} x_{ij} \geq u_{ij} \frac{p_i}{p_j} \quad \forall i,j: \; u_{ij} \neq 0 \\
& x_{ij} \geq 0, \; p_i > 0 \quad \forall i,j.
\end{array} \tag{23}
$$

Initially, $\theta > 1$, which is an inflated factor for the utility value. The problem is to drive θ to 1.

Let $y_j = \log p_j, \; \forall j$ and $\kappa = \log \theta$. Then problem (20) becomes

$$
\begin{array}{lc}
\text{minimize} & \kappa \\
\text{subject to} & \sum_i x_{ij} = 1 \quad \forall j \\
& \sum_j u_{ij} x_{ij} \geq u_{ij} e^{y_i - y_j - \kappa} \quad \forall i,j: \; u_{ij} \neq 0 \\
& x_{ij} \geq 0, \quad \forall i,j.
\end{array} \tag{24}
$$

Again, the new problem is a convex optimization problem since $e^{y_i - y_j - \kappa}$ is a convex function in y and κ, and the minimal value of the problem is 0.

3.5 Rounding to an exact solution

Eaves [12] showed that the Arrow-Debreu problem with linear utility can be formulated as a *linear complementarity problem.* Again, an optimal solution $y_{ij} := p_j x_{ij}$ and price vector p is the solution of the homogeneous system of *linear equarions and inequalities*:

$$
\begin{array}{lll}
u_{ij}\lambda_i = p_j, & y_{ij} > 0, & \forall (i,j) \in B^* \\
u_{ij}\lambda_i = p_j, & y_{ij} = 0, & \forall (i,j) \in Z^* \\
u_{ij}\lambda_i < p_j, & y_{ij} = 0, & \forall (i,j) \in N^* \\
& \sum_{j \in P} y_{ij} = p_i, & \forall i \\
& \sum_{i \in C} y_{ij} = p_j, & \forall j,
\end{array}
$$

where B^*, Z^* and N^* are identical to those defined in the Fisher model. We may normalize p such that $p_1 = 1$. Then, the system has a rational solution and the size of its each entry is bounded by the bit-length L of all input data u_{ij}. Thus, the same rounding technique can be applied to separate $B^*(\in P^k)$ from $N^*(\notin P^k)$ for a variable partition P^k generated from the interior-point algorithm, and to compute an exact solution of the system *linear equations and inequalities*:

$$\begin{array}{lll} u_{ij}\lambda_i = p_j, & y_{ij} \geq 0, & \forall (i,j) \in P^k \\ u_{ij}\lambda_i \leq p_j, & y_{ij} = 0, & \forall (i,j) \notin P^k \\ & \sum_{j \in P} y_{ij} = p_i, & \forall i \\ & \sum_{i \in C} y_{ij} = p_j, & \forall j, \\ & p_1 = 1 & . \end{array}$$

in $O(n^4 L)$ arithmetic operations. This implies that

Corollary 4. *There is an interior-point algorithm to compute an exact solution of problem (20) with n producers and n consumers in at most $O(n^4 L)$ arithmetic operations, where L is the bit-length L of the input data u_{ij}.*

Again, our result is a significant improvement over the ellipsoid method discussed by Jain [17].

4 A Path to an Arrow-Debreu equilibrium

Now, we move our attention to whether there is a direct interior-point algorithm in solving the Arrow-Debreu equilibrium problem, similar to the primal-dual path-following algorithm for linear programming and the Fisher equilibrium. Such an algorithm may have economical and practical applications.

Consider the convex optimization problem for a fixed scalar $0 \leq \mu \leq 1$ and a nonnegative weight vector w with $\sum_i w_i = n^2$:

$$\begin{array}{ll} \text{maxmize} & \mu \sum_{i,j} \log(x_{ij}) + \sum_i w_i(1-\mu) \log\left(\sum_j u_{ij} x_{ij}\right) \\ \text{subject to} & \sum_i x_{ij} = 1, \quad \forall j \\ & x_{ij} \geq 0, \quad \forall i, j. \end{array} \tag{25}$$

4.1 Economic interpretations

The objective of (25), when $\mu = 0$, is the same objective function which Eisenberg-Gale used for Fisher's model. We now present economic interpretations for $\mu > 0$. When $\mu = 1$, then the objective function becomes the logarithmic barrier function and the unique maximizer of (25) is the analytic center of the feasible set, namely, $x_{ij} = \frac{1}{n}$ for all i, j. This is probably an ideal socialist solution if all players are homogeneous.

In our setting, the combined objective function represents a balance between socialism and individualism. Here $w_i(1-\mu)$ is the weight for the log-utility value of player i. If again, w_i represents the amount of money player i possesses, $\sum_i w_i = n^2$ represents the total wealth of the players, and μ represents player i's tax-rate to be collected for social welfare. The leftover amount, $w_i(1-\mu)$, would be the weight used in Eisenberg-Gale to make the market clear. Here, the total collected tax amount is $n^2\mu$ and the tax-rate μ is uniformly applied among the payers. Mike Todd also pointed out that the objective function here is really the convex combination of the two different utility functions, one is un-weighted and the other is weighted, representing two different idealisms.

4.2 The fixed-point theorem

Unlike Fisher's problem, we really don't know how much money w each player possesses in Walras' model—it depends on the prices p, since they have to sell their goods at these prices for revenues. But the prices are the optimal dual variables or Lagrange multipliers of the n equality constraints in (25). Then, the natural question becomes, is there a vector w such that the optimal dual prices of (25) equal the w_i's, respectively. We give an affirmative answer in the following theorem.

Theorem 6. *For any scalar $0 < \mu \leq 1$, there exists a weight vector $w \geq n\mu$ and $\sum_i w_i = n^2$ such that the optimal dual price vector of (25) equals w.*

Proof. When $\mu = 1$, i.e., the tax-rate equals 1, the (unique) prices would be

$$w_i = p_i = n \quad \text{and} \quad x_{ij} = \frac{1}{n} \ \forall i, j.$$

Consider $0 < \mu < 1$. Denote the compact simplex by

$$S(\mu) = \{y \in R^n : \ \sum_i y_i = n^2, \ y_i \geq n\mu, \ \forall i\} \subset R^n_{++}.$$

Since $\mu > 0$, the objective function of (25) is strictly convex for any $w \geq 0$, and from convex optimization theory the optimal solution x and its Largrange (dual) solution p are *unique* and strictly positive and they satisfy the necessary and sufficient conditions:

$$\begin{array}{rcl} \frac{\mu}{x_{ij}} + \frac{w_i(1-\mu)u_{ij}}{\sum_j u_{ij}x_{ij}} & \leq & p_j, \ \forall i, j \\ x_{ij}\left(p_j - \frac{\mu}{x_{ij}} - \frac{w_i(1-\mu)u_{ij}}{\sum_j u_{ij}x_{ij}}\right) & = & 0, \ \forall i, j \\ \sum_i x_{ij} & = & 1, \ \forall j \\ x_{ij} & \geq & 0, \ \forall i, j. \end{array} \tag{26}$$

where p_j is the optimal dual price or Lagrange multiplier for equality constraint j. The first set of constraints is for the dual feasibility condition, the second set of constraints is for the complementarity condition, and the last two sets are for the primal feasibility condition.

Summing up the complementarity equations over i and noting $\sum_i x_{ij} = 1$, we have

$$p_j = n\mu + \sum_i \frac{w_i(1-\mu)u_{ij}x_{ij}}{\sum_j u_{ij}x_{ij}} \geq n\mu, \ \forall j.$$

Summing the above equations over j, we have

$$\sum_j p_j = n^2\mu + \sum_j \sum_i \frac{w_i(1-\mu)u_{ij}x_{ij}}{\sum_j u_{ij}x_{ij}}$$

$$= n^2\mu + \sum_i \frac{w_i(1-\mu)}{\sum_j u_{ij}x_{ij}} \sum_j u_{ij}x_{ij} = n^2\mu + \sum_i w_i(1-\mu) = n^2\mu + n^2(1-\mu) = n^2.$$

That is, $p \in S(\mu)$. For given u_{ij}'s and fixed $\mu > 0$, we may think $p \in S(\mu)$ being a mapping of $w \in S(\mu)$, that is, $p(w)$ is a mapping from the simplex to itself, and it is one-to-one, continuous and differentiable (see again, e.g., [16, 22, 21] and the proof of Theorem 4). From Brouwer's fixed-point theorem (see, e.g., [31, 32, 34]), there exists $w \in S(\mu)$ such that

$$p(w) = w,$$

which completes the proof. ∎

Note that summing up the complementarity equations in (26) over j when $w = p$, we have

$$\sum_j p_j x_{ij} = n\mu + \sum_j \frac{w_i(1-\mu)u_{ij}x_{ij}}{\sum_j u_{ij}x_{ij}} = n\mu + w_i(1-\mu) = n\mu + p_i(1-\mu).$$

That is, the individual payment spent by player i equals his net income (after tax) plus $n\mu$ which can be viewed as a tax amount refunded back to each player uniformly.

4.3 A path-following algorithm?

Let $p = w = p(w)$ in the optimality conditions of (25). Moreover, let $y_i = \sum_j u_{ij}x_{ij}$ and $q_i = \frac{p_i(1-\mu)}{\sum_j u_{ij}x_{ij}}$. Then we have

$$\begin{array}{rcl} x_{ij}(p_j - u_{ij}q_i) & = & \mu,\ \forall i,j \\ y_i q_i - p_i(1-\mu) & = & 0,\ \forall i \\ \sum_i x_{ij} & = & 1,\ \forall j \\ y_i - \sum_j u_{ij}x_{ij} & = & 0,\ \forall i \\ \sum_i p_i & = & n^2, \\ y_i,\ q_i & \geq & 0,\ \forall i \\ x_{ij},\ (p_j - u_{ij}q_i) & \geq & 0,\ \forall i,j. \end{array} \tag{27}$$

Since both the primal and dual solutions are unique and bounded interior points for any given $0 < \mu \leq 1$, they can be written as $(x_{ij}(\mu), y_i(\mu), q_i(\mu), p_i(\mu))$. Similar to the central path theory of linear programming (e.g., [22, 16]), we have, for $\mu \in (0,1]$, $(x_{ij}(\mu), y_i(\mu), q_i(\mu), p_i(\mu))$

form a continuous and bounded path ($\sum_i p_i(\mu) = \sum_i p_i(1) = n^2$ for all μ). Moreover, when $\mu \to 0^+$, any limit point converges to an Arrow-Debreu equilibrium solution. System (27) has linear and bilinear equations, which are similar to the central path equations for linear programming and primal-dual path-following Newton's methods might be applicable. This is a subject of further research.

5 Final Remarks

Consider a more general Arrow-Debreu exchange market problem where the market has n players and m type goods. Player i, $i = 1, ..., n$, has an initial bundle of goods $0 \leq w_i = (w_{i1}; w_{i2}; ...; w_{im}) \in R^m$ and has a linear utility function with coefficients $(u_{i1}; u_{i2}; ...; u_{im}) \geq 0 \in R^m$. The problem is how to price each good so that the market clears. We assume that $w_i \neq 0$ for every i, that is, every player brings some goods to the exchange market; and, again, as least one $u_{ij} > 0$ for every i, and at least one $u_{ij} > 0$ for every j; that is, every player in the market likes at least one good; and every good is valued by at least one player.

Note that, given the price vector $p = (p_1; p_2; ...; p_m) > 0$, the individual utility maximization problem is

$$\begin{array}{ll} \max & \sum_j u_{ij} x_{ij} \\ \text{s.t.} & \sum_j p_j x_{ij} \leq p^T w_i \\ & x_{ij} \geq 0. \end{array}$$

The optimality conditions of this individual utility maximization problem, besides x_{ij} is feasible, are

$$u_{ij} \leq \frac{\sum_j u_{ij} x_{ij}}{p^T w_i} p_j, \ \forall j.$$

It can be verified that if p and x satisfy the constraints

$$\sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} w_{ij}, \quad \forall j,$$
$$\sum_{k} u_{ik} x_{ik} \geq u_{ij} \cdot \frac{p^T w_i}{p_j}, \quad \forall i, j$$
$$x_{ij} \geq 0, \; p_j > 0, \quad \forall i, j,$$

then, p is an equilibrium price vector. This is because of that, rewriting the second set of inequalities as

$$p_j \cdot \sum_{k} u_{ik} x_{ik} \geq u_{ij} \cdot p^T w_i, \; \forall i, j$$

and multiplying x_{ij} to the both sides, we have

$$p_j x_{ij} \cdot \sum_{k} u_{ik} x_{ik} \geq u_{ij} x_{ij} \cdot p^T w_i, \; \forall i, j.$$

Summing them over j, we have

$$\sum_{j} p_j x_{ij} \cdot \sum_{k} u_{ik} x_{ik} \geq p^T w_i \cdot \sum_{j} u_{ij} x_{ij}, \; \forall i$$

or

$$\sum_{j} p_j x_{ij} \geq p^T w_i, \; \forall i.$$

Now summing them over i, we have

$$\sum_{j} p_j \left(\sum_{i=1}^{n} x_{ij} \right) \geq \sum_{j} p_j \left(\sum_{i=1}^{n} w_{ij} \right).$$

But $\sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} w_{ij}$ for all j, so that the above must hold as equality, and so is

$$\sum_{j} p_j x_{ij} = p^T w_i, \; \forall i.$$

That is, x is optimal for each of the individual utility maximization problems.

To find x and p, similar to our earlier discussion, we form a minimization problem:

$$\begin{array}{ll} \min & \theta \\ \text{s.t.} & \sum_{i=1}^n x_{ij} = \sum_{i=1}^n w_{ij} + \theta, \ \forall j, \\ & \sum_k u_{ik} x_{ik} \geq u_{ij} \cdot \frac{p^T w_i}{p_j}, \ \forall i, j, \\ & x_{ij} \geq 0, \ p_j > 0, \ \forall i, j. \end{array}$$

The problem is feasible and has a relative interior, and its minimal value is 0. For example, one can assign

$$x_{ij} = w_{ij} + \frac{\theta}{n} \quad \text{and} \quad p_j = 1, \ \forall i, j$$

which satisfy the equalities. Then, if we choose θ sufficiently large, all inequalities are strictly satisfied.

By introducing new variable $y_j = \log(p_j)$, $j = 1, ..., m$, the problem can be written

$$\begin{array}{ll} \min & \theta \\ \text{s.t.} & \sum_{i=1}^n x_{ij} = \sum_{i=1}^n w_{ij} + \theta, \ \forall j, \\ & \log\left(\sum_k u_{ik} x_{ik}\right) \geq \log(u_{ij}) + \log\left(\sum_{k=1}^m w_{ik} e^{y_k}\right) - y_j, \ \forall i, j, \ u_{ij} > 0, \\ & x_{ij} \geq 0, \ \forall i, j. \end{array}$$

Since $\sum_k u_{ik} x_{ik}$ is concave in x and $\log\left(\sum_k w_{ik} e^{y_k}\right)$ is convex in y, so that the inequalities are convex constraints. The rest of constraints are linear and so is the objective function. Therefore, the generalized Arrow-Debreu exchange market problem is also a convex minimization problem. I believe that an efficient barrier exists for solving this problem.

I also feel that the general self-dual weighted analytic center discussed in this paper seems to have more applications in matrix games and other fixed-point problems. We expect more problems can be transformed to convex optimization problems where efficient interior-point algorithms may apply.

Other questions remain, such as how to handle general concave utility functions and/or productions. Some answers have been given by Codenotti, Deng, Huang, Jain, Pemmaraju, Varadarajan and Vazirani [4, 5, 7, 19]. Are there direct primal-dual interior-point algorithms for finding an Arrow-Debreu equilibrium? The path developed in this paper may give an answer.

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