An interior-point path-following algorithm for computing a Leontief economy equilibrium

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March 8, 2008

Abstract

In this paper, we present an interior-point path-following algorithm for computing a Leontief economy equilibrium, that is, an exchange market equilibrium with Leontief utility functions, which is known to be in the complexity class of PPAD-complete. It is known that an equilibrium corresponds to a solution of a system of complementarities, so we construct a homotopy to tackle this system. We prove that there always exists a continuously differentiable path leading to a complementary solution of the nonlinear system and at the same time to a Leontief economy equilibrium associated with the solution. We also report preliminary computational results to show effectiveness of the path-following method.

1 Introduction

The Arrow-Debreu exchange market equilibrium problem was first formulated by Léon Walras in 1874 [15]. In this problem everyone in a population of n traders has an initial endowment of a divisible good and a utility function for consuming all goods—their own and others'. Every trader sells the entire initial endowment and then uses the revenue to buy a bundle of goods such

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that his or her utility function is maximized. Walras asked whether prices could be set for everyone's goods such that this is possible. An answer was given by Arrow and Debreu in 1954 [2] who showed that, under mild conditions, such equilibrium would exist if the utility functions were concave. In general, it is unknown whether or not an equilibrium can be computed efficiently.

Consider a special class of Arrow-Debreu's problems, where each of the n traders has exactly one unit of a divisible and distinctive good for trade, and let trader i, $i = 1, ..., n$, bring good i. This class of problems is called the *pairing* Arrow-Debreu model. For given prices p_j on good j, consumer i's maximization problem is

$$\begin{array}{ll} \text{maximize} & u_i(x_{i1}, ..., x_{in}) \\ \text{subject to} & \sum_j p_j x_{ij} \leq p_i, \\ & x_{ij} \geq 0, \quad \forall j. \end{array} \tag{1}$$

Let x_i^* denote a maximal solution vector of (1). Then, vector p is called the Arrow-Debreu price equilibrium if there exists an x_i^* for consumer i, $i = 1, ..., n$, such that

$$\sum_i x_i^* = e$$

where e is the vector of all ones representing available goods on the exchange market.

The Leontief exchange economy equilibrium problem is the Arrow-Debreu equilibrium when the utility functions are in the Leontief form:

$$u_i(x_i) = \min_{j:\ a_{ij}>0} \left\{ \frac{x_{ij}}{a_{ij}} \right\},$$

where the Leontief coefficient matrix is given by

$$A = \left(\begin{array}{cccc} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{array} \right). \tag{2}$$

Here, one may assume that

Assumption 1 *A has no all-zero row, that is, every trader likes at least one good.*

In this paper, we present an interior-point path-following algorithm for computing a Leontief economy equilibrium, that is, an exchange market equilibrium model with Leontief utility coefficient matrix; where the equilibrium set is non-convex or non-connected. First, we describe the characterization of

Leontief economy equilibria, which is basically a system of complementarity equations. Then we construct an almost surely regular homotopy for the nonlinear system, and show that there always exists a continuous differentiable path leading to a solution of the system, from which a Leontief economy equilibrium can be obtained. We also report preliminary computational results which show that the method is highly effective.

2 Characterization of a Leontief economy equilibrium

Let u_i be the equilibrium utility value of consumer i and p_i be the equilibrium price for good i, $i = 1, ..., n$. Also, let U and P be diagonal matrices whose diagonal entries are u_i's and p_i's, respectively. Then, any (quasi) Leontief economy equilibrium $p \in R^n$, together with $u \in R^n$, must satisfy

$$\begin{array}{rcl} UAp & = & p, \\ P(e - A^T u) & = & 0, \\ A^T u & \leq & e, \\ u,\ p & \geq & 0, \\ p & \neq & 0; \end{array} \tag{3}$$

where U and P are diagonal matrices whose diagonal entries are u and p respectively. The Arrow-Debreu theorem implies that nonzero p and u exist for this system of equalities and inequalities.

Let $B \subset \{1, 2, ..., n\}$, $N = \{1, 2, ..., n\} \setminus B$. And, A_{BB} is the principal submatrix of A corresponding to the index set B, A_{BN} is the submatrix of A whose rows in B and columns in N. Similarly, u_B and p_B are sub-vectors of u and p with entries in B, respectively. It was proved that

Theorem 1 *(Ye [17]) Let $B \subset \{1, 2, ..., n\}$, $N = \{1, 2, ..., n\} \setminus B$, A_{BB} be irreducible, and u_B satisfy the linear system*

$$A_{BB}^T u_B = e, \quad A_{BN}^T u_B \leq e, \quad \text{and} \quad u_B > 0.$$

Then the (right) Perron-Frobenius eigen-vector p_B of $U_B A_{BB}$ together with $p_N = 0$ will be a solution to System (3). And the converse is also true. Moreover, there is always a rational solution for every such B, that is, the entries of price vector are rational numbers, if the entries of A are rational.

The theorem implies that the traders in block B can trade among themselves and keep others' goods "free". In particular, if one trader likes his or her own good more than any other good, that is, $a_{ii} \geq a_{ij}$ for all j, then $u_i = 1/a_{ii}$, $p_i = 1$, and $u_j = p_j = 0$ for all $j \neq i$, that is, $B = \{i\}$, makes a Leontief economy equilibrium. The theorem also establishes a combinatorial algorithm to compute a Leontief economy equilibrium by finding a right block $B \neq \emptyset$, which is actually a non-trivial complementary solution ($u \neq 0$) to an LCP problem (see Cottle at al. [7] for more literature on linear complementarity problems)

$$A^T u + v = e,\ u^T v = 0,\ (u \neq 0, v) \geq 0. \tag{4}$$

Any complementary solution $u \neq 0$ and $B = \{j : \ u_j > 0\}$, as long as A_{BB} is irreducible, induces an equilibrium for system (3). The equivalence between the pairing Leontief economy model and the LCP also implies

Corollary 1 *LCP (4) always has a non-trivial complementary solution under Assumption 1 (i.e., A has no all-zero row).*

If Assumption 1 does not hold, the corollary may not be true; see the example below:

$$A^T = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}.$$

In general, the complementary solution set of (4) is not convex or even connected. This is true even when A is symmetric: consider a Leontief coefficient matrix given by

$$A^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then, there are three isolated non-trivial solutions.

$$u^1 = (1/2;\ 0), \quad u^2 = (0;\ 1/2), \quad u^3 = (1/3;\ 1/3).$$

In summary, a Leontief economy equilibrium can always induce a solution to system (3) and LCP problem (4), however the solution to system (3) or LCP problem (4) can induce a Leontief economy equilibrium only when A_{BB} is irreducible. Under certain structure conditions, for example all entries of A are positive, then they are in one-to-one correspondence.

3 Connection to bimatrix game equilibria

The Leontief economy equilibrium problem characterized by (3) is also connected to the bimatrix game equilibrium problem specified by a pair of $n \times m$ pay-off matrices C and R, with positive entries, one can construct a Leontief exchange economy with $n + m$ traders and $n + m$ goods as follows.

Theorem 2 *(Codenotti et al. [6]) Let $(C > 0, R > 0)$ denote an arbitrary bimatrix game, where assume, w.l.o.g., that the entries of the matrices C and R are all positive. Let*

$$A^T = \begin{pmatrix} 0 & C \\ R^T & 0 \end{pmatrix}$$

describe the Leontief utility coefficient matrix of the traders in a Leontief economy. There is a one-to-one correspondence between the Nash equilibria of the game (C, R) and the market equilibria A of the Leontief economy.

Therefore, computing a bimatrix game equilibrium is also equivalent to computing a solution of system (3). The reader may want to read Brainard and Scarf [3], Gilboa and Zemel [11], Chen, Deng and Teng [5], Daskalakis, Goldberg ans Papadimitriou [8], and Tsaknakis and Spirakis [14] on hardness and approximation results of computing a bimatrix game equilibrium.

In fact, Chen and Deng in [4] proved that the general Leontief economy equilibrium problem (or equivalently two-player Nash equilibrium problem) is complete for the complexity class PPAD (Polynomial Parity Argument, Directed version), a class of search problems defined by its completeness introduced by Papadimitriou in 1991. It implies that the general Leontief economy equilibrium does not have a full polynomial-time approximation scheme (FPTAS) unless every problem in PPAD-complete is solvable by a FPTAS.

4 Standard Lemke's algorithm may not work

Lemke's algorithm is well-known for solving classes of LCP problems, but it seems unable to find a non-trivial solution for LCP (4).

If we apply Lemke's algorithm directly to LCP (4), then it returns the trivial solution $u = 0,\ v = e$. To exclude the trivial solution, we rewrite LCP (4) into an equivalent homogeneous LCP as follows:

$$Mz + q = w,\ z^T w = 0,\ (z, w) \geq 0, \tag{5}$$

where $z, w \in R^{n+1}$,

$$M = \begin{pmatrix} -A^T & e_n \\ e_n^T & 0 \end{pmatrix} \in M^{n+1}, q = \begin{pmatrix} 0_n \\ -1 \end{pmatrix}.$$

Then, we can obtain a nontrivial solution of LCP (4) from any solution of LCP (5). However, the standard Lemke's algorithm can not solve LCP (5), indeed it will stop at the second iteration because it terminates with a secondary ray; see Lemke [13] and Cottle at al. [7].

5 Interior-point equilibrium path

For Arrow-Debreu equilibrium problems with linear utilities, a convex optimization setting is formulated in [16], from which an Arrow-Debreu equilibrium can be obtained by following a continuous path. Similarly, we consider the following convex optimization problem for Leontief economy equilibrium:

$$\begin{array}{ll} \text{maximize} & \sum_i tw_i \log u_i + \sum_j (1-t)\log y_j \\ \text{subject to} & A^T u + y = e + t(1-t)\alpha \\ & u, y \geq 0. \end{array} \tag{6}$$

for any given parameter $0 \leq t < 1$, $w \geq (1-t)e \in R^n$ with $e^T w = n$, and $\alpha \in R^n$ is a randomly generated small perturbation.

Let $p \in R^n$ be the optimal Lagrange multiplier vector of the constraints. Then, we have

$$\begin{array}{rcl} UAp &=& tw, \\ Yp &=& (1-t)e, \\ A^T u + y &=& e + t(1-t)\alpha, \\ u, y, p &\geq& 0, \\ p &\neq& 0; \end{array}$$

where U and Y are diagonal matrices whose diagonal entries are u and y respectively.

Clearly, $p \geq 1 - t > 0$ and $e^T p = n$ in the system. And, from the fixed point theorem, we have

$$\begin{array}{rcl} UAp &=& tp, \\ Yp &=& (1-t)e, \\ A^T u + y &=& e + t(1-t)\alpha, \\ u, y, p &\geq& 0, \\ p &\neq& 0; \end{array} \tag{7}$$

feasible for any $0 \leq t < 1$. In particular, $u = 0, y = p = e$ is the unique solution at $t = 0$. When $t = 1$, system (7) is exactly system (3) to represent a Leontief economy equilibrium.

Next, we will show that the solutions of (7) form a continuous and differentiable path leading to a Leontief economy equilibrium.

6 Homotopy and the primary path

Let us apply the homotopy approach to system (7) (see Garcia and Zangwill [10] for more literature on general homotopy methods and Eaves and Schmedders [9] for general equilibrium models and homotopy methods). For any given $\alpha \in R^n$, consider the homotopy $H_\alpha : D \rightarrow R^{3n}$ ($D = R^{3n}_+ \times [0, 1) \subset$

R^{3n+1}):

$$H_\alpha(x,t) = H_\alpha(u,y,p,t) = \begin{pmatrix} UAp - tp \\ Yp - (1-t)e \\ A^T u + y - e - t(1-t)\alpha \end{pmatrix} \tag{8}$$

where $x = (u,y,p) \in R^{3n}_+, u, y, p \in R^n_+$ (i.e. $(u,y,p) \geq 0$). Clearly, condition (7) is exactly $H_\alpha(x,t) = 0$.

Define

$$H_\alpha^{-1} = \{(x,t) \in D | H_\alpha(x,t) = 0\}$$

as the solutions $(x,t) \in R^{3n+1}$ to the system $H_\alpha(x,t) = 0$. In particular, define

$$H_\alpha^{-1}(t) = \{x \in R^{3n}_+ | H_\alpha(x,t) = 0\}$$

as the solutions for any fixed t. Then, $H_\alpha^{-1}(0)$ consists of the unique solution $x = (0,e,e)$ of $H_\alpha(x,0) = 0$, and $H_\alpha^{-1}(1)$ consists of all the solutions of $H_\alpha(x,1) = 0$, which solve system (3).

In [10], a fundamental theorem to characterize H_α^{-1} is proved by the Implicit Function Theorem:

Theorem 3 *(Path Theorem) Let $H : R^{n+1} \to R^n$ be continuously differentiable and suppose that H is regular (i.e. for every $(x,t) \in H^{-1} = \{(x,t) | H(x,t) = 0\}$, the Jacobian $H^{'}(x,t)$ is of full rank). Then H_α^{-1} consists only of continuously differentiable paths (including loops).*

Clearly, H_α is continuously differentiable. Further, the following Extended Sard Theorem states that regularity holds almost always:

Theorem 4 *(Extended Sard Theorem / Transversality Theorem) Let $L : B \times R^s \to R^n$ for $B \subset R^q$ be C^k, where $k \geq 1 + \max\{0, q-n\}$. Also let L be regular. Now suppose that we fix $e = \hat{e}$. Then $L(\cdot, \hat{e}) :\to R^n$ is regular for almost all $\hat{e}$.*

Therefore, we have the following

Theorem 5 *For almost all $\alpha \in R^n$ fixed, H_α^{-1} consists only of continuously differentiable paths. In other words, there is no splitting, bifurcation, fork, crossing, or infinite endless spiral in H_α^{-1}.*

Proof Let $L(x,t,\alpha) := H_\alpha(x,t) : D \times R^n \to R^{3n}$ for $D \subset R^{3n+1}$ be C^2. The Jacobian of L is

$$L^{'}(x,t,\alpha) = \begin{pmatrix} S & 0 & UA - tI & -p & 0 \\ 0 & P & Y & e & 0 \\ A^T & I & 0 & (2t-1)\alpha & -t(1-t)I \end{pmatrix} \tag{9}$$

Clearly, for any $t \in (0,1)$, $L^{'}(x,t,\alpha)$ has full rank because the first $2n$ and the last n column of $L^{'}(x,t,\alpha)$ forms a lower triangular matrix with non-zero diagonal entries, which of course is rank $3n$. For $t = 0$, we know that $H_\alpha(x,0) = 0$ has a unique solution $(0,e,e)$, and thus

$$L^{'}(x,0,\alpha) = \begin{pmatrix} S & 0 & 0 & -p & 0 \\ 0 & P & I & e & 0 \\ A^T & I & 0 & (2t-1)\alpha & -t(1-t)I \end{pmatrix} \tag{10}$$

which has full rank because the first $3n$ column of $L^{'}(x,0,\alpha)$ has full rank. Summarize the above, L is regular. As a result, $H_\alpha(x,t) = L(x,t,\alpha) : D \to R^{3n}$ is regular for almost all $\alpha \in R^n$ by Theorem 4.

Applying Theorem 3 to continuously differentiable and regular $H_\alpha(x,t)$, we have that H_α^{-1} consists only of continuous differentiable paths. ∎

Among those continuously differentiable paths in H_α^{-1}, we are particularly interested in the so-called primary path, which is defined as

Definition 1 *The primary path is a continuously differentiable path that starts from a solution at $t = 0$ and leads to a solution of $H_\alpha^{-1}(1)$ at $t = 1$.*

We can show that

Theorem 6 *There exists a unique (interior-point) primary path in H_α^{-1} for almost all $\alpha \in R^n$ sufficiently small, which starts from the unique solution $(0,e,e)$ in $H_\alpha^{-1}(0)$ at $t = 0$ and leads to a non-trivial solution of $H_\alpha^{-1}(1)$ at $t = 1$.*

Proof From Theorem 5, H_α^{-1} consists only of continuously differentiable paths. Consider the path starting from the unique solution $(0,e,e)$ of $H_\alpha^{-1}(0)$ at $t = 0$, it must be a primary path if we can exclude the following two situations: (a) The path returns to $t = 0$; (b) The path goes to the boundary or infinity before reaching a solution at $t = 1$. Indeed, we have

(a) The path can not go back to $t = 0$. If the path go back to $t = 0$, then it can only return to $(x_0, 0)$ because $H_\alpha(x,0) = 0$ has a unique solution $(x_0, 0)$ on R_+^{3n}. Therefore the path must be a circle tangent to the line $t = 0$, which contradicts with the fact that the Jacobian of H_α at $(x_0, 0)$ (i.e. the first $3n$ columns of (10)) is nonsingular.

(b) H_α^{-1} is bounded and boundary-free (H_α^{-1} is boundary-free if $H_\alpha^{-1} \cap \partial D = \emptyset$). Clearly, (u,y,p) are bounded, because $(u,p,y) \geq 0$, A is nonnegative, $A^T u + y = e + t(1-t)\alpha \leq (1 + \|\alpha\|_\infty)e$ and $e^T p = n - t\alpha^T p$ (implicitly implied by system (7)). Furthermore, u, y and p are bounded away from zero for all $0 < t < 1$, because $Yp = (1-t)e$, $UAp = tp$ and (u,y,p) are bounded. Therefore, H_α^{-1} is boundary-free except at the starting point $(x_0, 0)$ and the ending point when $t = 1$.

As a result, the path starting from the unique solution at $t = 0$ will converge to a solution of $H_\alpha(x,t) = 0$ at $t = 1$. Moreover, this solution is nontrivial because $e^T p = n - t\alpha^T p > \frac{n}{2}$ all along the path for sufficiently small α. Therefore, there exists a unique (interior-point) primary path in H_α^{-1} that approaches a nontrivial solution of $H_\alpha^{-1}(1)$ at $t = 1$. ∎

The "central path" for linear programming interior-point algorithms can be viewed as the "primary path" of a corresponding homotopy system– the KKT system of a logarithmic barriered problem for linear programming, where $(1-t)$ is the weight parameter for the barrier function term. The main difference between the primary path of H_α^{-1} and the central path of linear programming is that, in the latter, the homotopy (barrier) parameter t is monotone (increasing) along the central path, while it may not be monotone (increasing) along the primary path of (7), as one can see from the simulations. That is, the primary path of (7) can retrogress at "turning points", which are those points (x,t) such that $H_\alpha(x,t) = 0$ and the Jacobian matrix, $D_x H_\alpha(x,t)$, is singular.

7 Interior-point path-following algorithm

There are several methods to follow the primary path, including the simplicial algorithms and predictor-corrector algorithms (see [10]). A high level predictor-corrector algorithm of path-following for the Leontief economy problem can be described as:

Initialization

Let $\epsilon_1 > \epsilon_2 > 0$ be given, and set $u^0 = 0, y^0 = e, p^0 = e, t^0 = 0$, $k = 0$. Let $x = (u,y,p)$, and $z = (x,t) = (u,y,p,t)$, then $x^0 = (0,e,e)$, $z^0 = (0,e,e,0)$.

Predictor

Given $z^k = (x^k, t^k)$, take a predictor step from $z^k = (x^k, t^k)$ to obtain $z^{k+1} = (x^{k+1}, t^{k+1})$. Replace k by $k+1$. Repeat the predictor step if $\|H_\alpha(x^k, t^k)\| < \epsilon_1$. Otherwise, go to the corrector step.

Corrector

Let $b \in R^{3n+1}$ (column vector) describe a hyperplane

$$b^T(z - z^k) = 0,$$

which intersects the path at a nearby point farther along the path. Setting $z^k = z^{k,0}$ compute

$$z^{k,l+1} = z^{k,l} - \begin{pmatrix} H'_\alpha(z^{k,l}) \\ b^T \end{pmatrix}^{-1} \begin{pmatrix} H_\alpha(z^{k,l}) \\ 0 \end{pmatrix}, l = 0, 1, \cdots \tag{11}$$

Stop when $\|H_\alpha(z^{k,l+1})\| < \epsilon_2$.

If $z^{k,l+1}$ is near $t = 1$, terminate. Otherwise, let $z^{k+1} = z^{k,l+1}$ and go to the predictor step with $k + 1$ replacing k. ∎

The path-following algorithm contains three key ingredients in its loop: the predictor, the corrector and the step size control.

For predictor step, there are three types of commonly used predictors: tangent (or Euler) predictor, secant predictor and Cubic (or Hermite) predictor, each of which has its own merit. We implement a hybrid predictor as follows: secant predictor is used whenever possible because it requires least computation; if secant predictor fails or it is not applicable (such as at the first step), the tangent predictor is applied instead; if both secant and tangent predictor fail, the Hermite predictor based on cubic interpolation is the last shoot to drag it back to the primary path, which is more accurate than secant or tangent predictor especially when near the turning points. This heuristic strategy works well for our simulation, it is rare to have cases that all the predictor fail to follow the path.

For corrector step, we choose $b = z^k - z^{k-1}$, which means the correction is done within the hyperplane orthogonal to the secant direction. Let L be the first $3n$ columns of $H_\alpha^{'}$, w be the last column of $H_\alpha^{'}$ and $b = (b_L,\ b_w), r_L = H_\alpha(z^{k,l})$. Then, (11) becomes

$$z^{l+1} = z^l - \begin{pmatrix} L & w \\ b_L^T & b_w \end{pmatrix}^{-1} \begin{pmatrix} r_L \\ 0 \end{pmatrix}, l = 0, 1, \cdots \tag{12}$$

which is equivalent to solving

$$\begin{pmatrix} L & w \\ b_L^T & b_w \end{pmatrix} \begin{pmatrix} z_L \\ z_w \end{pmatrix} = \begin{pmatrix} r_L \\ 0 \end{pmatrix}, \tag{13}$$

at each step $l = 0, 1, \cdots$. Eliminating z_w, we get

$$(L - \frac{1}{b_w} w b_L^T) z_L = r_L.$$

When the equations of the form $Lx = y$ permit a fast linear solver, then the above perturbed problem (rank-one update) can be solved by Sherman-Morrison formula by one additional call of the fast solver and several scalar products. When the Leontief matrix A is sparse and/or has special structure, equations of the form $Lx = y$ may be solved faster by preconditioned iterative solvers such as conjugate gradient or other Krylov methods, see, e.g., Golub and Van Loan [12]. In our preliminary simulation, we only take advantage of the structure of $H_\alpha^{'}$ by taking the Schur complement, and then let the backslash in MATLAB solve the sparse linear equations.

We use simple line search method for step size control by shrinking the step size by a constant factor if the last step size fails.

In addition, we implement an early termination strategy by identifying the support of the final solution based on the current approximate iterative solution, and directly solve the system of linear equations with the given support. Thus, we obtain an "exact" solution (to the machine accuracy) at termination.

8 Preliminary Computational Results

The homotopy-based interior-point path-following algorithm (referred as HOMOTOPY) is coded in MATLAB script files, and run in the MATLAB environment on a desktop PC (2.8GHz CPU). We solve general Leontief economy equilibrium problem as well as bimatrix game equilibrium problem (formulated as an equivalent Leontief economy equilibrium problem) using HOMOTOPY.

For different size n ($n = 20 : 20 : 100,\ 100 : 100 : 1000$), we randomly generate 15 non-symmetric matrices A of four different types (uniform in $[0, 1]$ or binary $\{0, 1\}$ for both general Leontief problems and Leontief problems risen from bimatrix games). For Leontief problems risen from bimatrix games, we randomly generate two $\frac{n}{2} \times \frac{n}{2}$ matrices C, R, and solve the Leontief problems for

$$A^T = \begin{pmatrix} 0 & C \\ R^T & 0 \end{pmatrix}.$$

For $n = 2000$, we generate only 5 pairs of random matrices C, R to run the simulation.

In Tables 1-4, for problems corresponding to each size of n, "mean_sup" stands for the average support size of u, "max_sup" the maximum support size of u, "mean_turns" the average number of "turning points" along the path, "max_turns" the maximum number of "turning points", "mean_iter" the average number of iterations, and "mean_time" the average computing CPU time in seconds.

From the preliminary computational results, we can draw a few conclusions about the homotopy-based algorithm. First, the primary path does retrogress, that is, the path may reverse in t; see "mean_turns" and "max_turns" columns in Tables 1-4. Secondly, the method seems able to solve sizable problems although the computational complexity is not proven to be a FPTAS. Thirdly, it can solve both Leontief economy equilibrium problem and bimatrix game equilibrium problem with general Leontief matrices, and the solution time is not much different whether or not matrices are symmetric.

n	mean_sup	mean_iter	mean_turns	mean_time	max_sup	max_turns
20	2.5	58.5	0.2	0.3	7	2
40	6.5	120.6	0.5	0.9	9	2
60	4.8	123.7	0.5	1.6	13	2
80	8.8	128.9	1.3	2.5	15	4
100	11.4	149.7	1.1	4.0	22	4
200	20.0	260.4	5.6	29.5	33	22
300	26.5	319.6	6.7	99.0	40	14
400	33.2	398.1	10.9	242.2	55	22
500	40.8	456.4	13.3	446.2	59	24
600	66.0	685.7	26.0	999.1	84	48
700	75.8	603.2	20.4	1207.7	91	54
800	80.0	745.0	29.0	2759.5	109	48
900	92.0	1058.3	40.0	3459.5	129	50
1000	97.4	897.8	38.8	4900.8	134	74

Table 1: General Leontief economy equilibrium with uniform matrix

n	mean_sup	mean_iter	mean_turns	mean_time	max_sup	max_turns
20	6.5	64.0	0	0.3	10.0	0
40	9.8	80.2	0	0.5	13.0	0
60	14.6	108.2	0.4	1.2	23.0	2
80	16.6	119.4	0.4	2.0	24.0	2
100	22.6	164.8	1.6	4.4	34.0	4
200	33.2	265.4	4.0	28.3	44.0	6
300	46.6	452.8	12.4	134.2	60.0	20
400	56.6	394.0	7.6	224.6	71.0	16
500	73.8	626.8	21.2	633.1	88.0	32
600	68.5	652.0	22.0	974.8	84.0	56
700	75.0	539.3	18.0	1209.4	95.0	30
800	81.3	747.0	27.5	2145.7	110.0	46
900	83.8	826.5	25.0	3094.9	101.0	42
1000	102.0	964.5	37.5	3574.4	119.0	50

Table 2: General Leontief economy equilibrium with binary matrix

n	mean_sup	mean_iter	mean_turns	mean_time	max_sup	max_turns
20	4.0	121.2	0.8	0.6	6	4
40	7.2	146.0	2.0	0.9	12	4
60	9.6	120.4	0.8	1.1	14	2
80	10.8	137.6	1.2	2.1	16	4
100	12.4	168.2	1.6	3.3	26	4
200	24.4	315.8	6.4	22.2	38	14
300	45.2	400.4	10.4	78.5	62	22
400	39.2	453.6	11.2	184.6	48	20
500	34.0	436.6	8.0	308.4	44	14
600	41.0	480.8	14.5	518.0	52	38
700	54.0	653.7	22.0	1015.5	60	48
800	59.2	647.8	21.2	1505.0	82	38
900	62.7	718.3	22.7	2236.0	72	26
1000	60.8	873.0	27.6	3293.1	76	42
2000	193.4	2391.6	104	52831.0	208	126

Table 3: Bimatrix game equilibrium problems with uniform matrix

n	mean_sup	mean_iter	mean_turns	mean_time	max_sup	max_turns
20	4.0	45.3	0	0.3	8	0
40	15.0	118.0	0	1.0	15	0
60	10.7	183.3	0	2.5	28	0
80	10.0	95.0	0	1.6	26	0
100	6.7	87.0	1.0	2.2	21	2
200	39.3	333.0	8.0	28.8	46	10
300	51.6	714.8	19.6	133.3	62	58
400	51.3	901.0	24.7	339.8	56	32
500	72.5	1018.5	25.0	666.1	90	32
600	77.3	1103.7	40.7	921.1	98	82
700	90.5	975.5	32.0	1185.1	106	52
800	87.0	857.8	28.5	1519.6	98	48
900	92.0	1143.2	36.8	2726.7	130	74
1000	112.0	1490.8	51.6	4458.9	136	68
2000	233.2	2462.6	130.6	56148.1	260	142

Table 4: Bimatrix game equilibrium problems with binary matrix

References

[1] E. L. Allgower and K. Georg. Continuation and Path Following. Acta Numerica, (1992) 1-64.

[2] K. J. Arrow and G. Debreu. Existence of an equilibrium for competitive economy. *Econometrica*, 22 (1954) 265–290.

[3] W. C. Brainard and H. E. Scarf. How to compute equilibrium prices in 1891. *Cowles Foundation Discussion Paper 1270*, August 2000.

[4] X. Chen and X. Deng. Settling the Complexity of Two-Player Nash Equilibrium. FOCS 2006: 261-27.

[5] X. Chen, X. Deng and S. Teng. Computing Nash Equilibria: Approximation and Smoothed Complexity. In 47th Annual IEEE Symposium on Foundations of Computer Science, (2006) 603 - 612.

[6] B. Codenotti, A. Saberi, K. Varadarajan, and Y. Ye. Leontief Economies Encode Nonzero Sum Two-Player Games. SODA 2006.

[7] R. Cottle, J. S. Pang, and R. E. Stone. *The Linear Complementarity Problem*. Academic Press, Boston, 1992.

[8] C. Daskalakis, P.W. Goldberg, C.H. Papadimitriou. The complexity of computing a Nash equilibrium In 47th Annual IEEE Symposium on Foundations of Computer Science, (2006) 71 - 78.

[9] B. C. Eaves and K. Schmedders. General Equilibrium Models and Homotopy Methods. Journal of Economic Dynamics & Control, 23 (1999) 1249-1279.

[10] C.B.Garcia, W.I.Zangwill. *Pathways to Solutions, Fixed Points, and Equilibria*. Prentice-Hall, Englewood Cliffs, NJ, (1981).

[11] I. Gilboa, E. Zemel. Nash and Correlated equilibria: Some Complexity Considerations. *Games and Economic Behavior*, 1 (1989) 80-93.

[12] G. H. Golub and C. F. Van Loan. *Matrix Computations.* The Johns Hopkins University Press, Baltimore, MD, (1996)

[13] C. E. Lemke. On complementary pivot theory. In Dantzig and Veinott, eds., *Mathematics of Decision Sciences, Part 1*, American Mathematical Society, Providence, Rhode Island (1968) 95-114.

[14] H. Tsaknakis and P. Spirakis. An Optimization Approach for Approximate Nash Equilibria. Lecture Notes in Computer Science, Volume 4858 (2007), 42-56.

[15] L. Walras. *Elements of Pure Economics, or the Theory of Social Wealth.* 1874 (1899, 4th ed.; 1926, rev ed., 1954, Engl. Transl.)

[16] Y. Ye. A Path to the Arrow-Debreu Competitive market equilibrium. Mathematical Programming, Ser. B (2008) 111:315-348.

[17] Y. Ye. Exchange Market Equilibria with Leontief's Utility: Freedom of Pricing Leads to Rationality. WINE 2005 and Theoretical Computer Science, 378:2 (2007) 134-142.