A Unified Framework for Dynamic Pari-mutuel Information Market Design

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Information markets

- Aggregating information via markets for forecasting events
  - Learning beliefs by providing incentive to report

Collecting Wisdom of the Crowds

- "Under the right conditions groups can be remarkably intelligent and possibly smarter than the smartest person."

  James Surowiecki

- Success in many experiments: Iowa Electronic Market, Inkling Markets, etc.

Scoring rules

- Participants directly report a probability distribution $\{p_i\}$ on the event outcomes
- Reward is determined by a scoring rule
  - Reward $s_i(p_i)$ if outcome is *i*.

$$s_i(\mathbf{p}) = b\log(p_i) \ \forall i$$
$$(LMSR)$$

  - Proper scoring rule
    - Reporting true beliefs maximizes reward

$$r = \arg\max_p E_r[s(\mathbf{p})]$$

- How to get a single probability estimate from different distributions reported by people?
- People hesitate to state probability numbers

Pari-mutuel Information markets

- Traders buy assets of form "Pay 1$ if outcome is i"
- Organizer matches the bets, sets price, so that net amount to paid is 0 (or small) in the worst case
- Dynamic markets:
  - Traders arrive one by one and place bets
  - organizer must decide price and which bets to accept in an online fashion

Market Scoring Rules

- q_i^t is the total number of assets on outcome i
- New trader buys order **a** at price $C(\mathbf{q}^t+\mathbf{a})-C(\mathbf{q})$
- Probability estimate reported at time t,

$$p^t = \nabla C(\mathrm{q}^t)$$

- Cost function $C(\mathbf{q})$ given by some proper scoring rule $s_i(\mathbf{p})$ [Chen et al, 2007]:

$$s_i(\mathbf{p}) = q_i - \mathrm{C}(q) + \mathrm{K}, \qquad p_i = \frac{\partial C(q)}{\partial q_i}, \sum_i p_i = 1$$

➡ *Properness of scoring rule ensures that best strategy for trader at time t is to report his belief*

- Can we directly choose cost function C?
- Which cost functions correspond to a proper scoring rule

… or result in true reporting?

SCPM
(Sequential Convex Pari-mutuel Mechanism)

- Bids of form $(\pi, l, \vec{a})$
  - $\vec{a}$: vector of outcome states
  - π : limit or bid price for an order
  - l : limit on number of orders
- Market maker's problem [Peters et al, 2007]

$$\max_{\{x,\mathbf{s},r\}} \pi x - r + \sum_i b_i \ln(s_i)$$

$$\text{s.t.} \qquad r = (\mathbf{q}_i + \mathbf{a}_i x) + s_i, \quad \forall i$$

$$0 \le x \le l, s \ge 0$$

- Risk attitude of market maker
- effects on learning?

Motivation

- To unify several existing mechanisms with a single framework.

- Overall design considerations
  1) Truthfulness/Properness
  2) Bounded worst-case loss
  3) Understanding risk attitude of market maker
  4) Efficient price updating

- Insights for design of new efficient mechanisms.

Our Unified Framework

- The market maker solves the following problem

$$\max_{\{x,\mathbf{s},r\}} \ \pi x - r + u(\mathbf{s})$$

$$\text{s.t.} \qquad \mathbf{a}x + \mathbf{s} = r\mathbf{e} - \mathbf{q},$$

$$0 \leq x \leq l$$

- Dual prices will reflect aggregate belief
- By proper choosing u(), this mechanism has nice properties and subsumes many existing mechanisms.

Examples from scoring rule markets

- LMSR: $$u(\mathbf{s}) = -b\ln\left(\sum_i e^{-s_i/b}\right) = -C(-\mathbf{s})$$
- QMSR: $$u(\mathbf{s}) = \frac{\mathbf{e}^T\mathbf{s}}{N} - \frac{1}{4b}\mathbf{s}^T\left(1 - \frac{1}{N}\mathbf{e}\mathbf{e}^T\right)\mathbf{s} = -C(-\mathbf{s})$$
- *Theorem:*

  All proper market scoring rules are equivalent to a SCPM market with convex utility function

Other Utilities

- **Log-SCPM:** $$u(\mathbf{s}) = b\sum_i \ln(s_i)$$
- **Linear-SCPM:** $$u(\mathbf{s}) = \mathbf{c}^T\mathbf{s}$$
- **Min-SCPM:** $$u(\mathbf{s}) = \min(\mathbf{s}_i)$$
- **Exp-SCPM:** $$u(\mathbf{s}) = -b\sum_i e^{-s_i/b}$$

1 Truthful reporting of beliefs: Properness

- Condition on utility function
  - Proper: Derivative of u(s) spans the simplex
    - Reporting true belief is dominant strategy
  - Strictly proper: u is smooth
    - Only dominant strategy

- Linear SCPM not proper
- Min-SCPM proper but not strictly proper
- LMSR, QMSR, Exp-SCPM, Log-SCPM strictly proper

② Risk of the market maker

- Market maker considers random return in his optimization problem
- For decision x,
  - In state i, return $z_i = \pi x - a_i x$
  - Worst case return $\pi x - r$
  - SCPM $\pi x - r + u(s)$ => risk on return

Relation to convex risk minimization

- Theorem

SCPM with non-decreasing concave utility function *u*

⇕

Minimizing a convex risk measure *ρ* on random return

$$\rho(x) = \min_t \{t - u(\vec{z}(x) + t\vec{e})$$

Learning interpretation

- Market maker's problem is equivalent to

$$\max_x \left(\min_{\mathbf{p}\in S} E_{\mathbf{p}}[z(x)] + L(\mathbf{p})\right)$$

- expected return under worst case distribution, after considering a penalty function $L(\mathbf{p})$
- $L(\mathbf{p})$ represents divergence from a prior distribution

- LMSR $$L(\mathbf{p}) = bL_{KL}(\mathbf{p} \,|\, U)$$
- QMSR No risk measure (non-monotone u)
- Log-SCPM $$L(\mathbf{p}) = bL_{LL}(\mathbf{p} \,|\, U)$$
- Min-SCPM 0
- Exp-SCPM $$L(\mathbf{p}) = bL_{KL}(\mathbf{p} \,|\, U)$$

Other properties

3 Easily computable bounds on Worst case loss

4 Truthful VCG pricing:

- Since the objective is an affine maximizer, we can use the generalized VCG mechanism to price each bids.
- VCG price turns out to be $C(q+a)-C(q)$ for equivalent cost function!

Design of new mechanisms

No existing mechanism is “perfect”:

- LMSR has unbounded (log N) worst-case loss, log-SCPM has infinite
- QMSR has no controllable risk measure and it even leads to negative “probabilities”
- Min-SCPM is overly conservative.

Design of new mechanisms

- Is there a perfect mechanism?

Quad-SCPM

$$u(\mathbf{s}) = \max_{\mathbf{v} \leq \mathbf{s}} \frac{1}{N}\mathbf{e}^T\mathbf{v} - \frac{1}{4b}\|\mathbf{v}\|^2$$

Desirable Properties of Quad-SCPM

- **Efficient computation for price update: yes**
- **Truthfulness (Myopic): yes**
- **Properness: yes and strictly**
- **Bounded worst-case loss: identical to QMSR**

$$\mathrm{WCL} = b\frac{N-1}{N}$$

- **Controllable risk measure of market-maker: yes**

$$L(\mathbf{p}) = b\,\|\,\mathbf{p} - U\,\|^2$$

Summary

- We presented framework for the information market that unifies many existing mechanisms.
- The framework has many desirable properties including truthfulness, properness, computable worst case loss and a risk measure interpretation.
- It is useful in designing new powerful mechanisms

Thanks!

Pricing Strategy

- Since the objective is an affine maximizer, we can use the generalized VCG mechanism to price each bids.
- VCG price turns out to be $C(q+a)-C(q)$ for equivalent cost function!
- The traders will be incentive compatible under this pricing scheme