Calculemus! An extended example illustrating van Rooij (2003)'s analysis of questions

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Cigarettes. It is Sunday afternoon, five o'clock. I stand in front of Margaret Jacks Hall, talking to a seasoned forth-year smoker who we'll call LW. I know and she knows (and we both know that we know &c. ..., in short, it is common knowledge) that the only two places on campus that could possibly sell cigarettes are the convenience store in Tresidder Union (T) and the gas station in the east of the campus (G). We also know that G is approximately twice as far away from MJH than T. I am unaware which of the two places is open for business on a Sunday afternoon. So, against this background of common knowledge and one-sided uncertainty, I ask LW: *Where can I buy cigarettes now?*

We represent the statements *You can buy cigarettes at Tresidder* as $Cig(T)$, *You can buy cigarettes at the gas station* as $Cig(G)$ and *You can buy cigarettes at Tresidder and the gas station* as $Cig(T \oplus G)$ and take the latter to be truth-conditionally equivalent to $Cig(T) \wedge Cig(G)$.

Assume further that we have modal model M with four different worlds such that

- $[\![Cig]\!]_{M,w_\emptyset} = \emptyset$
- $[\![Cig]\!]_{M,w_T} = \{T\}$
- $[\![Cig]\!]_{M,w_G} = \{G\}$
- $[\![Cig]\!]_{M,w_{T+G}} = \{T, G, T \oplus G\}$

We assume that I am faced with the choice between three different actions a_0, a_t, a_g representing doing nothing, going to Tresidder and going to the gas station, respectively. We represent the fact that going somewhere where I can buy cigarettes satisfies my desire for them by awarding 10 utilities in this case, and 0 otherwise. The varying efforts involved in executed the three actions (i.e. the distance traveled), we model as *costs* that get subtracted from the reward. We end up with the decision problem in Figure 1.

In order to calculate the set of answers with respect to decision problem D, we have to calculate the utility value of the derived decision problems (see Figure 2-4). The last row in the figures gives the expected utilities of the actions, the maximum value is bolded (and gives the UV of the decision problem).

w	P(w)	a_0	a_t	a_g
$w_\emptyset$	$\frac{1}{4}$	0	-1	-2
w_T	$\frac{1}{4}$	0	9	-2
w_G	$\frac{1}{4}$	0	-1	8
w_{T+G}	$\frac{1}{4}$	0	9	8
EU(a)		0	**4**	3

Figure 1: Decision problem D

w	P(w)	a_0	a_t	a_g
w_{T+G}	1	0	9	8
EU(a)		0	**9**	8

Figure 2: Decision problem $D[\![Cig(T \oplus G)]\!]$

w	P(w)	a_0	a_t	a_g
w_T	$\frac{1}{2}$	0	9	-2
w_{T+G}	$\frac{1}{2}$	0	9	8
EU(a)		0	**9**	3

Figure 3: Decision problem $D[\![Cig(T)]\!]$

w	P(w)	a_0	a_t	a_g
w_G	$\frac{1}{2}$	0	-1	8
w_{T+G}	$\frac{1}{2}$	0	9	8
EU(a)		0	4	**8**

Figure 4: Decision problem $D[\![Cig(G)]\!]$

Thus we can calculate the utility values of assertions of the form $Cig(x)$ as:

$$\begin{aligned}
&UV_D([\![Cig(T)]\!]) &&= UV(D[\![Cig(T)]\!]) - UV(D) &&= 9 - 4 = 5\\
&UV_D([\![Cig(G)]\!]) &&= UV(D[\![Cig(G)]\!]) - UV(D) &&= 8 - 4 = 4\\
&UV_D([\![Cig(T \oplus G)]\!]) &&= UV(D[\![Cig(T \oplus G)]\!]) - UV(D) &&= 9 - 4 = 5
\end{aligned}$$

And thus we can rank the assertions in terms of their utility value (wRt D): $UV_D([\![Cig(T)]\!]) = UV_D([\![Cig(T \oplus G)]\!]) > UV_D([\![Cig(G)]\!])$.

With this, we can calculate $Op_D(Cig)$, defined as

$$Op_D(Cig) = \{\langle w, g\rangle \mid w \in [\![Cig(g)]\!] \ \&\ \neg\exists g' : w \in [\![Cig(g')]\!] \ \&\ UV_D([\![Cig(g')]\!]) > UV_D([\![Cig(g)]\!])\}$$

We get:

$$\begin{aligned}
&Op_D(Cig)(w_\emptyset) = \emptyset\\
&Op_D(Cig)(w_T) = \{T\}\\
&Op_D(Cig)(w_G) = \{G\}\\
&Op_D(Cig)(w_{T=G}) = \{T, T \oplus G\}
\end{aligned}$$

Finally, we can figure out the question denotation according to van Rooy (2003), which is defined as

$$[\![?x\ Cig(x)]\!]^R_D = \{\lambda v\,[g \in Op_D(Cig)(v)] \mid w \in W \ \&\ g \in Op_D(Cig)(w)\}$$

i.e. for all w, g: If $g \in Op_D(w)$, we collect all v such that $g \in Op(Cig)(v)$. This is done below:

$$\begin{aligned}
&\langle w_T, T\rangle : && \{w_T\}\\
&\langle w_G, G\rangle : && \{w_G\}\\
&\langle w_{T+G}, T\rangle : && \{w_T, w_{T+G}\}\\
&\langle w_{T+G}, T \oplus G\rangle : && \{w_{T+G}\}
\end{aligned}$$

The set of the sets on the right side is not a partition of the space of worlds. It would be if for all worlds v: If $v \in [\![Cig(g)]\!]$ and there is v' such that $g \in Op(v')$ then $g \in Op(v)$. Or, perhaps less opaque: If g is optimal anywhere, it is optimal whereever it is true.

In this case, it is also true that every *resolving* answer will be in the question denotation (where 'resolving' is understood as 'makes one action dominate all others'). But that is not true in general. Consider the problem D' in Figure 5, which results in the following utility values:

$$\begin{aligned}
UV_{D'}([\![Cig(T)]\!]) &= 9 - 8 = 1\\
UV_{D'}([\![Cig(G)]\!]) &= 18 - 8 = 10\\
UV_{D'}([\![Cig(G \oplus T)]\!]) &= 9 - 8 = 1
\end{aligned}$$

Resulting in

$$
\begin{aligned}
Op_{D'}(Cig)(w_T) &= \{T\} \\
Op_{D'}(Cig)(w_G) &= \{G\} \\
Op_{D'}(Cig)(w_{T+G}) &= \{G\}
\end{aligned}
$$

With this:

$$[\![?x\ Cig(x)]\!]^R_{D'} = \{\{w_T\}, \{w_G, w_{T+G}\}\}$$

Thus, even though $Cig(T)$ is a *resolving* answer, it is not an element of the question denotation. Is this a virtue, or a vice, of van Rooij's account? **Discuss.**

w	P(w)	a_0	a_t	a_g
$w_\emptyset$	$\frac{1}{4}$	0	-1	-2
w_T	$\frac{1}{4}$	0	9	-2
w_G	$\frac{1}{4}$	0	-1	28
w_{T+G}	$\frac{1}{4}$	0	9	8
EU(a)		0	4	**8**

Figure 5: Decision problem D'