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Simplified Virtual Source Model for Unstable Resonators

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This memo outlines an alternative formulation of the virtual source model for unstable resonators using a symmetric unstable resonator model rather than on the canonical model employed by Horwitz and others. This alternative model contains no new physics and should be mathematically equivalent to the canonical model, but it clarifies the physical basis of the geometrical as well as edge wave analyses.

Figure 1(a) shows the widely used "canonical model" for an unstable resonator, consisting of a zero-length beam expanding telescope having transverse magnification M located immediately after the output mirror or lensguide aperture, plus a free space section of length MB, where B is the B element of the $ABCD$ matrix from aperture to aperture in the actual unstable system.

This model demonstrates the geometrical features of the unstable resonator and shows how diffraction effects within the resonator can be handled in a single propagation step through a collimated beam section with a specified collimated Fresnel number N_c. Note that the beam-expanding telescope can equally well have a negative geometric magnification $M < -1$, corresponding to a so-called negative branch unstable resonator.

Figure 1(b) shows an alternative model for the same physical system in the form of an axially symmetric periodic lensguide with a diverging (or, potentially, over-converging) lens filling each aperture of the periodic lensguide so as to produce an equivalent geometric magnification M per period. Either of these models can represent any unstable resonator having arbitrary paraxial intracavity elements plus a single output coupling mirror.

In either of these models it is convenient to consider the eigenmodes at a reference plane just inside the output coupling plane, as shown by the dashed lines in both figures. If one looks backward down the lensguide from this reference plane, in either case one can see all of the earlier apertures of the lensguide, with each earlier aperture seen through an increasing number of lenses or other optical elements contained within each period of the lensguide. Each earlier aperture will then appear to have a magnified or demagnified apparent size, and to be located at a different apparent distance back down the guide. Considering the edges of each of these earlier apertures as potential scattering points or radiation sources for "edge waves" which will be added into the radiation propagating along the unstable system is the essence of the virtual source approach. The basic concept of the virtual source theory is that each eigenmode function at any reference plane in an unstable lensguide can be viewed as the summation of a series of edge-wave functions coming from the edges of the nearer apertures seen looking back down the lensguide, plus a spherical or geometric component which comes from a superposition of all the more distant apertures. The set of edge wave functions coming from the nearer apertures provides the most effective set of basis functions for expanding the complicated eigenwaves of an unstable system.

The first step in the virtual source analysis is, therefore, to determine the apparent locations and sizes of the earlier apertures as seen looking back from the reference plane through the intervening optics. This can be easily done using an $ABCD$ matrix formalism. If one considers a fan of geometrical rays emerging with different slopes x_1' from a source point having a transverse

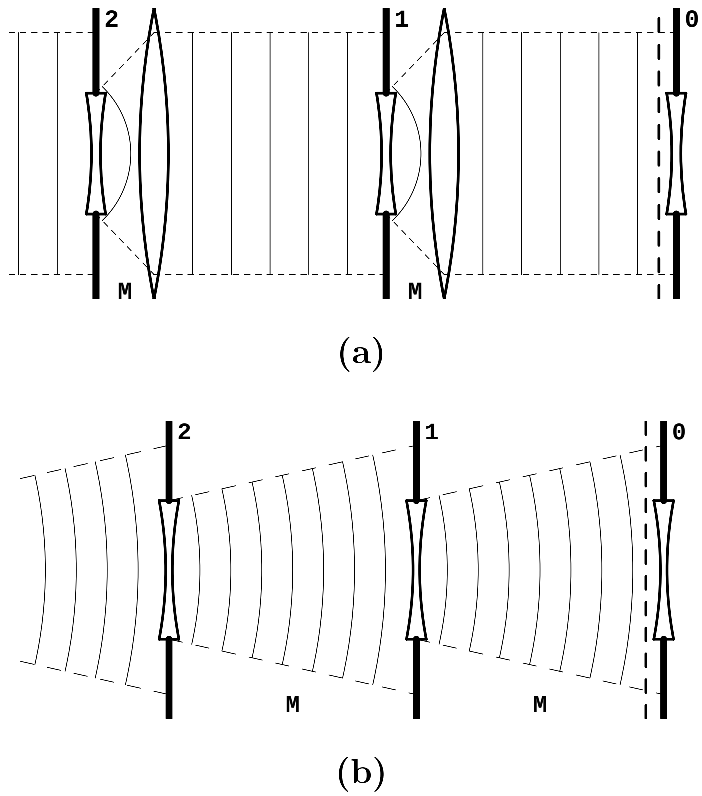

Figure 1. (a) Canonical lensguide model for a general one-aperture unstable resonator or lensguide. (b) Alternative symmetrical model for the same case.

displacement x_1 and passing through an $ABCD$ system representing for example multiple periods of an unstable lensguide, the ray displacements x_2 and slopes x_2' at the output plane from this $ABCD$ system will be given by $x_2 = Ax_1 + Bx_1'$ and $x_2' = Cx_1 + Dx_1'$. Eliminating the input slope between these equations gives

$$x_2 = (1/D)\, x_1 + (B/D)\, x_2' \ . \tag{1}$$

In physical terms this says that the rays emanating from point x_1 at the input plane appear at the output plane as if they emanated from a virtual source point displaced off the axis by displacement x_1/D and located at a distance B/D behind the output plane.

If we apply this formula to the canonical model of Figure 1(a), for which the $ABCD$ matrix going from one aperture to the next is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} M & B \\ 0 & 1/M \end{bmatrix} \ , \tag{2}$$

the successive apertures seen looking back down the lensguide in this model appear to be located at increasing distances given by

$$L_N \equiv \frac{B_N}{D_N} = \frac{M^{2N} - 1}{M^2 - 1/M} \times MB \ . \tag{3}$$

and to have increasing effective sizes given by

$$c_N = \frac{a}{D_N} = M^N\, a\ . \tag{4}$$

The apparent aperture distances L_N in this model grow considerably more rapidly with increasing N than do the aperture diameters cN, and so the distant apertures appear to subtend a steadily decreasing solid angle. The more distant apertures in this model in fact converge to an apparent point source located in this case at minus infinity.

A similar analysis for the symmetric lensguide of Figure 1(b) leads to slightly more complex algebraic expressions, but a more physically interesting result. We can first note that the $ABCD$ matrix for one complete period of the symmetric system, going from a midplane halfway between the lenses to a midplane one period later, can be written in the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} m & (m+1)L/2 \\ 2(m-1)/L & m \end{bmatrix} \equiv \begin{bmatrix} \cosh\theta & R_0 \sinh\theta \\ (1/R_0)\sinh\theta & \cosh\theta \end{bmatrix} \tag{5}$$

where $m \equiv (A+D)/2 \equiv \cosh\theta$ is the half-trace of this $ABCD$ matrix. The geometric magnification M for a positive-branch unstable system is then given by

$$M = m + \sqrt{m_1^2} = e^{\theta} \tag{6}$$

and the "characteristic radius" R_0 is given by

$$R_0 = \sqrt{\frac{m+1}{m-1}} \times \frac{L}{2} = \frac{M+1}{M-1} \times \frac{L}{2} = \frac{L}{2\tanh(\theta/2)} \tag{7}$$

This characteristic radius turns out to be just the radius of curvature of the geometrical wave at the midplane of the unstable resonator. The transformation through N cascaded midplane-to-midplane sections in sequence can then be immediately written as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \begin{bmatrix} \cosh N\theta & R_0 \sinh N\theta \\ R_0^{-1} \sinh N\theta & \cosh N\theta \end{bmatrix} \tag{8}$$

The total transformation from a plane just behind the N-th aperture to the reference plane than consists of the lens associated with the N-th aperture back; free-space propagation through length $L/2$ from this lens to the next nearest midplane; $N-1$ midplane-to-midplane sections evaluated according to the matrix just above; and finally an additional half-period of free-space propagation from the final midplane to the reference plane, leading to the overall $ABCD$ matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cosh(N+1/2)\theta/\cosh\theta/2 & L\sinh N\theta/\sinh\theta \\ 2\tanh(\theta/2)\sinh N\theta/L & \cosh(N-1/2)\theta/\cosh\theta/2 \end{bmatrix} \tag{9}$$

Using these $ABCD$ values shows that each earlier aperture appears to lie at a distance behind the

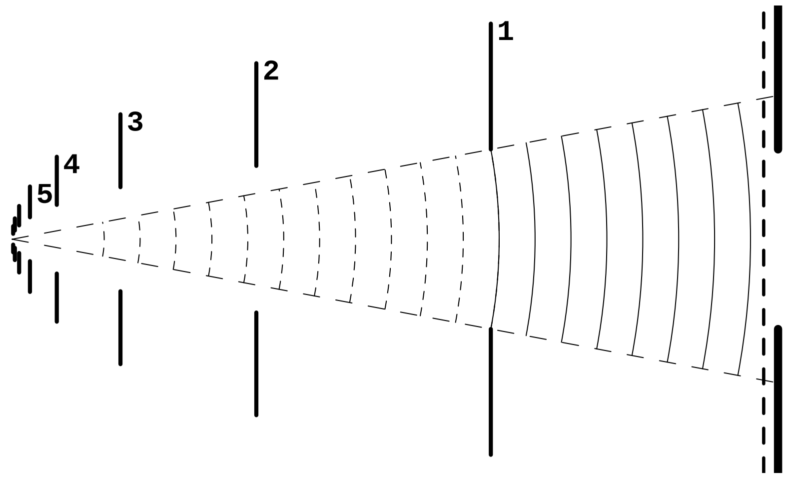

Figure 2. Virtual source model derived from the symmetric unstable resonator model.

reference plane given by

$$L_N = \frac{B}{D} = \frac{L}{2}\left[\frac{\tanh(N-1/2)\theta}{\tanh\theta/2} + 1\right]. \tag{10}$$

and to have a successively smaller halfwidth a_N given by

$$a_N = \frac{a}{D} = \frac{a\cosh\theta/2}{\cosh(N-1/2)\theta} \tag{11}$$

Figure 2 illustrates the apparent sizes and locations of the earlier apertures for a typical symmetric unstable resonator case.

A striking feature of Figure 2 is that the axial locations for all the more distant apertures converge to a limiting value given by

$$L_\infty \equiv \lim_{N\to\infty} L_N = \frac{M}{M-1}L = R_g \tag{12}$$

where R_g is just the radius of curvature of the geometrical wave at the ouput plane of the unstable lensguide. In other words, in this symmetric-lensguide model the earlier apertures appear to all converge down to a single point on axis at a finite distance $L_\infty \equiv R_g$ behind the output plane; and this convergence point corresponds exactly to the source point of the divergent geometrical wave that characterizes the symmetric lensguide of Figure 2. In physical terms, the geometrical part of the beam in the lensguide can be described as coming from the superposition of all the most distant apertures, all of which appear to be very small and to be located to a fixed apparent distance behind the reference plane.

Another striking feature of Figure 2 is how rapidly the apparent apertures converge down to the geometrical source point. We can inquire how far back along this chain of apertures one must go until the apparent apertures become so small that the waves scattered from two edges effectively merge into a single point source. The phase difference between spherical wavelets coming from the upper and lower edges of aperture N, as observed at the outer edge of the reference plane aperture, will be given by

$$\Delta\phi \approx \frac{\pi\, a\, a_N}{L_N \lambda} \ . \tag{13}$$

The condition for this phase shift to be a small fraction of a cycle, or $\Delta\phi \leq \epsilon 2\pi$ with $\epsilon \ll 1$can then be written in the form

$$N \geq \ln\!\left(\frac{4\epsilon M^{1/2}}{M+1} N_{eq}\right) \Big/ \ln M \tag{14}$$

where N_{eq} is the equivalent Fresnel number for the resonator. If we set $\epsilon \approx 1\%$ this is not significantly different from Horwitz's original criterion requiring that $N \geq \ln(250 N_{eq})/\ln(M)$.

Appendix: Derivation of the Asymptotic Polynomial Equation

For reference, we append here an outline of how the polynomial equation for the asymptotic analysis originally developed by Horowitz can be obtained using the virtual source model described in this memo.

Assuming a one-transverse-dimensional model for simplicity, let $f(x)$ denote any one of the transverse eigenwaves of the unstable lensguide as observed at the reference plane selected above, and let $f_g(x)$ denote the elementary spherical wave characteristic of the geometrical solution for the same resonator at the same plane. We expect to find in general some set of multiple lowest and higher-order eigenwaves $f_n(x)$—possibly only a finite set?—having corresponding eigenvalues γ_n, but for the moment we will leave off the index n that identifies individual eigenwaves.

We then assume that any one eigenwave of the unstable system is launched into the lensguides of Figs. 1 or 2 with unit amplitude at the N-th aperture back from the output reference plane, where N has a sufficiently large value that this aperture approaches the convergence point shown in Fig. 2. Let $f^{(N)}(x)$ denote the wave pattern of this same eigenwave when it reaches the output aperture N periods later. Based on the physical model just discussed, we can assume that this output wave will consist of the geometric wave $f_g(x)$ produced by the wave coming through the apertures located more than N periods back, plus edge waves $F_n(x)$ scattered from each of the intervening N aperture edges seen looking back from the output plane. We assume therefore that we can write the output wave (for symmetric eigenmodes) in the form

$$f^{(N)}(x) = \gamma_g^N c_g f_g(x) + \sum_{n=1}^{N} c_n F_n(x) \tag{$A-1$}$$

The first term in this expansion is the geometric part of the solution which we assume was launched with an initial amplitude c_g at the aperture N periods back. This geometrical wave as it radiates out and spreads will be attenuated by the geometrical attenuation factor γ_g for each period it passes through, or a total attentuation of γ_g^N by the time it reaches the output plane., where the geometical attenuation if given by $\gamma_g = 1/M^{1/2}$ for one transverse dimension or $1/M$ for two transverse dimensions. The functions $F_n(x)$ represent the edge-wave contributions at the output plane produced by a wave of unit amplitude striking the edges of each of the apertures as seen looking back from the output plane. Various exact or approximate expressions for these edge-wave functions for slit and circular apertures are available in the literature.

Now, since the actual wave passing down the lensguide and illuminating the successive apertures is one of the system eigenwaves, the actual amplitude striking an aperture edge n periods back will be just the output function $f^{(N)}(x)$ increased by the ratio γ^{-n}. Hence the actual amplitude c_n of each edge wave at the earlier aperture will be given by

$$c_n = \gamma^{-n} f^{(N)}(a) \qquad (A-2)$$

and Eq. (A-1) becomes

$$f^{(N)}(x) = \gamma_g^{-N} c_g f_g(x) + f^{(N)}(a) \sum_{n=1}^{N} \gamma^{-n} F_n(x) \ . \qquad (A-3)$$

Suppose now that we launch the same eigenwave, with the same input amplitude, except we launch it $N+1$ periods back. By the same arguments the output wave is now given by

$$\begin{aligned} f^{(N+1)}(x) &= \gamma_g^{N+1} c_g f_g(x) + f_{N+1}(a) \sum_{n=1}^{N+1} \gamma^{-n} F_n(x) \\ &= \gamma f^{(N)}(x) \qquad (A-4) \\ &= \gamma \gamma_g^{N} c_g f_g(x) + f^{(N)}(a) \sum_{n=1}^{N} \gamma^{1-n} F_n(x) \end{aligned}$$

Note that the second line in this equation must be true because $f^{(N+1)}(x)$ is, by definition, just the same eigenwave as $f^{(N)}(x)$ but propagated one period further along the lensguide. By comparing the summations in the first and third lines of Eq. (A-4), we can see that the individual $F_n(x)$ terms in the first and second lines are automatically matched in amplitude for $n = 1$ to $n = N$, since $f^{(N+1)}(a) \equiv \gamma f^{(N)}(a)$. Equating the remaining terms on the right-hand sides of the first and third lines then gives

$$(\gamma - \gamma_g)\gamma_g^N c_g f_g(x) = \gamma^{1-N} f^{(N)}(a) F_{N+1}(x) \qquad (A-5)$$

Both sides of this equation will have the same x dependence, as required, because the distant edge-wave term $F_{N+1}(x)$ will in fact have exactly the same form as the geometrical wave term $f_g(x)$ for a large enough value of N (i.e., for a distant enough and small enough aperture). This equation thus reduces to

$$c_g f_g(x) = \frac{(\gamma\gamma_g)^{-N}}{\gamma - \gamma_g} F_{N+1}(x) \qquad (A-6)$$

and this expression serves to scale the amplitude c_g of the initial geometric wave launched through the distant aperture. Combining the above results and setting $x = a$ then gives the polynomial equation

$$(\gamma - \gamma_g)\gamma^N = F_{N+1}(a) + (\gamma - \gamma_g \sum_{n=1}^{N} F_n(a) \gamma_g^{N-n} \qquad (A-7)$$

Since the F_n's are known functions, and the eigenwave amplitude $f(a)$ has cancelled out, this becomes an $(N+1)$-th order polynomial equation which can be used to determine $N+1$ eigenvalues γ_n for the eigenwaves $f_n(x)$ in the lensguide. This is in fact the same polynomial derived by Horwitz and later authors using more mathematical and less physical arguments.

We inquire finally how large the order of this polynomial must be to obtain accurate results. In physical terms this is the same as asking how far back along the chain of apertures one must go until the apparent apertures become so small that their two edges effectively merge on axis, so that the edge waves from the N-th aperture back will have essentially the same form as the geometrical wave at the output plane. The phase front for the geometrical wave has the form

$$f_g(x) \propto \exp\left[-j\frac{\pi x^2}{L_\infty \lambda}\right] \qquad (A-8)$$

while the phase fronts from the two edge waves of aperture N will have the form

$$F_N(x) \propto \exp\left[-j\frac{\pi}{L_N\lambda}(x \mp c_N)^2\right] \qquad (A-9)$$

Suppose enough constant phase shift is added to one of these waves to make these two wavefronts coincide at one edge ($x = +a$) of the output aperture. The phase difference between the two waves at the other edge ($x = -a$) will then be given by

$$\Delta\phi \approx \frac{\pi}{L_N\lambda} a c_N \qquad (A-10)$$

and we desire that this phase shift be a small fraction of a cycle, or $\Delta\phi \leq \epsilon \times 2\pi$ with $\epsilon \ll 1$. Using the formulas for c_N and L_N for large N then leads to the criterion

$$N \geq \ln\left(\frac{4\epsilon M^{1/2} F_{eq}}{M+1}\right) \Big/ \ln M \qquad (A-11)$$

If we set $\epsilon = 1/125$ this is not significantly different from Horwitz's original criterion requiring that $N \geq \ln(250F_{eq})/\ln(M)$.

Annotated Reference List

J. B. Keller, "Geometrical theory of diffraction" *J. Opt. Soc. Am.*, vol. 52, pp. 116–119, February 1962. Summarizes Keller's "geometrical" or "edge wave" approach to diffraction theory which provides one basis for (among many other things) the "virtual source" approach to unstable resonator modes.

A. E. Siegman, "Unstable optical resonators for laser applications," *Proc. IEEE* **53**, 277 (March 1965). First paper introducing the unstable resonator concept.

P. Horwitz, "Asymptotic theory of unstable resonator modes" *J. Opt. Soc. Am.*, vol. 63, pp. 1528–1543, December 1973. This paper presents a very important early development in unstable resonator mode analysis. Horwitz's "asymptotic analysis" of unstable resonator modes, as presented in this paper, was apparently done "out of the blue" based entirely on a mathematical asymptotic or stationary-phase approach to Huygens integral, rather than on any physical insight. (Horwitz was actually I believe a theoretical physicist). His results, however, are really essentially the same as the very useful "virtual source" method developed later. on

G. L. James, *Geometrical Theory of Diffraction for Electromagnetic Waves*, Peter Peregrinus, 1976. Possible discussion of edge waves? (I haven't examined this one).

P. Horwitz, "Modes in misaligned unstable resonators" *Appl. Opt.*, vol. 15, pp. 167–178, January 1976. This is a further extension of Horwitz's classic early "asymptotic" unstable resonator analysis to misaligned unstable resonators.

A. E. Siegman, "A canonical formulation for multi-element unstable resonator calculations," *IEEE J. Quantum Electron.* **QE–12**, 35 (January 1976). Original source for the canonical formulation.

G. T. Moore and R. J. McCarthy, "Theory of modes in a loaded strip confocal unstable resonator" *J. Opt. Soc. Am.*, vol. 67, pp. 228–241, February 1977. I've keyworded this paper under "virtual source", but I don't think this group was the first to apply the virtual source method to unstable resonators. I need to go back and have another look at this.

R. R. Butts and P. V. Avizonis, "Asymptotic analysis of unstable laser resonators with circular mirrors" *J. Opt. Soc. Am.*, vol. 68, pp. 1072–1078, August 1978. This paper presents an extension and application of Horwitz's original asymptotic analysis of unstable resonators. I believe this is one of the first publications, if not the first, that points out the physical interpretation of that analysis in terms of "virtual sources" or edge waves seen looking back into the unstable resonator or lensguide.

W. H. Southwell, "Asymptotic solution of the Huygens-Fresnel integral in circular coordinates" *Opt. Lett.*, vol. 3, pp. 100-102, September 1978. Early (is this actually the earliest?) discussion of the virtual source or edge wave approach to unstable resonator analysis.

W. H. Southwell, "Virtual-source theory of unstable resonator modes" *Opt. Lett.*, vol. 6, pp. 487–489, October 1981. Another early exposition of the edge-wave analysis of unstable resonators.

W. H. Southwell, "Unstable-resonator-mode derivation using virtual-source theory" *J. Opt. Soc. Am. A*, vol. 3, pp. 1885–1891, November 1986. Another Southwell article on virtual source analysis.

A. E. Siegman, "New developments in laser resonators (invited paper)" in *Optical Resonators: Proc. SPIE*, vol. 1224, January 1990. Invited talk which described the symmetric-lensguide virtual source model for unstable resonators, the PARAXIA software package, the M-squared beam propagation factor definition, and ray-pulse matrices.

Y. Q. Li and C. C. Sung, "Extended asymptotic theory of unstable resonator modes" *Appl. Opt.*, vol. 29, pp. 4462–4467, 20 October 1990. Apparently an extension of the Horwitz papers. I need to go back and reexamine this one.

J.-l. Doumont, *Laser Beam and Resonator Calculations on Desktop Computers*, Ph.D. dissertation, Department of Applied Physics, Stanford University, June 1991. Stanford University, Describes the PARAXIA software package which Jean-luc developed and programmed. Contains background material (sometimes rather brief) for most of the PARAXIA programs including the FFT and FHT (Fast Hankel Transform) approaches to diffraction used in the FRESNEL program, and the symmetric-lensguide virtual source approach used in the calculation of unstable resonator modes. Includes some results for ENF values for VRM and hard-edged unstable resonators. Also presents extensive formulas for normalization and overlap integrals of complex hermit-gaussian functions.

J.-l. Doumont and A. E. Siegman, “Laser beam and resonator calculations on the Macintosh,” presented at First Annual Conference on Scientific and Engineering Applications of the Macintosh (SEAM ’92),, San Francisco CA, January 1992. Describes the PARAXIA package, part of which is based on the virtual source theory of unstasble resonators..

A. R. Larson, “Unstable resonator modes for lasers with circular mirrors and high Fresnel numbers” *Appl. Opt.*, vol. 32, pp. 5872–5884, 20 October 1993. Compares results of what he claims to be numerical calculations, Horowitz approach, Butts-Avizonis approach, and virtual source approach for circular confocal unstable resonator calculations. So far as I can se, the last three of these are all the same.

P. Mussche, *Excess Quantum Linewidth in Lasers with Nonorthogonal Eigenmodes*, Ph.D. dissertation, Department of Electrical Engineering, Stanford University, December 1994. Stanford University, Paul Mussche’s dissertation in my group. These results and Yuh-Jen Cheng’s subsequent results are among the first definitive experimental confirmations of the ENF for transverse modes in unstable resonator lasers. I believe Paul also put some still unpublished results on the virtual source theory of unstable resonators in this dissertation, but I need to go back and check the text further.

G. H. C. New, “The origin of excess noise” *J. Mod. Opt.*, vol. 42, pp. 799–810, April 1995. This is basically a pretty good review/summary article by Geoff New, giving a slightly different viewpoint on how the excess noise factor should be written and interpreted, and also giving some additional numerical results for the confocal unstable resonator case. Gives a numerical comparison of ENF and excess initial wave excitation. Points out the utility in virtual source calculations of using a few additional Fox-and-Li type round trips after the virtual source calculation, to clean up the calculated wavefronts.

M. V. Berry, C. Storm, and W. van Saarloos, “Fractal (?) lasers” *Opt. Commun.*,, pp. submitted for publication, 2001. Interesting extension of the