Pari-mutuel Betting on Permutations

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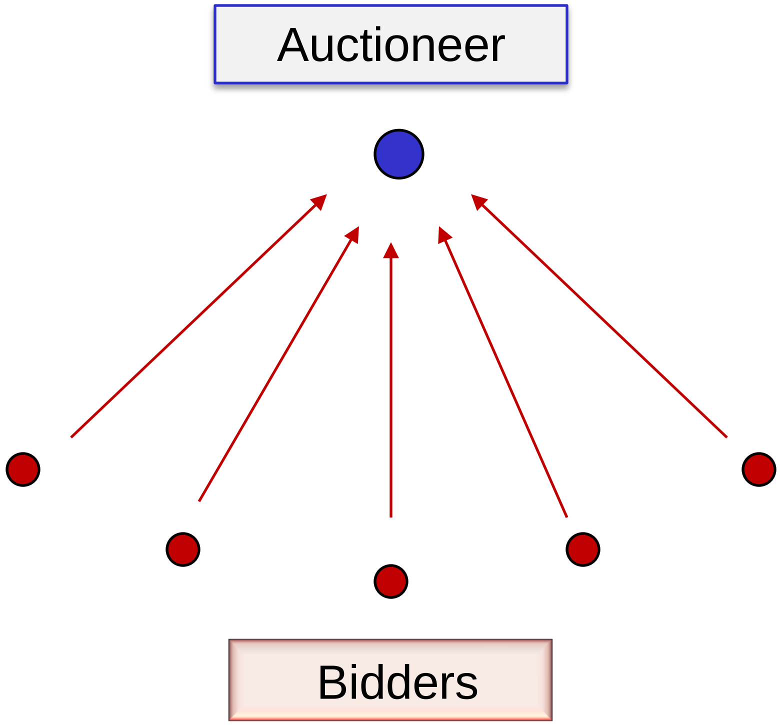
Betting as Prediction Markets

- Prediction markets
  - Primary purpose is forecasting events, or the probability that an event will occur
- Betting markets
  - Traders speculate the outcome and bet on events they consider probable
- Prices of bets reflect traders’ beliefs
  - Price of a bet is high if many traders placed similar bets (high price for low risk bets)
  - Price discovery is the end goal of the market organizer

Betting on Permutations

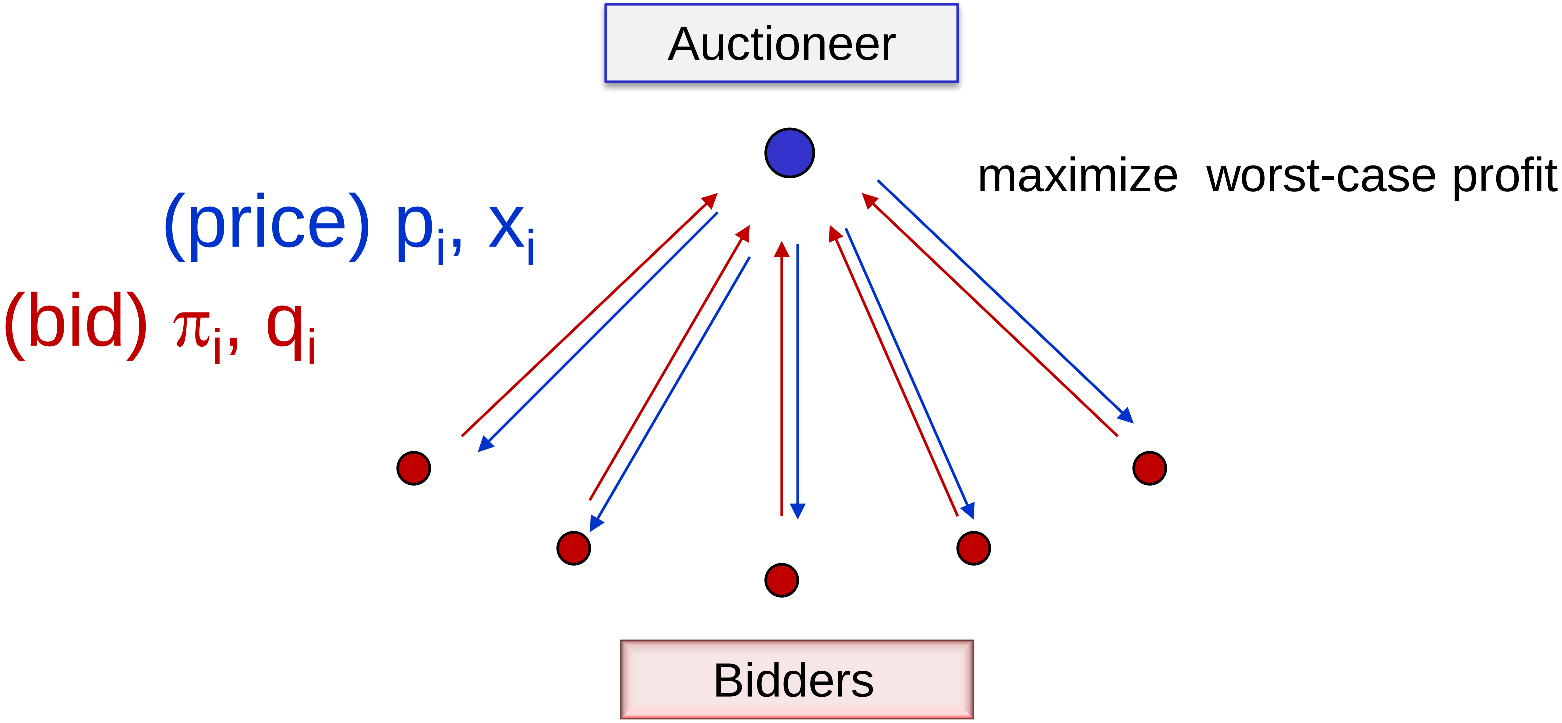
- Horse race
  - n! outcomes, $2^{n!}$ subset of events
  - Need for restrictive betting languages

Pari-mutuel call auction model



(bid) π_i, q_i

Pari-mutuel call auction model



Auctioneer

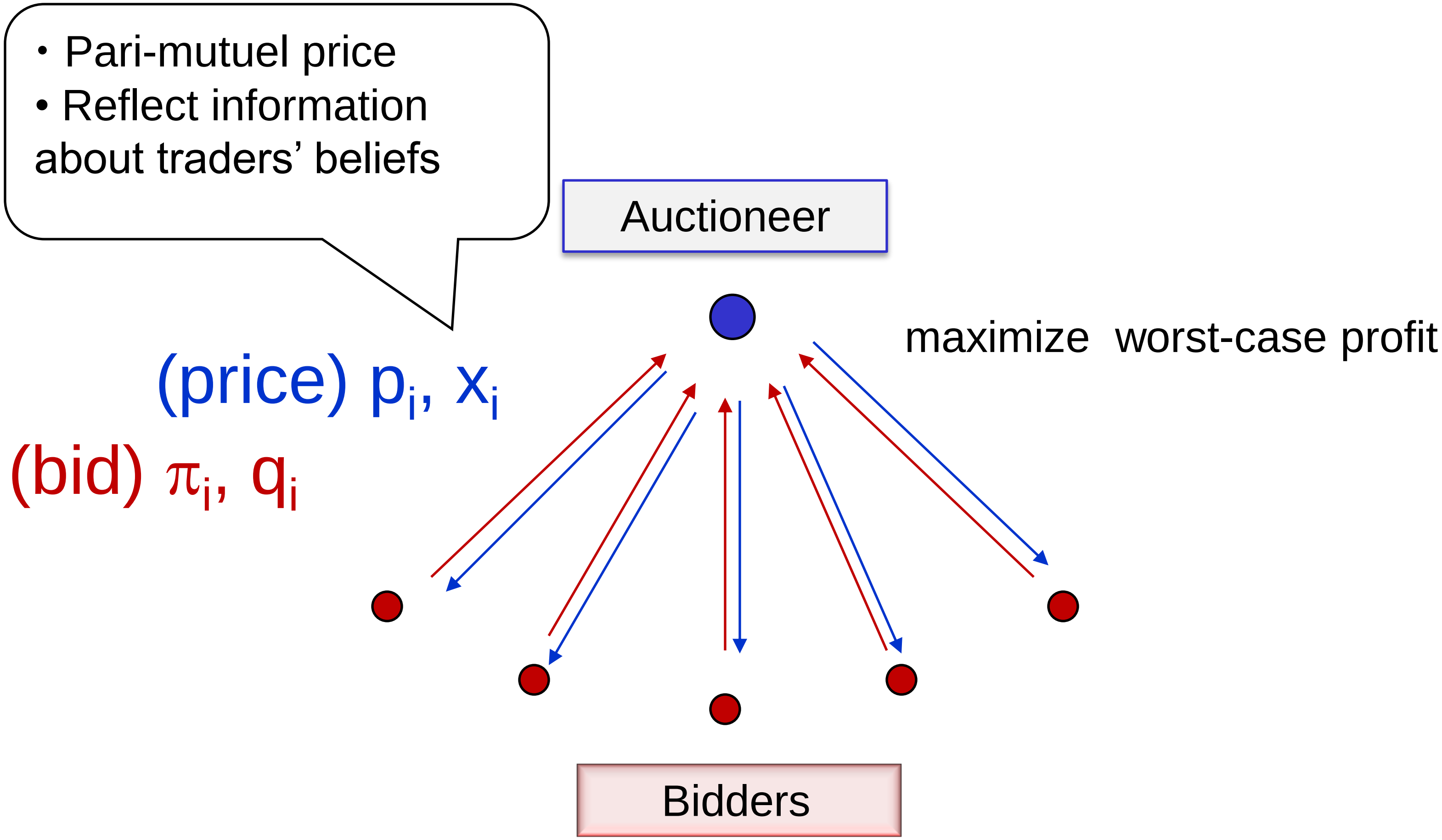
maximize worst-case profit

(price) p_i, x_i

(bid) π_i, q_i

Bidders

Pari-mutuel call auction model



- Pari-mutuel price
- Reflect information about traders' beliefs

Auctioneer

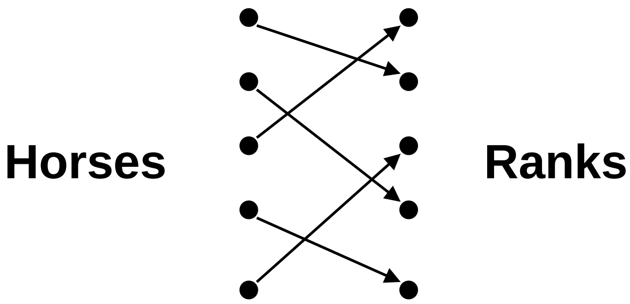
maximize worst-case profit

(price) p_i, x_i

(bid) π_i, q_i

Bidders

Permutation Betting Mechanism



Outcome

Permutation matrix

Horses

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Ranks

Bid

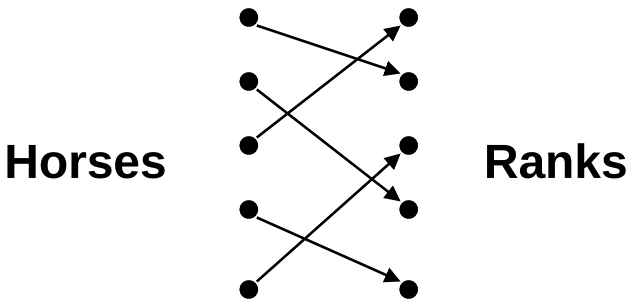
Horses

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Ranks

Permutation Betting Mechanism

Outcome



Permutation matrix

Horses

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Ranks

Bid

Horses

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

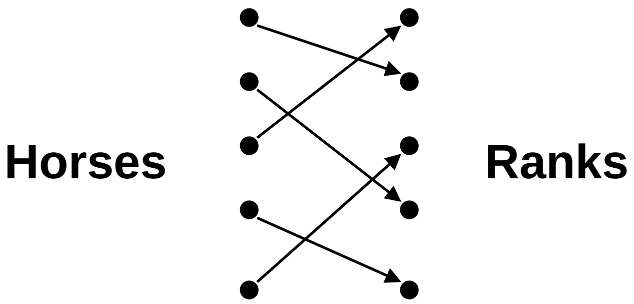
Ranks

Fixed reward Betting

Reward = $1

Harder than maximum satisfiability problem!

Permutation Betting Mechanism



Outcome

Permutation matrix

$$\begin{array}{c} \textbf{Horses} \end{array} \begin{bmatrix} 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 0 \end{bmatrix}$$

Ranks

Bid

$$\begin{array}{c} \textbf{Horses} \end{array} \begin{bmatrix} 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 1 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 1 \end{bmatrix}$$

Ranks

Proportional Betting

Reward = $3

Mathematical Formulation

n! prices

p_σ

$$
\begin{array}{lll}
\max_x & \pi^T x - r & \\
\text{s.t.} & r \geq (\sum_k x_k A_k) \bullet M_\sigma & \text{for all permutations } \sigma \\
 & 0 \leq x \leq q &
\end{array}
$$

Mathematical Formulation

n! prices

p_σ

$$\begin{array}{ll} \max_x & \pi^T x - r \\ \text{s.t.} & r \geq (\sum_k x_k A_k) \bullet M_\sigma \quad \text{for all permutations } \sigma \\ & 0 \leq x \leq q \end{array}$$

$$\begin{array}{lll} r = & \max_M & (\sum_k x_k A_k) \bullet M \\ & \text{s.t.} & M^T e = e \\ & & Me = e \\ & & M_{ij} \geq 0 \end{array}$$

Mathematical Formulation

n! prices

$\mathbf{p_\sigma}$

$$\begin{array}{lll} \max_x & \pi^T x - r & \\ \text{s.t.} & r \geq (\sum_k x_k A_k) \bullet M_\sigma & \text{for all permutations } \sigma \\ & 0 \leq x \leq q & \end{array}$$

$$r = \begin{array}{ll} \max_M & (\sum_k x_k A_k) \bullet M \\ \text{s.t.} & M^T e = e \\ & Me = e \\ & M_{ij} \geq 0 \end{array} = \begin{array}{ll} \min_{v,w} & e^T v + e^T w \\ \text{s.t.} & v_i + w_j \geq (\sum_k x_k A_k)_{ij} \end{array}$$

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n² prices

$\mathbf{Q_{ij}}$

$$\begin{array}{lll} \max_x & \pi^T x - e^T v - e^T w & \\ \text{s.t.} & v_i + w_j \geq (\sum_k x_k A_k)_{ij} & 1 \leq i,j \leq n \\ & 0 \leq x \leq q & \end{array}$$

Marginal Prices

$$\mathbf{Q} \quad \begin{array}{cc} & \textbf{Ranks} \\ \textbf{Horses} & \left[\begin{array}{ccccc} 0.2 & 0.1 & 0.3 & 0.3 & 0.1 \\ 0.1 & 0.3 & 0.1 & 0.2 & 0.3 \\ 0.2 & 0.2 & 0.1 & 0.3 & 0.2 \\ 0.3 & 0.3 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.3 & 0.1 & 0.3 \end{array}\right] \begin{array}{c} \\ =1 \\ \\ \\ \\ \end{array} \\ & \begin{array}{ccccc} & & & =1 & \end{array} \end{array}$$

Marginal Distributions

Theorem:

- One can compute in polynomial-time, an n × n marginal price matrix Q which is sufficient to price the bets in the Proportional Betting mechanism.
- Further, the price matrix is unique, pari-mutuel, and satisfies the desired price-consistency constraints.

Pricing the permutations

$$\begin{array}{ccc} & \textbf{Ranks} & \\ \textbf{Horses} & \begin{bmatrix} 0.2 & 0.1 & 0.3 & 0.3 & 0.1 \\ 0.1 & 0.3 & 0.1 & 0.2 & 0.3 \\ 0.2 & 0.2 & 0.1 & 0.3 & 0.2 \\ 0.3 & 0.3 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.3 & 0.1 & 0.3 \end{bmatrix} & =1 \\ & \qquad\qquad\quad =1 & \end{array}$$

$$p_1\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} + p_2\begin{bmatrix} & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} + p_2\begin{bmatrix} & & & 1 & \\ & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & & 1 \end{bmatrix} + \ldots\ldots\ldots + p_{n!}\begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}$$

Joint Distribution p over permutations

Maximum entropy criteria

$$\begin{aligned} &\min_p \ \textstyle\sum_\sigma p_\sigma \log p_\sigma \\ &\text{s.t.} \quad \textstyle\sum_\sigma p_\sigma M_\sigma = Q \\ &\qquad\ \ p_\sigma \geq 0 \end{aligned} \quad \Longrightarrow \quad p_\sigma = e^{Y \bullet M_\sigma}$$

- Closest distribution to uniform prior
- Completely specified by n^2 parameters
- Maximum likelihood estimator

Complexity of computing

$$\begin{aligned} &\max_Y && Q \bullet Y - 1 \\ &\text{s.t.} && \textstyle\sum_\sigma e^{Y \bullet M_\sigma} M_\sigma \geq Q \\ & && Y \leq 0 \end{aligned}$$

- #P-hard to compute parameter Y
  - Reduction from the problem of computing permanent of a non-negative matrix
  - Permanent of matrix $[e^{Yij}]$ is $\sum_\sigma e^{Y \bullet M_\sigma}$

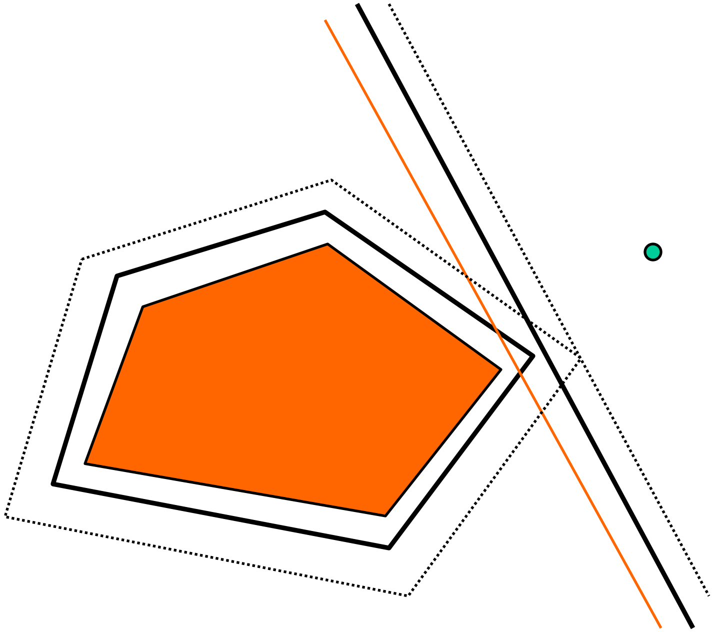
Polynomial-time approximation

$$\begin{aligned} \max_Y \quad & Q \bullet Y - 1 \\ \text{s.t.} \quad & \textstyle\sum_\sigma e^{Y \bullet M_\sigma} M_\sigma \geq Q \\ & Y \leq 0 \end{aligned}$$

- Use ellipsoid method for convex optimization
  - In conjunction with sampling to estimate the separating hyperplane at each step.
  - MCMC method for estimating permanent of a matrix used for sampling

Illustration

- Example: problem of finding feasible point in convex set



- We are guaranteed to find a feasible point in the outermost convex set

Ongoing and future work

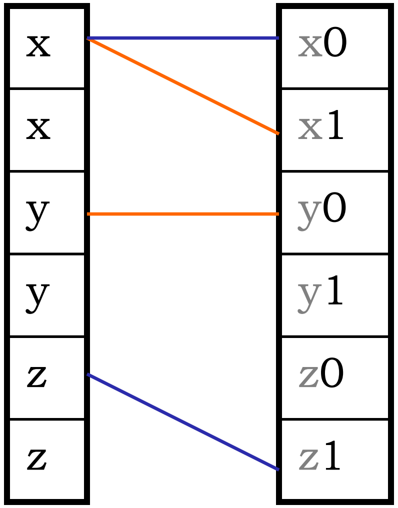
- Sequential auction model
  - Traders come one by one
  - Sequential decisions
- Truthfulness of the price vector
  - How well price captures the beliefs?
  - Empirical studies exploring this aspect
- Betting mechanisms
  - What are the gains of fixed reward betting from information collection perspective?
  - What other betting mechanisms are solvable/useful?

Questions?

Reduction

$(x \vee \neg y) \wedge (\neg x \vee z) \wedge \ldots$

Clauses



Bets

- A bet is won if and only if a clause is satisfied
- Assignment satisfying K clauses satisfies K or more bets and vice-versa