# A brief introduction to Koenig's eigenfunctions to Schroeder's functional equation

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## 1 Functional equations and Schröder's equation of iteration

A functional equation is an equation where the unknowns include not only variables (like scalars) but also include functions. An example is that the Gamma function (or the more familiar integral factorial function) satisfies

$$\Gamma(x+1) = x\Gamma(x)$$

Here the variable $x$ can take any real value, but equations of this type are of interest usually when the functions satisfying them are unknown. The theory behind solving these types of equations is too involved to cover here.

In the late nineteenth century, Ernst Schröder studied the functional equation of the composition operator: $\circ$, where if $f(z) = z^2$ and $g(z) = z + 1$, $(f \circ g)(z) = (z+1)^2$. The functional equation named after him is

$$f \circ g = \lambda f$$

where $g$ is a given complex function, and we want to find $f$ and $\lambda$ to satisfy the equation. The applicable results in literature apply to situations where $g$ is a function mapping the unit disc in the complex plane onto itself. There must be some point $a$ in the unit disc that is actually mapped onto itself. Gabriel Koenigs discovered the existence-uniqueness theory for solutions to Schröder's functional equation by an iterative method. A Koenigs eigenfunction $k$ is given by

$$k \circ g = g'(a)k$$

where

$$k(z) = \lim_{n\to\infty} \frac{1}{g'(a)^n} \left( \underbrace{g \circ g \circ \ldots \circ g}_{n}(z) - a \right)$$

To relate this back to the original function $f$, $f = Ck^n$ and $\lambda = g'(a)^n$ for some constant $C$. The beautiful part about this theory is that the iterates of the above limit for finite $n$ converge geometrically toward the actual solution $k(z)$, so after about a dozen iterations, (usually) no discernable differences exist between the iterate and the actual solution.

The screensaver solves for $k(z)$ by applying the iteration formula, and uses $g(z) = K\frac{z(z-A)}{(z-C)(z-D)}$, where $K$, $A$, $C$, and $D$ are complex constants with real and imaginary parts in $[-\frac{1}{2}, \frac{1}{2})$. The user is allowed to set the range of acceptable eigenvalues $g'(a)$, but it is limited to (0,1) on the real axis.

Notice that the origin is the fixed point ($a = 0$), so any interesting behavior is "centered" about the origin. There is no guarantee that $g(z)$ is a self-map of the unit disc onto itself, but the solutions found still seem to converge.

The mathematically inclined are invited to read Shapiro's "Composition Operators and Schroder's Functional Equation" in Contemporary Mathematics 213, Bourdon and Shapiro's "Mean Growth of Koenigs Eigenfunctions" in the Journal of the American Mathematical Society volume 10 number 2.

## 2 Plotting complex functions

There are several coloring schemes which have been borrowed from various websites and publications. Unlike traditional fractal plotting programs, such as those for the Mandlebrot set, the generated images are plots of the actual function values rather than the number of iterations taken. Each point in the complex plane is assigned a (mostly) unique color. Each pixel represents some sample points of the complex plane, which are fed into the iterated function to obtain some final point. The color of the pixel is determined by the complex plane coordinates of the final point. In all coloring schemes, the hue value usually gives the complex argument of the point, and the complex modulus of the point modules the lightness or saturation.

The fiery coloring scheme I called "Halun" from the url of Hans Lundmark's page: `http://www.mai.liu.se/~halun/complex/complex.html`, and Crone is from my original inspiration: Larry Crone's page at `http://www.american.edu/cas/mathstat/People/lcrone/ComplexPlot.html`. The "Plain" coloring scheme is my own which is a variant of Crone's for more saturated colors.

## 3 Things to try

Interesting ways of modifying the program largely involve changing the functions used in iteration, such as using higher order functions, trancendental functions, etc. Interesting things happen with complex exponentials (or trigonometric functions) but require zooming in on certain regions to capture the intricate details.