Unusual Properties of Modes in Nonnormal Optical Systems

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Modes in Normal and Nonnormal Systems

- Some common optical systems have “normal modes” in the usual sense
- The modes of other common optical systems are not normal modes as usually understood
- The modes of these nonnormal (or nonhermitian) systems are not orthogonal to each other
- Leads to major changes in the mathematical, physical and quantum properties of systems having nonnormal modes

Optical Systems with Normal Modes

Microwave and optical systems that have normal propagating or resonant modes include:

- Closed metal structures (such as microwave waveguides and microwave cavities)
- Dielectric resonators
- Index-guided fibers and optical waveguides
- Idealized stable optical resonators and stable lensguides

Systems with Not So Normal Modes

Optical systems that have distinctly <u>nonnormal</u> modes include:

- Loss-guided and gain-guided ducts
- Gain-guided semiconductor lasers
- Open-sided optical resonators of all types
- Unstable optical resonators
- Birefringent systems having optical "twist"

Unusual Properties of Nonnormal Systems

1. Total power or energy no longer given by the sum of powers in individual modes.

2. Second quantization lost; basic concept of “photons” seriously muddied.

3. Matched coupling replaced by adjoint coupling: more power into one mode than total power in whole system .

4. Major changes in eigenmode expansion procedures

5. Excess quantum noise, potentially large increase in Schawlow-Townes linewidth

Mode Equations

- Modes of optical systems are eigensolutions of linear equations:

  1) Wave equation for propagating modes in optical waveguides

  $$\left[\nabla_x^2 + k^2(x)\right] \tilde{u}_n(x) = \beta_n^2\, \tilde{u}_n(x)$$

  2) Integral equation for resonant modes in optical cavities

  $$\int K(x, x')\, \tilde{u}_n(x')\, dx' = \tilde{\gamma}_n\, \tilde{u}_n(x)$$

Operator Formulation

- These equations can be written in a generalized operator formalism:

$$\mathcal{L}\,\tilde{u}_n(x) = \tilde{\gamma}_n\,\tilde{u}_n(x)$$

- Eigenfunctions must also satisfy boundary conditions
- Operators for many physical systems are <u>hermitian</u>

$$\mathcal{L} \equiv \mathcal{L}^\dagger \equiv (\mathcal{L}^T)^*$$

$$\mathcal{L}^\dagger \equiv \text{hermitian conjugate or adjoint}$$

$$\mathcal{L}^T \equiv \text{transposition of variables}$$

$$\mathcal{L}^* \equiv \text{ordinary complex conjugation}$$

Hermitian Operators

- Hermitian operators always have eigenfunctions (normal modes) $\tilde{u}_n$ such that

$$\mathcal{L}\,\tilde{u}_n(x) = \tilde{\gamma}_n\,\tilde{u}_n(x)$$

- These eigenfunctions are orthogonal

$$\int \tilde{u}_n^*(x)\tilde{u}_m(x)\,dx = \delta_{nm}$$

- and form a complete basis set such that

$$\tilde{f}(x) = \sum_n \tilde{c}_n \tilde{u}_n(x)$$

  for any possible state $\tilde{f}(x)$ of the system

Nonhermitian (Nonnormal) Operators

- Nonhermitian operators ($\mathcal{L} \neq \mathcal{L}^\dagger$) not guaranteed to even have eigensolutions

$$\mathcal{L}\,\tilde{u}_n(x) \overset{?}{=} \tilde{\gamma}_n\,\tilde{u}_n(x)$$

- Eigenfunctions, if they exist, are not orthogonal

$$\int \tilde{u}_n^*(x)\tilde{u}_m(x)\,dx \neq 0$$

- If nonhermitian operator $\mathcal{L}$ does have eigenfunctions $\tilde{u}_n$, adjoint operator $\mathcal{L}^\dagger$ will have matching set of adjoint functions $\tilde{v}_n$

$$\left.\begin{aligned} \mathcal{L}\,\tilde{u}_n(x) &= \tilde{\gamma}_n\,\tilde{u}_n(x) \\ \mathcal{L}^\dagger\,\tilde{v}_n(x) &= \tilde{\gamma}_n^*\,\tilde{v}_n(x) \end{aligned}\right\} \quad \left\{\begin{array}{l} \text{eigenmodes of } \mathcal{L}\text{, and} \\ \text{adjoint modes of } \mathcal{L}^\dagger, \\ \text{with same } \tilde{\gamma}_n \text{ values} \end{array}\right\}$$

Biorthogonality

- Eigenmodes $\tilde{u}_n$ and adjoint modes $\tilde{v}_n$ will be biorthogonal

$$\int_{-\infty}^{\infty} \tilde{v}_n^*(x)\, \tilde{u}_m(x)\, dx \equiv \delta_{nm}\,, \qquad (\delta_{nn} = 1,\ \delta_{nm} = 0)$$

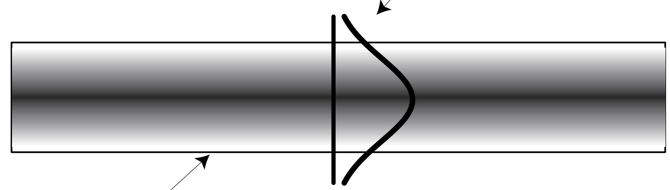
- Eigenmodes are nonorthogonal, but normalizable:

$$M_{nm} \equiv \int_{-\infty}^{\infty} \tilde{u}_n^*(x)\, \tilde{u}_m(x)\, dx\,, \qquad (M_{nn} = 1,\ M_{nm} \neq 0)$$

- Adjoint modes also nonorthogonal, with normalization$>$1

$$K_{nm} \equiv \int_{-\infty}^{\infty} v_n^*(x)\, \tilde{v}_m(x)\, dx\,, \qquad (K_{nn} > 1,\ K_{nm} \neq 0)$$

Example #1: Gaussian Gain Guided Duct



- Assume index guiding plus gain or loss guiding

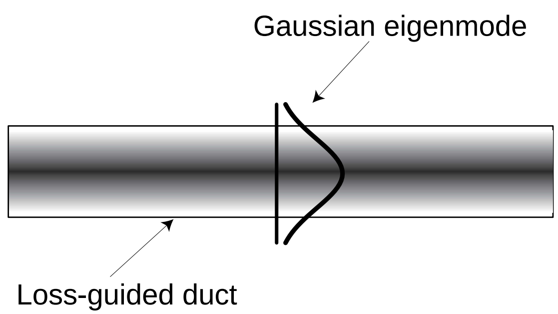
$$n(x) = n_0 - \frac{n_2 x^2}{2}\,, \qquad g(x) = g_0 - \frac{g_2 x^2}{2}$$

- Eigenmodes and adjoint modes are Hermite-Gaussians

$$\tilde{u}_n(x) = \tilde{u}_{n0}\, H_n(\tilde{a}x)\, \exp[-\tilde{a}^2 x^2/2]$$

$$\tilde{v}_n(x) = \tilde{v}_{n0} H_n(\tilde{a}^* x) \exp[-\tilde{a}^{*2} x^2/2]$$

Complex-Valued Hermite Gaussians



- Hermite-Gaussian eigenmodes

$$\tilde{u}_n(x) = \tilde{u}_{n0}\, H_n(\tilde{a}x)\, \exp[-\tilde{a}^2 x^2/2]$$

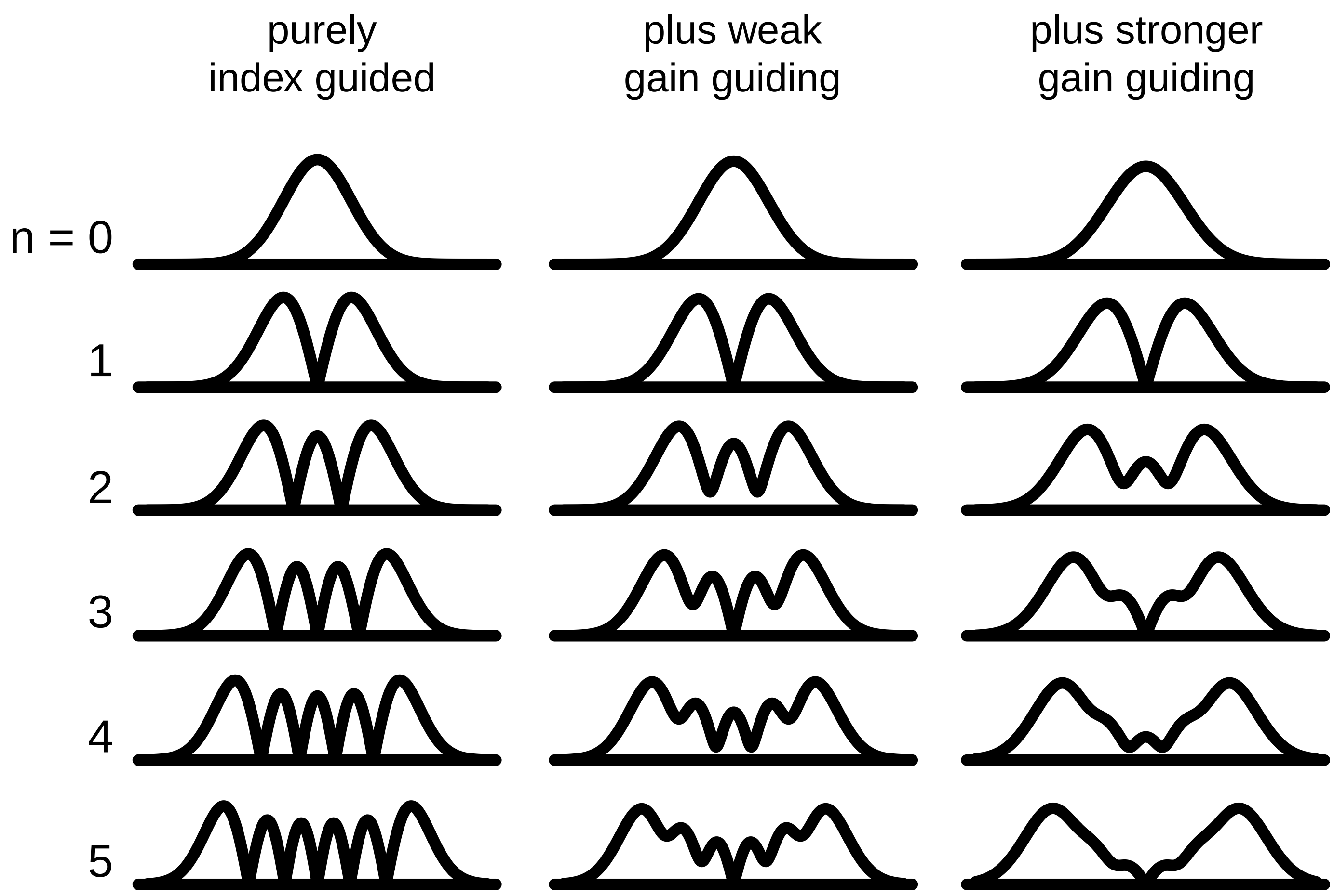
- have complex-valued scale factor

$$\tilde{a} = \left(\frac{2\pi}{\lambda_0}\right)^{1/2} \left(n_0 n_2 + j\, \frac{\lambda_0}{2\pi} g_2\right)^{1/4} = |\tilde{a}|\; e^{j\theta}$$

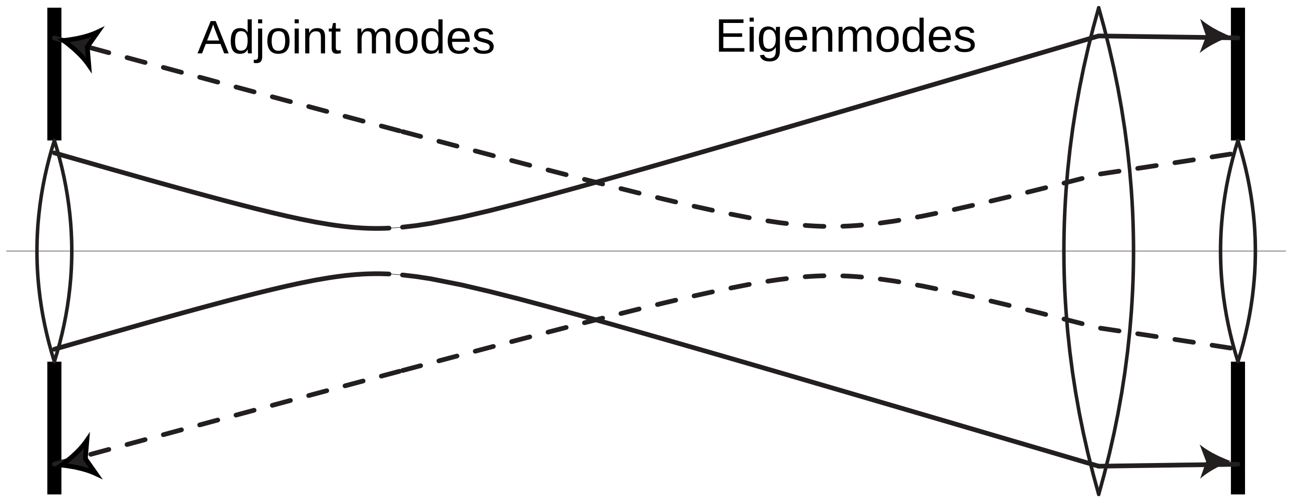
- with angle $0 < \theta < 45$ degrees for positive gain guiding $(g_2 > 0)$

Complex HG Mode Profiles

- Intensity profiles of first 6 lowest-order HG modes with increasing amounts of gain guiding

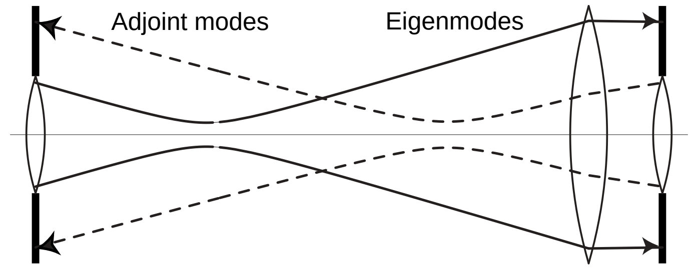


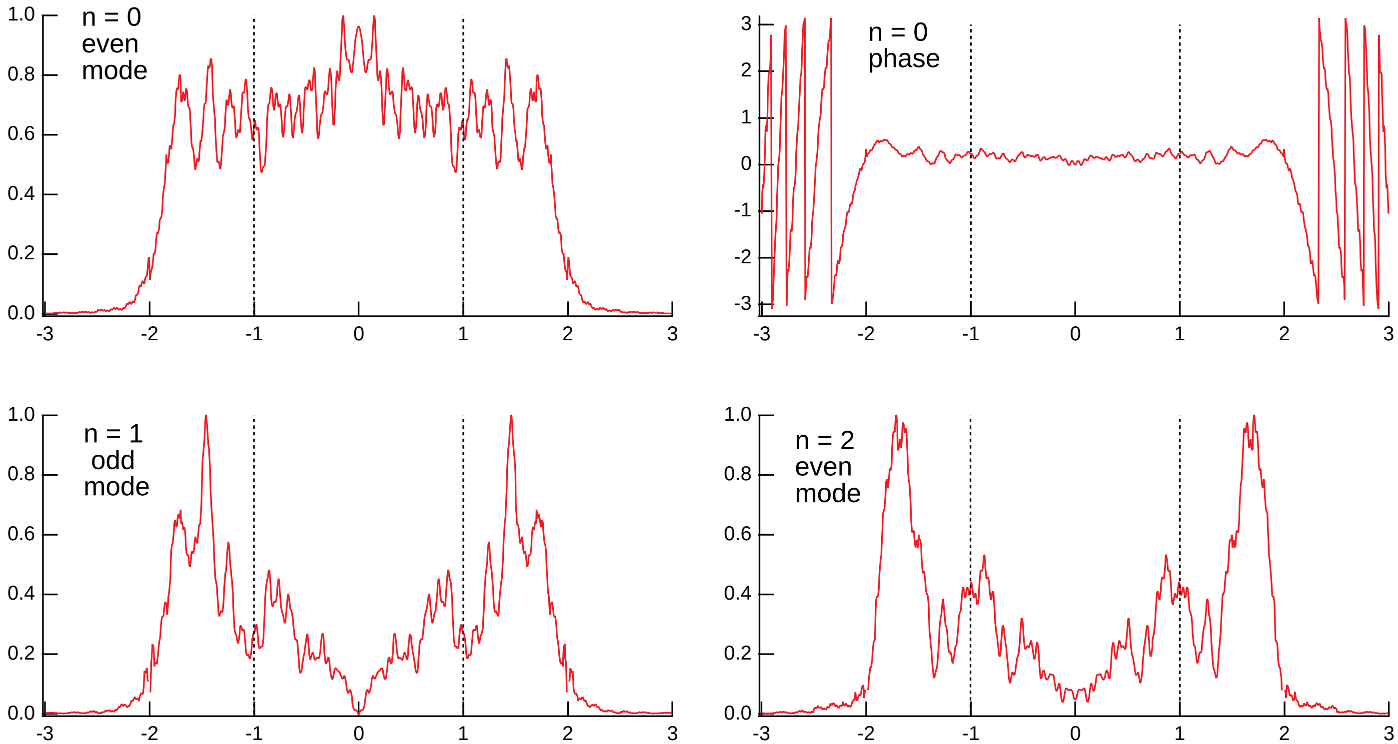
Example #2: Unstable Resonator Modes



- Right-going eigenmodes (solid lines) and corresponding adjoint modes (dashed lines) in an unstable lensguide
- Adjoint modes corresponding to right-going eigenmodes have the same form as reversed left-going eigenmodes

Exact Unstable-Resonator Mode Profiles





Why Are These Systems Nonnormal?

- Gain-guided systems

$$\left[\nabla_x^2 + \tilde{k}^2(x)\right] \tilde{u}_n(x) = \beta_n^2 \, \tilde{u}_n(x)$$

  - Gain guiding makes wave vector $\tilde{k}$ complex valued
  - Wave equation operator is no longer hermitian

- Open-sided optical resonators:

$$\int K(x, x') \, \tilde{u}_n(x') \, dx' = \tilde{\gamma}_n \, \tilde{u}_n(x)$$

  - Wave equation remains fully hermitian
  - But boundary conditions at ∞ are not hermitian
  - Huygens integral operator becomes nonhermitian

Unusual Properties of Nonnormal Modes

1) Total energy $\neq$ sum of mode energies
2) Second quantization lost; photon concept muddied
3) Matched coupling vs. adjoint coupling
4) Changes in eigenmode expansion procedure
5) Excess quantum noise leads to potentially large increase in Schawlow-Townes linewidth of laser oscillators

1) Energies Per Mode

- Expand field in nonnormal eigenmodes, evaluate total energy

$$\mathcal{E}(x) = \sum_{n=0}^{N} \tilde{c}_n \, \tilde{u}_n(x)$$

$$\begin{aligned}
\text{Energy} &= \int_{-\infty}^{\infty} |\mathcal{E}(x)|^2 \, dx \\
&= \sum_{n=0}^{N} |\tilde{c}_n|^2 \; + \; \sum_{n \neq m} \tilde{c}_n^* c_m M_{nm} \\
&= \sum_{n} \text{Energies/mode} + \sum_{n \neq m} \text{``cross-mode terms''}
\end{aligned}$$

- Energies per mode can be greater than <u>total</u> energy; cross-mode terms can be negative.

2) "Lasers Without Photons"?

- Classical energy in laser cavity fields

$$\int_{-\infty}^{\infty} |\mathcal{E}(x)|^2 \, dx = \sum_{n=0}^{N} |\tilde{c}_n|^2 \; + \; \sum_{n \neq m} \tilde{c}_n^* c_m M_{nm}$$

- transforms into quantum Hamiltonian (coefficients $\tilde{c}_n$ become quantum operators $\mathbf{a}_n$)

$$\mathcal{H} = \sum_{n=0}^{N} \mathbf{a}_n^\dagger \mathbf{a}_n \, \hbar\omega_{qn} + \sum_{n \neq m} \mathbf{a}_n^\dagger \mathbf{a}_m M_{nm} \, \hbar\sqrt{\omega_{qn}\omega_{qm}} \; .$$

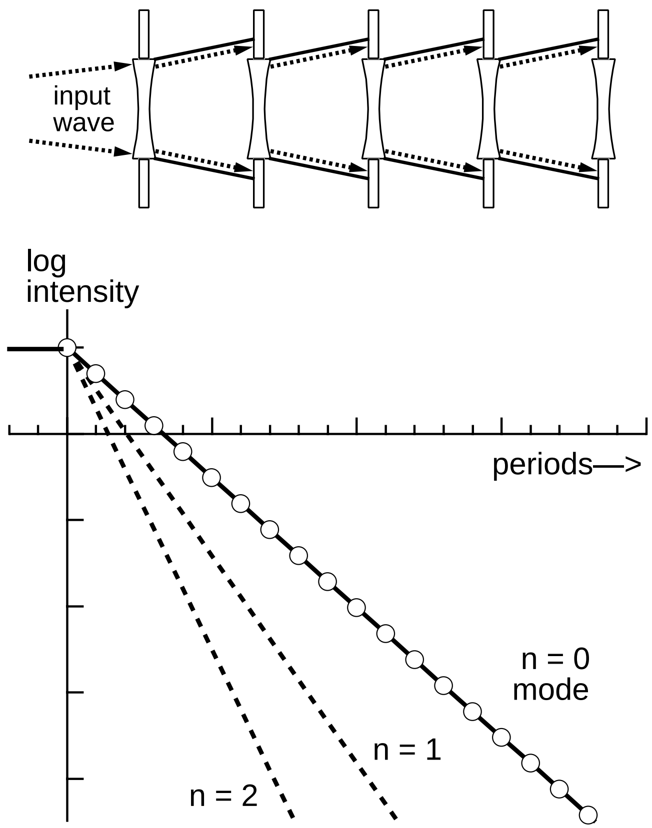
- With normal modes each $\mathbf{a}_n^\dagger \mathbf{a}_n$ term becomes a quantized simple harmonic oscillator (second quantization)
- With nonnormal modes second quantization no longer possible because of cross-modal terms

3) Matched vs. Adjoint Coupling

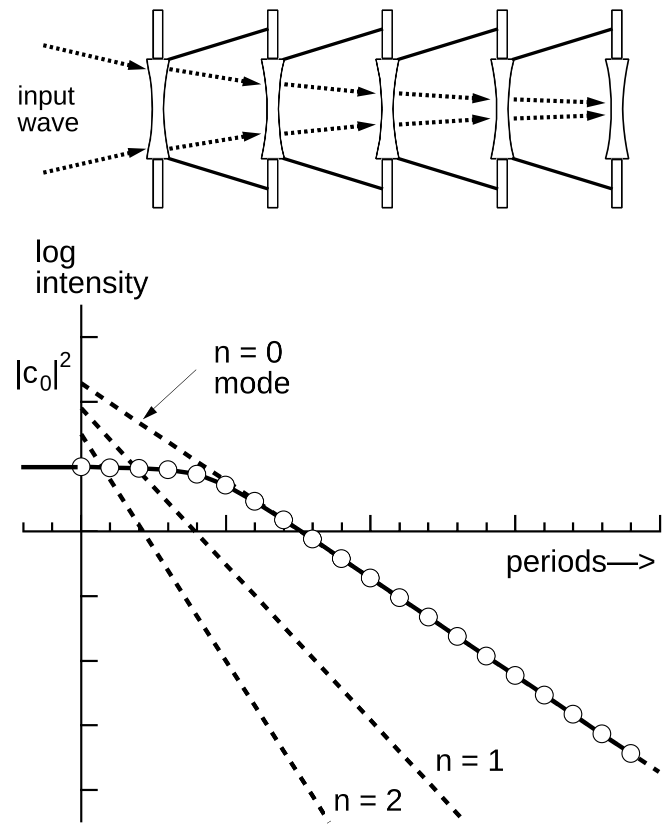
- Matched coupling
  - Input wavefront matched to one selected eigenmode of lensguide or cavity
  - Delivers all the energy in the input beam into that one selected mode
- Adjoint Coupling
  - Injected wavefront matched to selected adjoint mode
  - Excites the corresponding eigenmode with greater than unity coupling
  - Excess coupling factor $= K_{nn}$
  - Simultaneously excites multiple other eigenmodes so that cross-power terms conserve energy

Adjoint Coupling

Matched coupling



Adjoint coupling

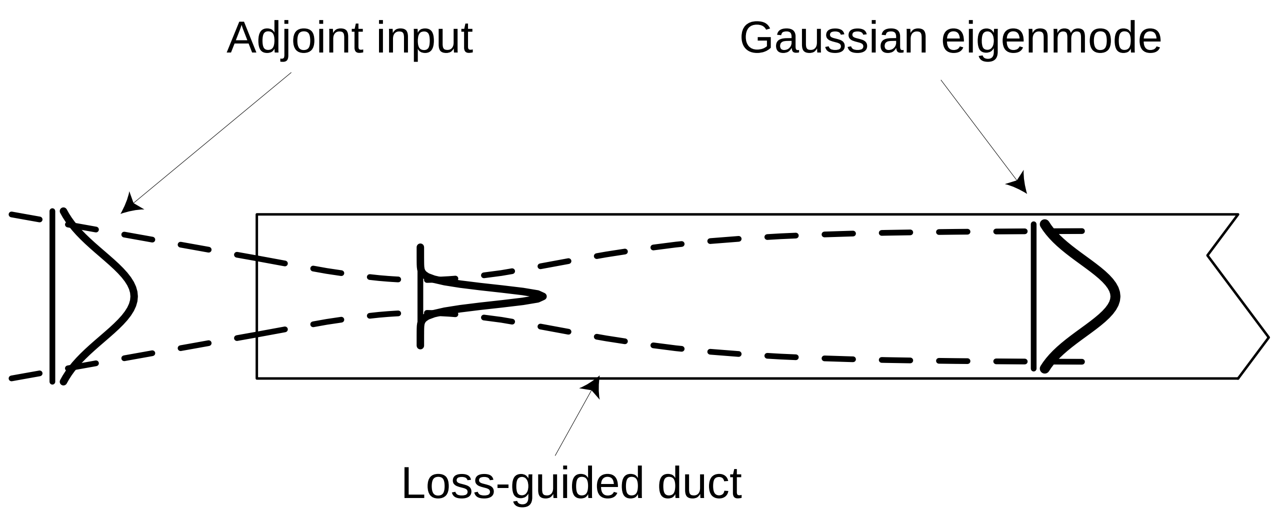


4) Expansion in Nonnormal Eigenmodes

- Can fields of nonnormal optical systems still be expanded in superposition of nonnormal eigenmodes?

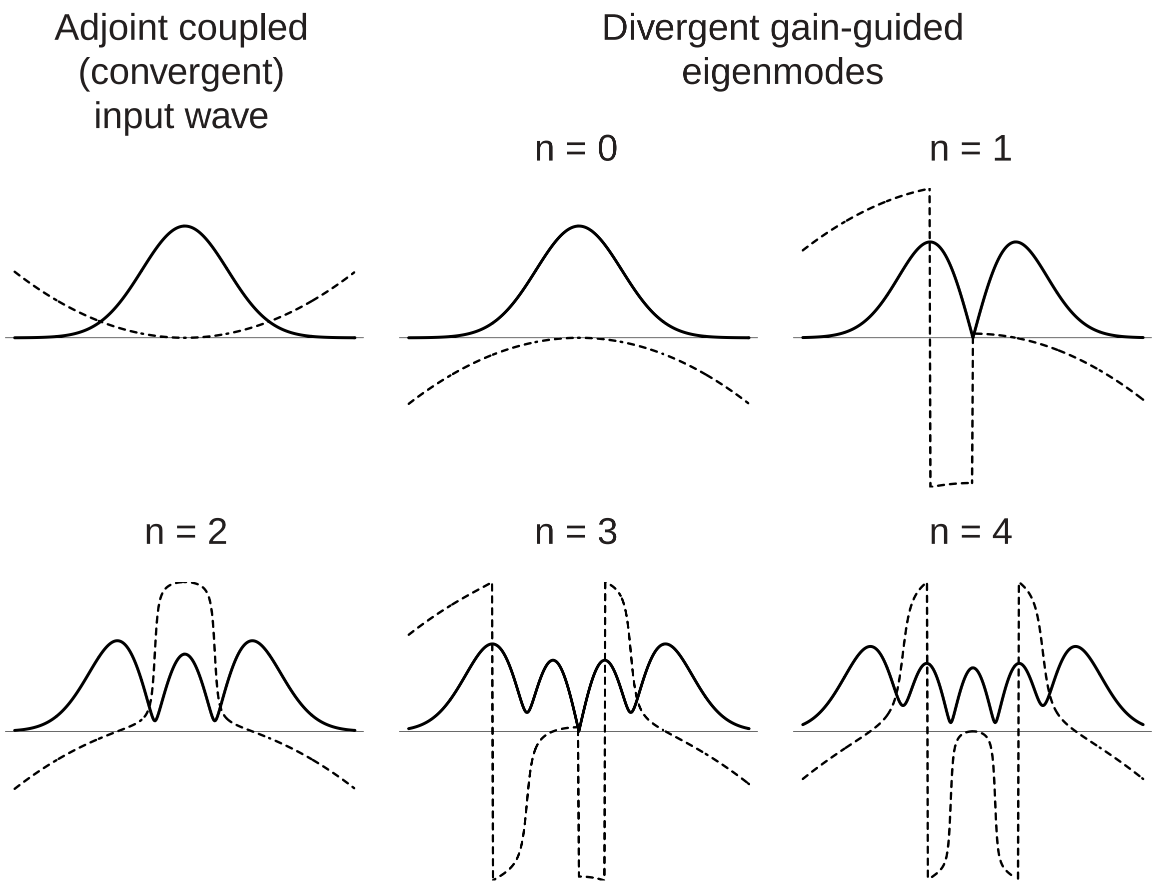
$$\tilde{f}(x) = \sum_{n=0}^{\infty} \tilde{c}_n \, \tilde{u}_n(x)$$

- Test example: eigenmode expansion of adjoint coupling into HG modes of loss–guided duct



Expansion in Nonnormal Eigenmodes

- Problem is to expand converging TEM_{00} gaussian adjoint wave in terms of diverging complex -HG eigenmodes



Quadrature Expansion

- How to find coefficients $\tilde{c}_n$ in the expansion?

$$\tilde{f}(x) = \sum_{n=0}^{\infty} \tilde{c}_n \, \tilde{u}_n(x)$$

- Quadrature method: multiply both sides by $\tilde{v}_n^*(x)$ and use biorthogonality relation

$$\int \tilde{v}_n^*(x) \tilde{u}_m(x) \, dx = 0$$

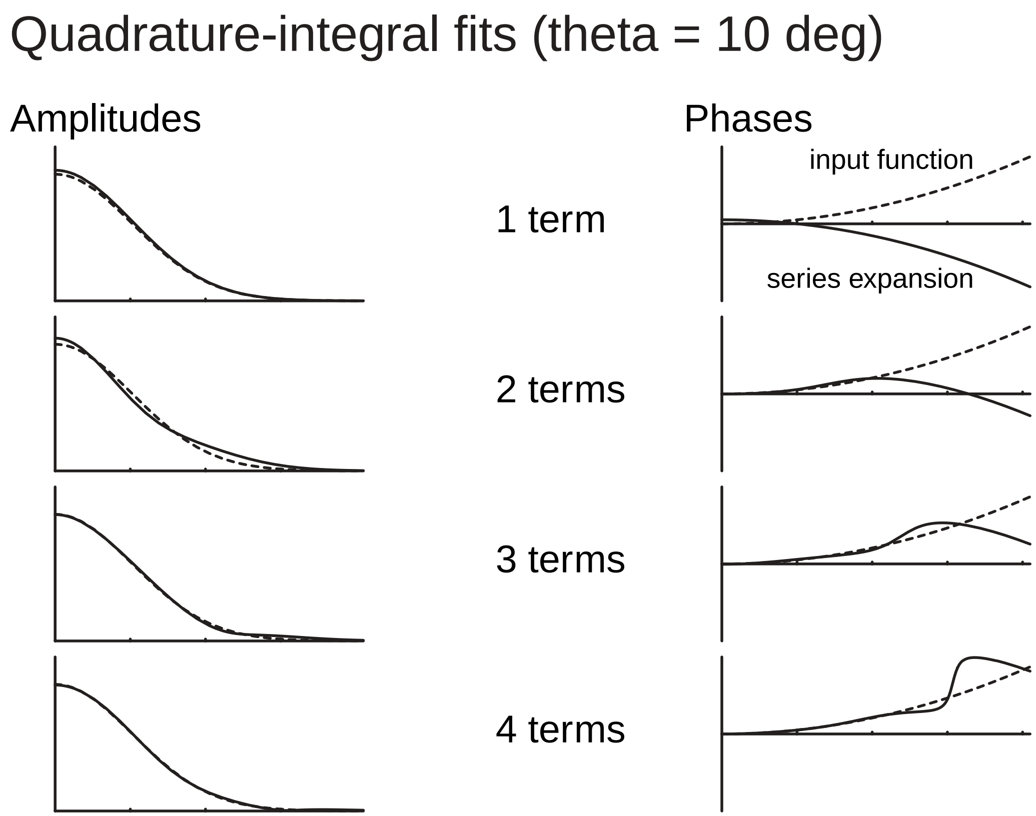
- This gives "quadrature coefficients"

$$\tilde{c}_n = \int_{-\infty}^{\infty} \tilde{v}_n^*(x) \, \tilde{f}(x) \, dx$$

$$= \text{“Quadrature coefficients”}$$

Quadrature Fit (Weak Gain Guiding)

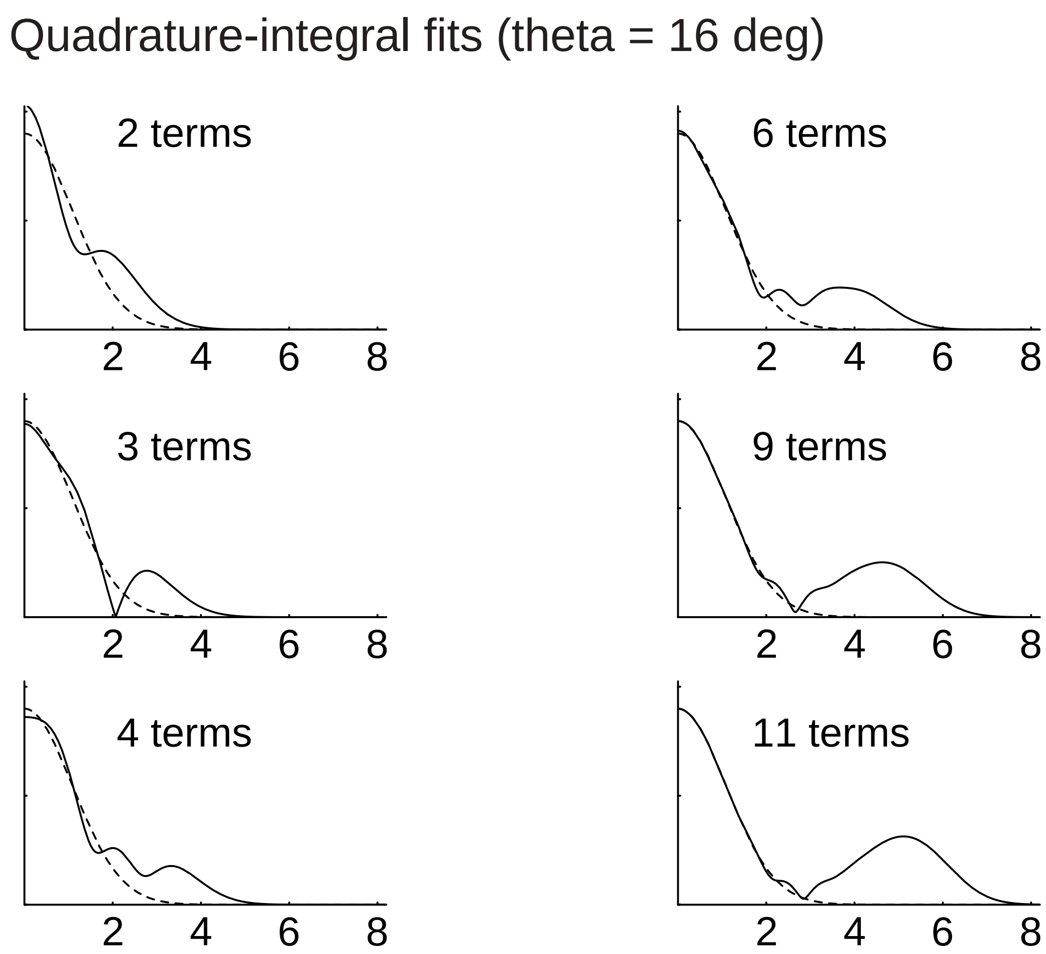
- Trial expansion using quadrature coefficients for weakly gain-guided example:



- Converges (slowly) with increasing number of terms

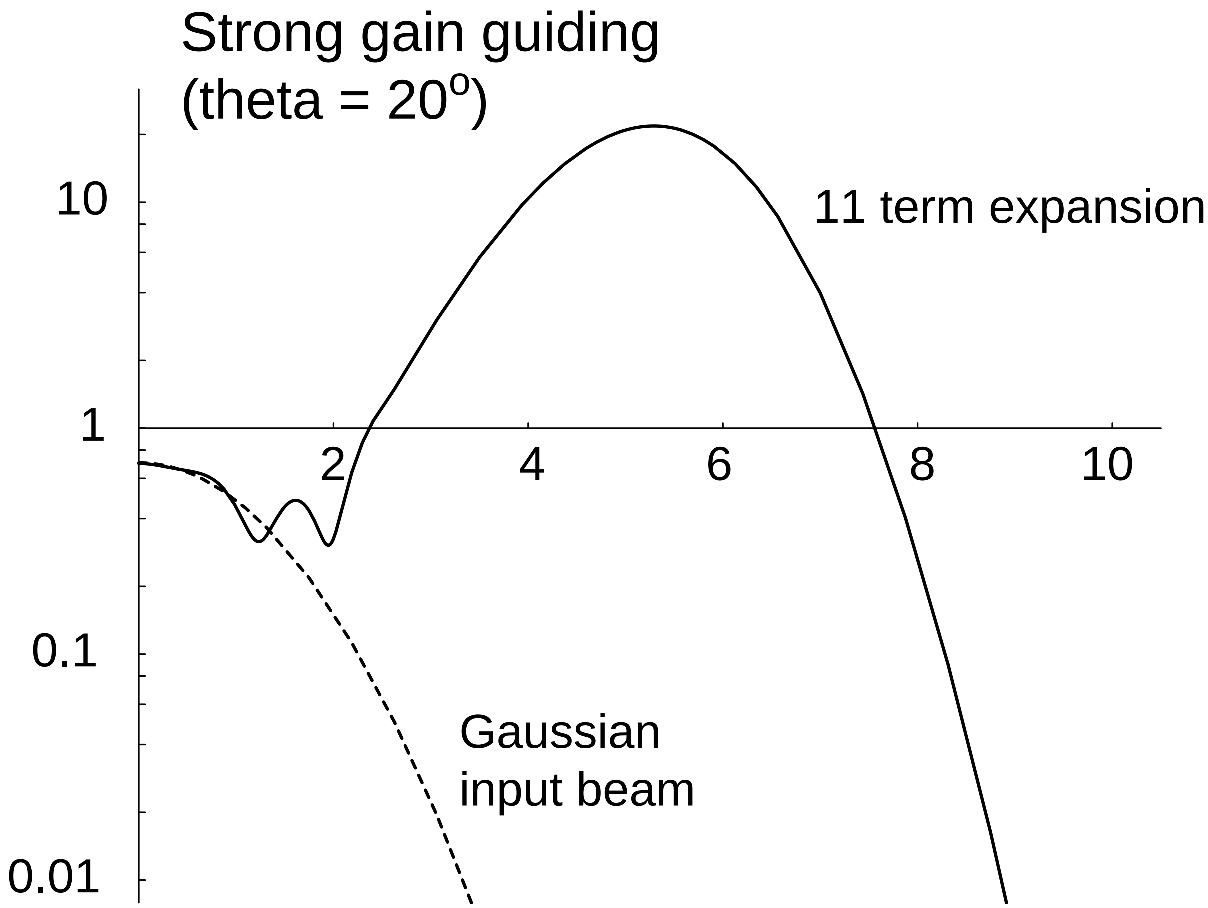
Quadrature Fit (Stronger Gain Guiding)

- Similar expansion using quadrature coefficients <u>diverges</u> for slightly stronger gain guiding



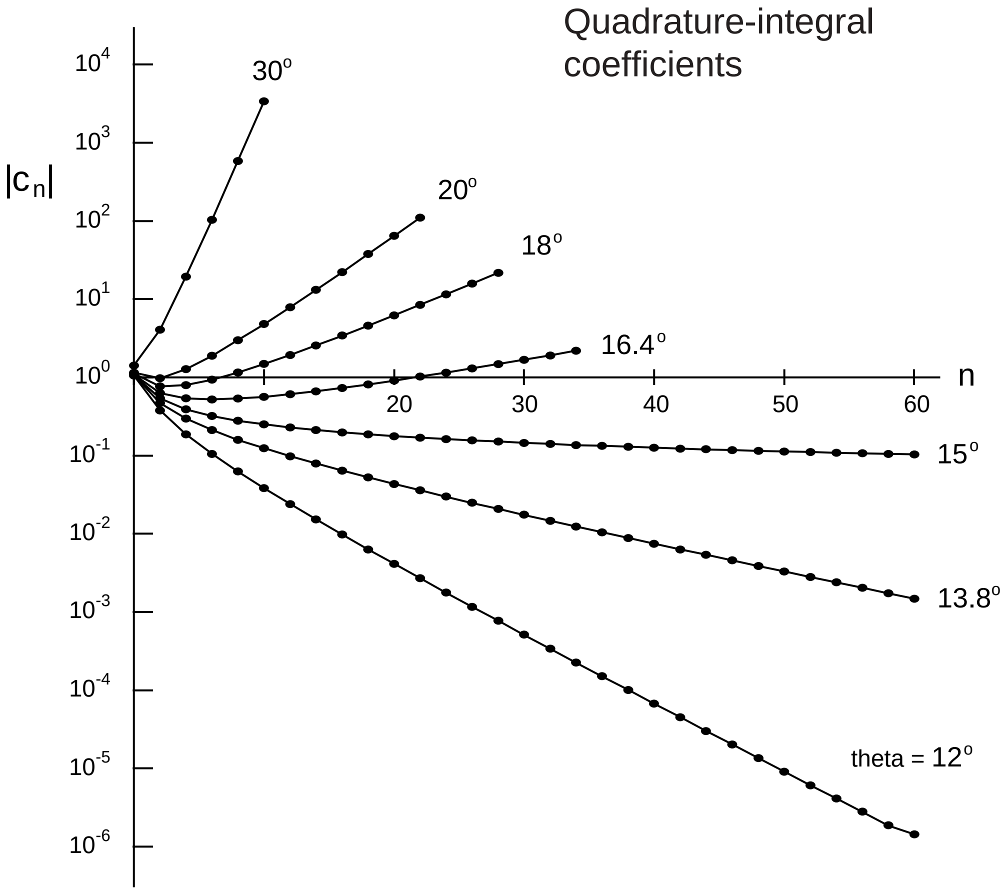
Quadrature Fit (Strong Gain Guiding)

- Quadrature expansions diverge wildly for still stronger gain guiding



Quadrature Expansion Coefficients

- Quadrature coefficients themselves diverge for gain guiding stronger than a threshold value



Minimum Error Expansions

- Is there a better way? Try writing mean-square error for a finite N-term eigenmode expansion as

$$\epsilon_N = \int_{-\infty}^{\infty} \left| \tilde{f}(x) - \sum_n \tilde{c}_n \tilde{u}_n(x) \right|^2 dx$$
$$= 1 - \sum_n \tilde{c}_n^* f_n - \sum_n \tilde{c}_n f_n^* + \sum_n \sum_m \tilde{c}_n^* \tilde{c}_m M_{nm}$$

where

$$f_n \equiv \int_{-\infty}^{\infty} u_n^*(x)\, \tilde{f}(x)\, dx$$

- Note that f_n's are usual (normal) eigenexpansion coefficients, but M_{nm} is nondiagonal matrix

Minimum Error Expansions (2)

- Define two vectors

$$\mathbf{c}_N \equiv \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \vdots \\ \tilde{c}_N \end{bmatrix} \qquad \text{and} \qquad \mathbf{f}_N \equiv \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_N \end{bmatrix}$$

- Minimum-error expression is converted into matrix form

$$\begin{aligned} \epsilon_N &= 1 - \sum_n \tilde{c}_n^* f_n - \sum_n \tilde{c}_n f_n^* + \sum_n \sum_m \tilde{c}_n^* \tilde{c}_m M_{nm} \\ &= 1 - \mathbf{c}_N^\dagger \mathbf{f}_N - \mathbf{f}_N^\dagger \mathbf{c}_N + \mathbf{c}_N^\dagger \mathbf{M}_N \mathbf{c}_N \end{aligned}$$

Minimum Error Expansions (3)

- Minimizing the error function

$$\delta\epsilon_N = [\mathbf{M}_N\,\mathbf{c}_N - \mathbf{f}_N]^\dagger\;\delta\mathbf{c}_N + \delta\mathbf{c}_N^\dagger\;[\mathbf{M}_N\,\mathbf{c}_N - \mathbf{f}_N] = 0$$

- gives matrix expressions for the minimum-error coefficients:

$$\mathbf{c}_N = \mathbf{M}_N^{-1}\,\mathbf{f}_N \qquad \text{(minimum-error coefficients)}$$

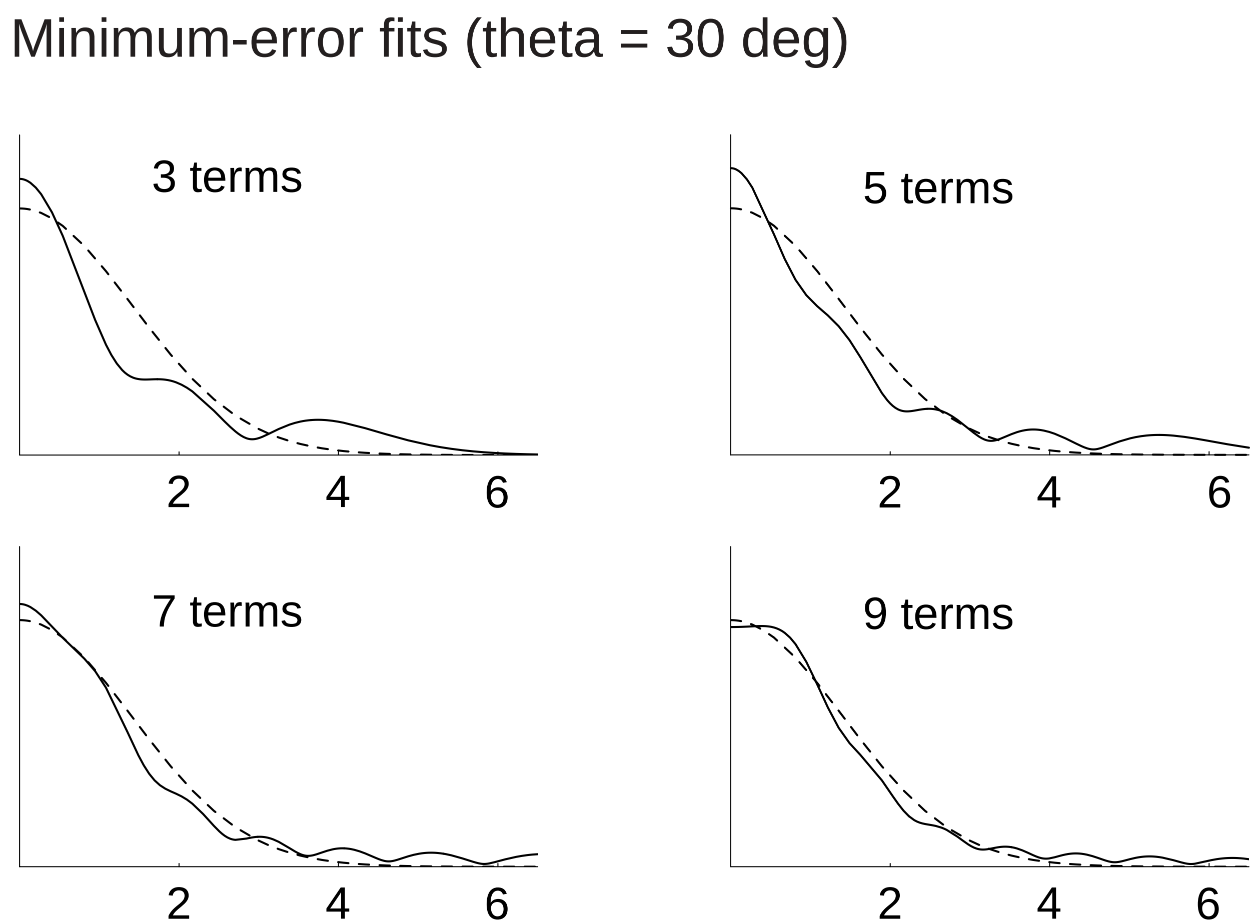
- and for the rms error itself

$$\epsilon_N = 1 - \mathbf{f}_N^\dagger\,\mathbf{M}_N^{-1}\,\mathbf{f}_N \qquad \text{(minimum error value)}$$

- Note that minimum-error coefficients depend on truncation index N as well as coefficient index n

Minimum-Error Example (Strong Gain Guiding)

- Minimum-error expansions always converge, even for very strong gain guiding



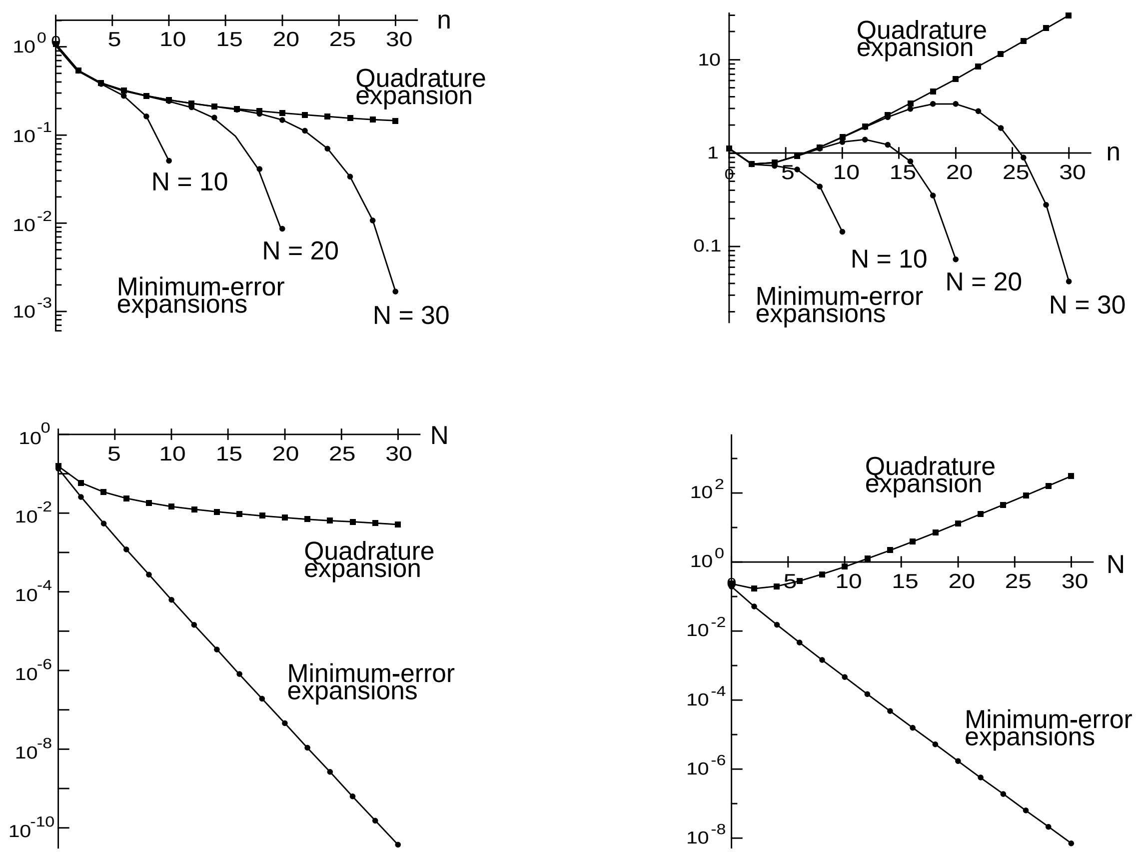
Quadrature vs. Minimum-Error

theta = 15 deg (quadrature expansion converges)

theta = 18 deg (quadrature expansion diverges)

Expansion coefficients

Expansion errors



5) Quantum Noise in Conventional Lasers

- Spontaneous emission rate in conventional lasers:
  "one noise photon per mode"

$$\frac{dn}{dt} = \kappa\,(n+1)\,N_2 \;-\; \kappa\,n\,N_1$$

- Resulting Schawlow-Townes linewidth for lasers
  having normal cavity modes

$$\Delta f_L = \frac{N_2}{N_2 - N_1} \;\times\; \frac{\pi\,hf\,\Delta f_c^2}{P_{osc}}$$

Excess Quantum Noise in Nonnormal Lasers

- Spontaneous emission rate in nonnormal lasers increases to "K_p noise photons per mode"

$$\frac{dn}{dt} = \kappa\,(n + K_p)\,N_2 \;-\; \kappa\,n\,N_1$$

$$K_p = \int \tilde{v}_0^*(x)\,\tilde{v}_0(x)\,dx$$

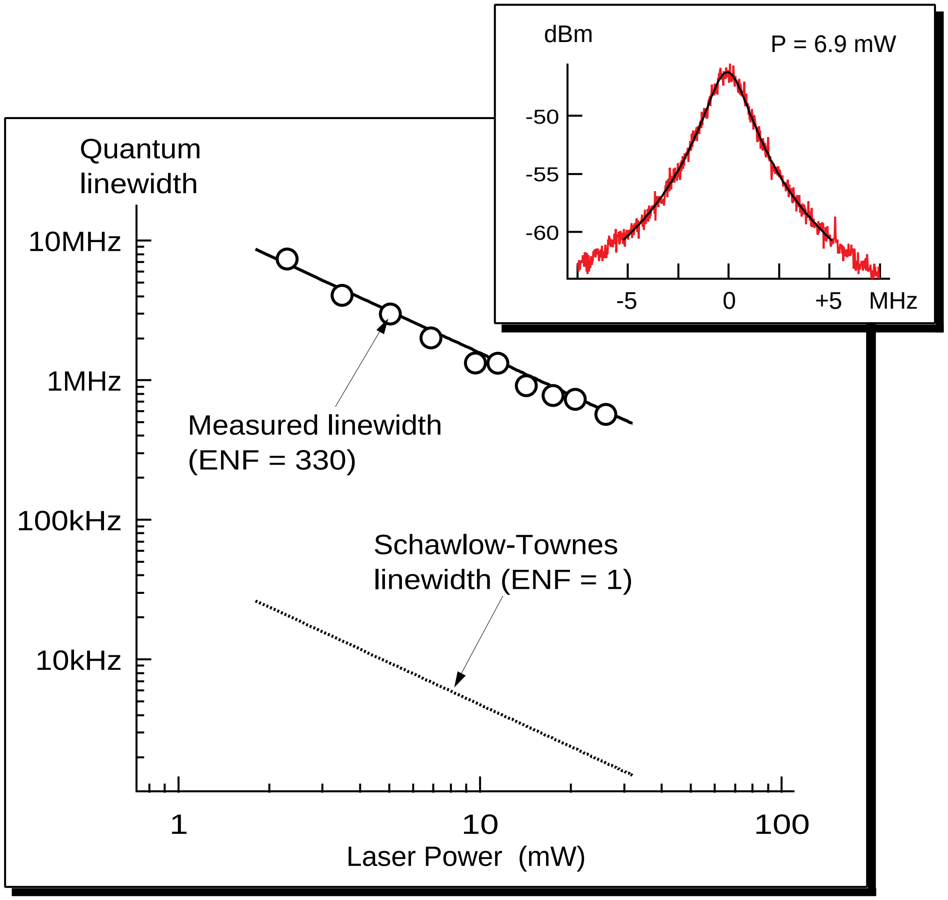
$$= \text{Petermann excess noise factor } (>1)$$

- Increased Schawlow-Townes linewidth for lasers having nonnormal cavity modes

$$\Delta f_L = K_p \times \frac{N_2}{N_2 - N_1} \times \frac{\pi\,hf\,\Delta f_c^2}{P}$$

Experimental Confirmation

- Excess noise factor $K_p > 300$ in miniature Nd:vanadate laser with hard-edged unstable resonator



Summary

- Not all physical systems (or quantum observables?) have a complete set of normal modes
- There are significant changes in the mathematical, experimental and quantum properties of nonnormal systems
- Loss of orthogonality is the driver for all of these effects
- All nonnormal systems are also lossy systems—but not all lossy systems are nonnormal systems
- The mathematics behind all this could use more attention

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