RATE-DISTORTION WITH COMMON RATE-LIMITED SIDE INFORMATION TO THE ENCODER AND DECODER

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ABSTRACT

We consider a family of problems involving source coding in the presence of limited-rate side information. We give complete single-letter characterizations of the fundamental limits for these problems. We also work out the explicit form of these fundamental limits for specific cases such as Gaussian and binary sources.

***Index Terms*—** Helper, rate-distortion, rate-limited side information.

1. INTRODUCTION

In this paper, we consider the problem of source encoding with a fidelity criterion in a situation where both the decoder and the encoder receive a common message from a helper. The problem is presented in Fig. 1. The source and helper's sequences, denoted as $\{X_i\}_{i\geq 1}$ and $\{Y_i\}_{i\geq 1}$, respectively, are i.i.d. random variables distributed according to some joint distribution $p(x,y)$.

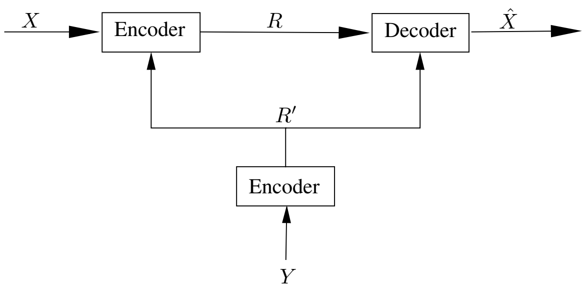

Fig. 1. The rate distortion problem with rate limited side information known at the encoder and decoder.

The encoding and decoding is done in block of length n. The communication protocol is that first, the helper sends a common message with rate R' to the source-encoder and decoder and then the source-encoder sends a message at rate R to the decoder. The decoder receives both messages and reconstructs the source. The fidelity (or distortion) criterion is only with respect to the source, i.e., $1/n\sum_{i=1}^n d(X_i,\hat{X}_i)$, where $\hat{\mathcal{X}}$ is the reconstruction alphabet, $d:\ \mathcal{X}\times\hat{\mathcal{X}}\to[0,\infty)$ is a single letter distortion measure, $\hat{X}_i$ is the reconstruction at time i, and the goal is to obtain a distortion that is less than D.

Our main result in this paper is that the achievable region of this problem is $\mathcal{R}(D)$, which is defined as the set of all rate pairs (R,R') that satisfies

$$\begin{aligned} R &\geq I(X;\hat{X}|U), \\ R' &\geq I(Y;U), \end{aligned} \tag{1}$$

for some joint distribution of the form

$$p(x,y,u,\hat{x}) = p(x,y)p(u|y)p(\hat{x}|u,x), \quad \mathbb{E}d(X,\hat{X})\leq D, \tag{2}$$

where U is an auxiliary random variable with cardinality $|\mathcal{U}|\leq|\mathcal{Y}|+2$.

This region can be achieved by a simple communication scheme in which the helper sends a message to generate a sequence u^n at the source-encoder and decoder. The communication rate R' needs to be high enough such that the joint-type of u^n and y^n is arbitrary close to the joint distribution $p(u,y)$ for n large enough. Then the sequence u^n is used as side information that is known both to the encoder and decoder and therefore a rate $R > I(X;\hat{X}|U)$ suffices to achieve the fidelity criterion.

Several settings of encoding two correlated sources (a.k.a multi-terminal source coding) have been solved in the literature. The first case was solved by Slepian and Wolf [1], in which the goal is to reproduce both sources losslessy, and the encoders are ignorant to each other messages. A similar setting was considered by Wyner [2] and by Ahlswede and Körner [3], also known as WAK, and there only one source needs to be reconstructed losslessy; the other source is acting as a helper. Wyner and Ziv [4] characterized the rate distortion region of correlated sources when one of the rates is unlimited and therefore known perfectly to the decoder.

Kaspi and Berger [5] and Kaspi [6] derived an achievable scheme for a general case that contains the regions for all the cases above. In particular, case C of Theorem 2.1, in [5, 6]

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provides an achievable scheme for the problem in Fig. 1 with a general distortion $d(x, y, \hat{x})$; however the converse is yet to be shown[1].

Berger and Yeung [7] solved the multi-terminal source problem where one of the two sources needs to be reconstructed perfectly and the other source needs to be reconstructed with a fidelity criterion. Oohama solved the multi-terminal source coding case for two [8] and $L+1$ [9] Gaussian sources, in which only one source needs to be reconstructed with a mean square error, that is, the other L sources are helpers. More recently, Wagner, Tavildar, and Viswanath, characterized the region where both sources [10] or $L+1$ sources [11] need to be reconstructed at the decoder with a mean square error criterion.

2. PROBLEM DEFINITIONS AND MAIN RESULTS

Here we formally define the problem of rate-distortion with a helper known both to the encoder and decoder and present a single-letter characterization of the achievable region.

We are using the regular definitions of rate distortion and we follow the notation of [12]. The source sequence $X_i \in \mathcal{X},\ i = 1, 2, \cdots$ and the side information sequence $Y_i \in \mathcal{Y},\ i = 1, 2, \cdots$ are discrete random variables drawn from finite alphabets $\mathcal{X}$ and $\mathcal{Y}$, respectively. The random variables (X_i, Y_i) are i.i.d $\sim p(x, y)$. Let $\hat{\mathcal{X}}$ be the reconstruction alphabet, and $d:\ \mathcal{X} \times \hat{\mathcal{X}} \to [0, \infty)$ a single letter distortion measure. Distortion between sequences is defined in the usual way

$$d(x^n, \hat{x}^n) = \frac{1}{n}\sum_{i=1}^{n} d(x_i, \hat{x}_i).$$

Definition 1 *An (n, M, M', D) code for source X with side information Y consists of two encoders*

$$\begin{aligned} f' &: \mathcal{Y}^n \to \{1, 2, ..., M'\} \\ f &: \mathcal{X}^n \times \{1, 2, ..., M'\} \to \{1, 2, ..., M\} \end{aligned} \tag{3}$$

and a decoder

$$g \ : \ \{1, 2, ..., M\} \times \{1, 2, ..., M'\} \to \hat{X}^n \tag{4}$$

such that

$$\mathbb{E}d(X^n, \hat{X}^n) \le D, \tag{5}$$

where $\mathbb{E}$ denotes the expectation operation.

The rate pair (R, R') of the (n, M, M', D) code is

$$\begin{aligned} R &= \frac{1}{n}\log M \\ R' &= \frac{1}{n}\log M'. \end{aligned} \tag{6}$$

[1] In [6, p.25-26] there is an attempt to prove the converse of region provided in [5, 6], but the Markov Chain $U - (X, W) - Y$, which is one of the conditions that characterize the region, is not proved.

Definition 2 *Given a distortion D, a rate pair (R, R') is said to be achievable if for any $\delta > 0$, $\epsilon > 0$, and sufficiently large n, there exists an $(n, 2^{n(R+\delta)}, 2^{n(R'+\delta)}, D + \epsilon)$ code for the source X with side information Y.*

Definition 3 *The (operational) achievable region $\mathcal{R}^O(D)$ of rate distortion with a helper known at the encoder and decoder is the closure of the set of all achievable rate pairs.*

We proceed to state some simple properties of the region $\mathcal{R}(D)$.

Lemma 1 *1. The region $\mathcal{R}(D)$ is convex*

2. To exhaust $\mathcal{R}(D)$, it is enough to restrict the alphabet of U to satisfy

$$|\mathcal{U}| \le |\mathcal{Y}| + 2.$$

The next theorem is the main result of this paper.

Theorem 2

$$\mathcal{R}^O(D) = \mathcal{R}(D). \tag{7}$$

The region $\mathcal{R}(D)$ is computed for the doubly-symmetric binary source with Hamming distortion measure, and for the Gaussian source with square error distortion measure. In addition, extensions to models with uncoded side information are presented, and the case where the helper sends different messages to the encoder and decoder (with possibly different rates) is also treated and solved to a limited extent.

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