A Bayesian Approach to Real Options: The Case of Distinguishing Between Temporary and Permanent Shocks*

Steven R. Grenadier
Graduate School of Business, Stanford University and NBER

Andrei Malenko
Graduate School of Business, Stanford University

September 2009

Abstract

Traditional real options models of investment timing demonstrate the importance of the "option to wait" when there is evolving uncertainty over future shocks to the value of the project. However, in many real world applications, there is also another important source of uncertainty: uncertainty over the permanence of past shocks. By adding Bayesian uncertainty over the permanent versus temporary nature of past cash flow shocks, the traditional "option to wait" is augmented by an additional "option to learn." The implied Bayesian learning environment evolves over time, as new shocks arrive and old shocks revert. We solve for the optimal investment rule and show that the implied investment behavior differs significantly from that predicted by standard real options models. For example, investment may occur at a time of stable or decreasing cash flows, investment may respond sluggishly to positive cash flow shocks, and the investment rule will critically depend on the timing of the project's cash flows.

Keywords: irreversible investment, real options, Bayesian updating, learning, temporary and permanent shocks, mean reversion

JEL Classification Numbers: D81, D83, G13, G31

*We thank Geert Bekaert, Jonathan Berk, Thomas Chemmanur, Barney Hartman-Glaser, Nadya Malenko, Paul Pfleiderer, Lukas Schmid, Ilya Strebulaev, Neng Wang, Jeffrey Zwiebel, seminar participants at BYU Marriott School and Stanford GSB, and participants at the UNC/Duke 2008 Corporate Finance Conference, 2008 European Finance Association Meeting in Athens and the 8$^{\text{th}}$ Trans-Atlantic Doctoral Conference at LBS for their helpful comments. We are responsible for all remaining errors. Address for correspondence: Graduate School of Business, Stanford University, 518 Memorial Way, Stanford, CA, 94305. Grenadier: sgren@stanford.edu; Malenko: amalenko@stanford.edu.

1 Introduction

During the past two decades the real options approach to valuation of irreversible investment opportunities has become part of the mainstream literature in financial economics. The central idea is that the opportunity to invest is equivalent to an American call option on the underlying investment project. As a consequence, the problem of optimal investment timing is analogous to the optimal exercise decision for an American option. Applications of the real options approach are now numerous.[1]

One feature which is common to virtually all real options models is that the underlying uncertainty is only about future shocks. Since the future cash flows are uncertain, there is an important opportunity cost of investing today: the value of the asset might go up so that tomorrow will be an even better time to invest. This opportunity cost is often referred to in the literature as the “option to wait.” Conversely, there is also a benefit of immediate investment in that the project’s cash flows begin being received immediately. At the point at which the costs and benefits of waiting an additional instant are equalized, investment is made. The implied investment strategy in this standard setting is to wait until the expected value of the asset reaches some upper threshold value and then invest.

In this paper we propose a very different kind of real options problem. While uncertainty about future shocks is important, it may not be the most important, let alone unique uncertainty faced by a firm. Indeed, another critical source of uncertainty is uncertainty about *past* shocks. This is the case when the firm observes the past shock, but fails to identify its exact properties. As time passes, the firm updates its beliefs about the past shock. Therefore, unlike the standard models, the evolving uncertainty in our setting is a combination of uncertainty over future shocks and Bayesian updating, or learning about the nature of past shocks. As a result, there is now an additional benefit to waiting: the ability to learn more about the nature of past shocks and use this information to make a better investment decision. This gives rise to an additional trade-off between investing now and waiting, embodied by what we call “the option to learn.”

Specifically, this paper focuses on the case where uncertainty about past shocks comes from the firm’s inability to distinguish between temporary and permanent shocks to the cash flow process. To gain intuition on the trade-off between investing now and waiting, consider a positive shock to the underlying cash flows of the project. In the standard setting the firm’s strategy would be rather simple: the firm should invest if and only if

[1] The early literature is well summarized in Dixit and Pindyck (1994). McDonald and Siegel (1986) provided the (now standard) setting, which was later extended to account for a time-to-build feature (Majd and Pindyck (1987)) and strategic interactions among several option holders (Grenadier (1996, 2002), Lambrecht and Perraudin (2003), Novy-Marx (2007)). Real options modeling is used to study specific industries such as natural resources (Brennan and Schwartz (1985)) or real estate (Titman (1985) and Williams (1991)). Recent developments include incorporating agency conflicts (Grenadier and Wang (2005)) and behavioral preferences (Grenadier and Wang (2007), Nishimura and Ozaki (2007)) into the standard setting.

the shock is high enough so that the value of the cash flow process exceeds some threshold level. However, when the firm does not know if the shock is temporary or permanent, it may want to wait in order to learn more about the identity of the shock. Indeed, if the firm waits until tomorrow and the value of the cash flow process is still at a high level, then the past shock is more likely to represent a positive fundamental change. On the other hand, if the cash flow process decreases, then there is a greater chance that the past positive shock was simply a result of temporary, non-fundamental fluctuations.

To account for the two fundamentally different notions of uncertainty, we model the evolution of the cash flow process in the following way. First, the cash flow process is subject to standard Brownian innovations, as in traditional real options models. Second, the cash flow process is subject to jump shocks, which may be permanent or temporary. If a shock is permanent, the cash flow process is changed forever. In contrast, if a shock is temporary, the cash flow process is changed only temporarily until the shock reverts at some random point in the future. Importantly, one does not know with certainty the true nature of these jump shocks. While learning about Brownian innovations is done by simply observing the increments of the cash flow process, learning about the permanent or temporary nature of past jump shocks involves waiting to see if they revert.

Our argument is based on two building blocks that distinguish this model from the real options literature. The first building block is the presence of both temporary and permanent shocks to cash flows. While virtually all real options models focus solely on permanent shocks, the presence of temporary shocks is a natural feature of many real-world economic environments. The focus on only permanent shocks is clearly a simplification that permits one to model the cash flow process as a geometric Brownian motion. However, as Gorbenko and Strebulaev (2009) show, assuming a geometric Brownian motion cash flow process leads to a number of undesirable empirical properties.[2]

The second building block is the inability of the firm to distinguish between permanent and temporary shocks. This is an especially important feature of investments in natural resources, where it is often unclear when a change in the commodity price represents a fundamental or ephemeral shift. Indeed, as the World Bank's Global Economic Prospects annual report for the year 2000 states:

> "Most important, distinguishing between temporary and permanent shocks to commodity prices can be extraordinary difficult. The swings in commodity prices can be too large and uncertain to ascertain their causes and nature. The degree of uncertainty about duration of a price shock varies. For example, market participants could see that the sharp jump in coffee prices caused by the Brazilian frost of 1994 was likely to be reversed, assuming a return to more normal weather. By contrast, most analysts assumed that the high oil

[2]For example, it implies that the volatilities of the cash flow and asset growth are equal, while empirically volatilities of cash flow growth are much higher than volatilities of asset value growth.

prices during the mid-1970s and early 1980s would last indefinitely."[3]

We show that augmenting the traditional Brownian uncertainty of standard real options models with Bayesian uncertainty over past shocks gives rise to a number of important and novel implications. First, in contrast to the standard real options setting in which investment occurs when cash flows rise to a constant trigger, we find that once we account for Bayesian uncertainty about past shocks, the investment trigger is a function of the firm's changing beliefs about the nature of past shocks. This distinction is due to the fact that while in standard real options models Brownian uncertainty is constant over time, in our Bayesian framework uncertainty depends on the timing of past shocks. More specifically, uncertainty is high soon after the arrival of a large shock since the firm is very unsure whether it is permanent or temporary. However, as time after the shock goes by, uncertainty declines as the firm becomes progressively more confident in the past shock's permanence. Thus, in addition to the standard option to wait for realizations of future shocks, there is now also a valuable option to wait in order to learn more about the nature of past shocks. We find that these two options to delay investment are additive, and appear to be of similar magnitudes when past shocks are significant. Ignoring either option would appear to lead to a similar investment timing error. Because uncertainty increases after a shock arrives, the model rationalizes a sluggish response of investment to cash flow shocks. Intuitively, after a new shock arrives, the firm might find it optimal to postpone investment in order to wait and see if the past shock is indeed permanent.

A second important implication of the Bayesian model is that investment may occur in the face of stable or even decreasing cash flows. Since standard real options models imply that investment will be triggered when shocks push the underlying cash flow level up to a fixed threshold, the firm can rationally exercise the investment option only when the underlying cash flows increase. However, when the traditional real options model is extended to include the Bayesian feature, the firm may invest even when the cash flow process is stable or declining simply because it becomes sufficiently certain about the permanent nature of past shocks. While investment in the face of stable or decreasing cash flows can be obtained in other models with more than one state variable, our argument is special since it links these features to the timing of observable shocks and thereby offers an intuitive explanation for firms (and industries) investing in markets where cash flows are stable, or even declining. For example, as we discuss below, learning about the persistence of the 1973 oil price shock was one likely reason for the rapid office construction in Denver and Houston during the late 1970's and early 1980's when the underlying property cash flows were declining.

Finally, in contrast to standard real options models with only uncertainty over future shocks, in the context of Bayesian uncertainty over past shocks the exercise strategy depends on the timing structure of the project's cash flows. In standard real options models,

[3] World Bank, Global Economic Prospects and Developing Countries 2000, p. 110.

the net present value that triggers exercise is constant, regardless of whether the project's cash flows are "front-loaded" or "back-loaded." That is, whether the project's cash flows occur mostly up-front or mostly later in the future, exercise will be triggered when the net present value of future cash flows equals the same premium over the cost. However, with Bayesian uncertainty over past shocks, the exercise strategy critically depends on the maturity structure of the project's cash flows. The greater the "front-loadedness" of the option payoff, the less important is the assessment of the relative likelihood that a shock is temporary or permanent. As a result, the "option to learn" is less valuable, so the firm invests at a lower option trigger. Thus, with Bayesian uncertainty, while the firm might invest in a short-maturity project when the value of the future cash flows exceeds the investment cost by a relatively small amount, the firm might only invest in a longer-maturity project if the value of the future cash flows exceeds the investment cost by a large premium.

Several papers deserve mentioning as being the most closely related to our model. Gorbenko and Strebulaev (2009) incorporate temporary shocks into a contingent claims framework of capital structure and show that the presence of temporary shocks provides a potential explanation of several puzzles in corporate finance. Our paper differs from theirs in two important respects. First, and most importantly, our argument relies heavily on the firm's imperfect information and learning about the permanence of past shocks, while Gorbenko and Strebulaev (2009) assume perfect information about past shocks. Second, while we consider firms' investment decisions, their focus is on financing policy.

Our paper is also related to models of parameter uncertainty such as Décamps et al. (2005) and Klein (2007, 2009), who solve for the optimal investment timing when the firm is uncertain about the drift parameter of the state process. These models of parameter uncertainty are based on very different notions of uncertainty and learning. First, in these models, over time the firm learns more about the true parameter. In the fullness of time, full information can be achieved. In contrast, our model deals with uncertainty about the state of the economy, which is an evolving one, unlike uncertainty about a parameter of the model. As new shocks arrive and old ones revert, the learning problem changes dynamically over time, and even in the fullness of time the true nature of uncertainty is never revealed. Second, in these models of parameter uncertainty all learning is driven by evolutions in the Brownian component of uncertainty.[4] Thus the Brownian and Bayesian components are fundamentally intertwined. This results in difficulty in decomposing the two. For example, a sequence of positive Brownian innovations not only lowers the option

[4]See also other papers on optimal experimentation that deal with problems of optimal control under learning (e.g., Bolton and Harris (1999), Keller and Rady (1999), and Moscarini and Smith (2001)). These papers focus on active learning through experimentation, in contrast to passive learning through observation found in the present paper. Similarly to Décamps et al. (2005) and Klein (2007, 2009), these papers feature Brownian uncertainty and learning, with the Brownian motion providing the noise that makes learning necessary in the first place. In the present paper, by contrast, Brownian uncertainty is irrelevant to learning and simply drives the standard option to wait.

to wait but also affects the option to learn as it implies a greater likelihood of a higher drift parameter. In our model's setting, the Brownian and Bayesian components are decoupled. The Brownian uncertainty is identical to that from standard real options models, while learning is driven by a separate shock process. Thus, our model is more appropriate for settings in which learning is driven by the evolution of economic shocks and not by the volatility of the process itself. These distinctions result in important differences in the resulting investment behavior. First, Klein (2007, 2009) shows that it may be the case that Bayesian learning about parameter uncertainty may actually lead to earlier investment than in the full-information setting. In contrast, our model of Bayesian uncertainty over past shocks always leads to delay due to the option to learn. Second, since the arrival of a shock increases uncertainty and thus the value of waiting to learn, our model implies that investment responds sluggishly to shocks, an implication not present in models of parameter uncertainty.

Another related paper is Miao and Wang (2007) who solve for the entrepreneur's optimal decision to exit a business in the presence of idiosyncratic nondiversifiable risk. Specifically, Miao and Wang (2007) consider an entrepreneur with incomplete information about his entrepreneurial abilities who chooses between continuing entrepreneurial activity and taking a safe job. While their paper contains Bayesian uncertainty over entrepreneurial ability, it does not allow for Brownian uncertainty of cash flows, as does ours. In addition, in their setting there is only a single source of Bayesian uncertainty, while in our model new shocks can arrive, and old shocks can revert. Finally, the focus of their paper is on risk aversion and imperfect hedging of idiosyncratic risks, while ours is on the joint impact of various sources of uncertainty on investment timing.[5]

Finally, our paper also shares the learning feature with Lambrecht and Perraudin (2003) who study competition between two firms for a single investment opportunity when information about investment costs is private. Because of this, as time goes by, while the competitor has not invested yet, each firm updates its belief about the competitor's investment costs upward.[6]

The remainder of the paper is organized as follows. Section 2 provides a set-up of the model and for purposes of providing intuition solves the most simple case of a pure jump process with only one shock. Section 3 focuses on incorporating Bayesian uncertainty over the past shock, developed in Section 2, into the standard geometric Brownian motion setting. Section 4 provides a discussion of some important implications of the model. Section 5 generalizes the model to the case of multiple shocks. Finally, Section 6 concludes.

[5]Other papers that include learning based on a Poisson process include Malueg and Tsutsui (1997), Bergemann and Hege (1998), and Keller et al. (2005). Similar to Miao and Wang (2007), but unlike the present paper, there is only Bayesian uncertainty and no Brownian uncertainty. In these models, firms learn about profitability by observing (their own and sometimes other firms') payoff outcomes.

[6]Our paper is also related to Moore and Schaller (2002), who extend the neoclassical q theory of investment by allowing for permanent and temporary shocks to interest rates, and uncertainty about them.

2 Model Set-Up and a Simple Case

In this section we describe the set-up of the model, which incorporates the essential features of investment timing into a Bayesian framework. Then, to show the main intuition behind the argument, we consider the simplest possible setting as a special case. In subsequent sections we solve for the more general cases.

2.1 The Investment Option

Consider a standard real option framework in which a risk-neutral firm contemplates irreversible investment. By paying the investment cost I, the firm obtains the perpetual cash flow $X(t)$. The firm is free to invest in the project at any time it chooses. The firm has a constant discount rate $r > 0$.

The evolution of the cash flow process consists of two components. As usual in the literature, the cash flow process evolves according to the geometric Brownian motion with drift α and diffusion σ, where $\alpha < r$ to ensure finite values.[7] In addition to this, the cash flow process is subject to jump shocks, which can be temporary or permanent. Upon the arrival of a shock, the cash flow process jumps from $X(t)$ to $X(t)(1+\varphi)$, with $\varphi > 0$.[8] A permanent shock changes the value of the cash flow process forever, while a temporary shock changes the value of the cash flow process only temporarily until it reverts at some random point in the future. Once the shock reverts, the cash flow process jumps down by the multiple of $1+\varphi$. Importantly, the firm is unable to distinguish between permanent and temporary shocks, and uses Bayesian updating to assess their relative likelihoods. We assume that the arrival and reversal (of temporary shocks) follow jump processes. Specifically, a permanent shock arrives with intensity λ_1, a temporary shock arrives with intensity λ_2, and each outstanding temporary shock reverts with intensity $\lambda_3 > \lambda_2$. Reversal of each outstanding temporary shock is independent of arrivals of new shocks and reversals of other outstanding temporary shocks. Arrival of a new permanent (temporary) shock is independent of arrival of a new temporary (permanent) shock and reversals of outstanding temporary shocks.

Let us formally define the evolution of the cash flow process, conditional on information about the identities of the shocks. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t) = \{\mathcal{F}_t : t \geq 0\}$. The evolution of the cash flow process $X(t)$ is given by

$$dX(t) = \alpha X(t) + \sigma X(t)\, dB(t) + \varphi X(t)\left(dM_1(t) + dM_2(t)\right) - \frac{\varphi X(t)}{1+\varphi} dN(t) \qquad (1)$$

[7] To be more precise, the upper bound on α that guarantees finite values depends on the particular specification of the model. In the model of Section 3 and the simple case of Section 2, we assume $\alpha < r$. In the model of Section 5, we assume that $\alpha + \lambda_1\varphi < r$, where λ_1 and φ are defined below.

[8] The model can be extended to the case of stochastic jump amplitudes, including the possibility of both positive and negative jumps, albeit at the cost of additional mathematical complexity. However, the basic intuition and results remain unchanged.

where $X(0) = X_0$, and $B(t)$, $M_1(t)$, $M_2(t)$, and $N(t)$ are $(\mathcal{F}_t)$-adapted. In (1), $dB(t)$ is the increment of a standard Wiener process, and $M_1(t)$, $M_2(t)$ and $N(t)$ are counting processes corresponding to the arrival of permanent shocks, arrival of temporary shocks, and reversal of outstanding temporary shocks, respectively. Specifically, $M_1(t)$ and $M_2(t)$ are Poisson processes with intensities λ_1 and λ_2, respectively, and $N(t)$ is a nonexplosive counting process with intensity $k(t)\lambda_3$, where $k(t)$ is the number of outstanding temporary shocks at time t: $k(t) = M_2(t) - N(t)$. Conditional on $\mathcal{F}_t$, the increments of $M_1(t)$, $M_2(t)$, and $N(t)$ over the next instant are independent. Defining the total number of outstanding shocks at time t by $n(t) = M_1(t) + M_2(t) - N(t)$, we can say that the triple $(X(t), n(t), k(t))$ represents the state under the full-information filtration $(\mathcal{F}_t)$.[9]

Now, let us consider the evolution of the cash flow process from the firm's point of view. The firm observes the evolution of $X(t)$, but cannot distinguish whether jumps in $X(t)$ were caused by permanent shocks $dM_1(t)$ or temporary shocks $dM_2(t)$. Mathematically, the information available to the firm is represented by the partial-information filtration $(\mathcal{G}_t)$ generated by (B, M, N), where $M(t) = M_1(t) + M_2(t)$. Hence, at any time t, the firm knows the total number $n(t) = M(t) - N(t)$ of outstanding shocks, but does not know the number $k(t) = M_2(t) - N(t)$ of outstanding temporary shocks. However, given the Bayesian learning process described in detail in the next subsection, the firm possesses posterior probabilities on the number $k(t)$ of outstanding temporary shocks:

$$p_k(t) \equiv \mathbb{P}(k(t) = k|\mathcal{G}_t). \tag{2}$$

Thus, the firm attaches probability $p_1(t)$ that there is exactly one outstanding temporary shock, $p_2(t)$ that there are two outstanding temporary shocks, all the way to $p_{n(t)}(t)$ that all of the $n(t)$ outstanding shocks are temporary. Therefore, at time t, the firm estimates that over the next instant $(t, t+dt)$ a new shock arrives with probability $(\lambda_1 + \lambda_2)dt$ and an existing shock reverts with probability $\sum_{k=1}^{n(t)} p_k(t) k\lambda_3 dt$.

Let $S(t)$ denote the expected (under the firm's information set) present value of the project's underlying cash flows:

$$S(t) = \mathbb{E}\left[\int_t^\infty e^{-r(s-t)} X(s)\, ds|\mathcal{G}_t\right]. \tag{3}$$

Then, the firm's investment problem is to find a stopping time $\tau^* \in (\mathcal{G}_t)$ such that

$$\sup_{\tau\in(\mathcal{G}_t)} \mathbb{E}\left[e^{-r\tau}(S(\tau) - I)|\mathcal{G}_t\right] = \mathbb{E}\left[e^{-r\tau^*}(S(\tau^*) - I)|\mathcal{G}_t\right]. \tag{4}$$

[9]Because the dynamics of $X(t)$ is driven by both Brownian motion and jump processes, it is a jump-diffusion process (see Chapter 11.4.1 in Shreve (2004) for a discussion) and, more generally, a semimartingale (see Chapter 2 in Protter (2004) for an overview). Mathematically, the set-up of the model is similar to models of correlated default in which default of one claim causes changes in the default intensity of other claims. For example, see the primary-secondary default model in Jarrow and Yu (2001). Both in Jarrow and Yu (2001) and in our model, there is a "primary" jump process with a constant intensity, whose arrivals influence the intensity of a "secondary" jump process.

2.2 The Bayesian Learning Process

Since the firm is unsure whether past shocks are permanent or temporary, an important state variable is the firm's belief about the identity of past shocks. Consider moments t and $t+dt$ for any t and infinitesimal positive dt. Depending on the history between t and $t+dt$, there are three cases to be analyzed:[10]

1. no new shocks occur or outstanding shocks reverse between t and $t+dt$;
2. a new shock occurs;
3. an outstanding shock reverses.

Consider the first case where there are no changes in the number of outstanding shocks between t and $t+dt$. Suppose that there are n outstanding shocks and at time t the firm assesses the probability that k of them are temporary at $p_k(t)$. Then, the beliefs of the firm are defined by the vector $p(t) = (p_0(t)\ \ p_1(t)\ \ ...\ p_n(t))$. If k of the outstanding shocks are temporary, over a short period of time dt a shock reverts back with probability $k\lambda_3 dt$. Similarly, a new shock arrives with probability $(\lambda_1+\lambda_2)\,dt$. Thus, conditional on k of the shocks being temporary, the probability that there are no new shocks or reversions over dt equals $1-(\lambda_1+\lambda_2)\,dt - k\lambda_3 dt$. Using Bayes rule, the posterior probability $p_k(t+dt)$ is given by

$$p_k(t+dt) = \frac{p_k(t)\left(1-(\lambda_1+\lambda_2)\,dt - k\lambda_3 dt\right)}{1-(\lambda_1+\lambda_2)\,dt - \lambda_3\sum_{i=1}^{n} p_i(t)\,idt}. \tag{5}$$

Eq. (5) is a direct application of Bayes rule. The numerator is the joint probability of having k temporary outstanding jumps and observing no jumps or reversions between t and $t+dt$. The denominator is the sum over $i=0,1,...$ of joint probabilities of having i temporary outstanding jumps and observing no jumps or reversions between t and $t+dt$.

We can rewrite (5) as

$$\frac{p_k(t+dt)-p_k(t)}{dt} = -\frac{\lambda_3 p_k(t)\left(k-\sum_{i=1}^{n} p_i(t)\,i\right)}{1-(\lambda_1+\lambda_2)\,dt - \lambda_3\sum_{i=1}^{n} p_i(t)\,idt}. \tag{6}$$

Taking the limit as $dt \to 0$,

$$\frac{dp_k(t)}{dt} = -\lambda_3 p_k(t)\left(k-\sum_{i=1}^{n} p_i(t)\,i\right). \tag{7}$$

[10] Note that the probability of observing more than one new shock or reversion between t and $t+dt$ has the order $(dt)^2$. Since dt is infinitesimal, we can ignore these cases. The same is true for the likelihood of both a new shock and a reversal occurring at the same instant.

The dynamics of $p_k(t)$ while there are no new shocks or reversions has two interesting properties. First, $p_k(t)$ increases in time when $k < \sum_{i=1}^{n} p_i(t)\, i$, and decreases in time, otherwise. Intuitively, when $k < \sum_{i=1}^{n} p_i(t)\, i$, the likelihood of reversion conditional on having k outstanding temporary jumps is lower than the unconditional likelihood of reversion. Therefore, when the firm does not observe a new shock or a reversion, it updates its beliefs $p_k(t)$ upward. The opposite is true when $k > \sum_{i=1}^{n} p_i(t)\, i$. Second, the speed of learning is proportional to λ_3. In other words, if temporary shocks are more short-term, then the firm updates its beliefs faster than if they are more long-term.

Now, consider the second case. If a new shock occurs between t and $t + dt$, then the updated beliefs equal

$$p_k(t + dt) \equiv \hat{p}_k(p(t)) = p_{k-1}(t)\frac{\lambda_2}{\lambda_1 + \lambda_2} + p_k(t)\frac{\lambda_1}{\lambda_1 + \lambda_2}, \tag{8}$$

The intuition behind (8) is relatively simple. When a new shock occurs, it can be either permanent or temporary. After the shock, there can be k temporary shocks outstanding either if there were $k - 1$ temporary shocks and the new shock is temporary or if there were k temporary shocks and the new shock is permanent.

Finally, considering the third case, if an outstanding shock reverses between t and $t + dt$, by Bayes rule the firm's updated beliefs equal

$$p_k(t + dt) \equiv \tilde{p}_k(p(t)) = \frac{p_{k+1}(t)(k+1)}{\sum_{i=1}^{n} p_i(t)\, i}. \tag{9}$$

When a shock reverses, the firm learns that it was temporary. Hence, there are k outstanding temporary shocks after the reversal if and only if there were $k + 1$ outstanding temporary shocks before that. The joint probability of having $k + 1$ temporary shocks at time t and observing a reversion between t and $t + dt$ equals $p_{k+1}(k+1)\lambda_3 dt$. The probability of observing a reversion between t and $t + dt$ conditional only on being at time t equals $\sum_{i=1}^{n} p_i(t)\, i\lambda_3 dt$. Dividing the former probability by the latter yields (9).

Eqs. (7)-(9) fully characterize the dynamics of the firm's beliefs. Notice that reversion of a shock decreases the number of outstanding shocks by one, while arrival of a new shock increases this number by one.

To keep the model both intuitive and tractable, in the early sections of the paper we focus on the case when there is only one shock, which can be either permanent or temporary.[11] In this case, the firm's beliefs are characterized by a single state variable $p(t) \equiv p_1(t)$, which is the probability that the outstanding shock is temporary. Before the arrival and after the reversion, there is no uncertainty about past shocks. At the moment t_0 of the arrival of the shock, the firm assesses the prior probability of the shock being temporary as $p(t_0) \equiv \pi_0 = \lambda_2/(\lambda_1 + \lambda_2)$. After the shock arrives, the firm continuously updates its assessment of this probability according to (7):

[11] Section 5 considers the multi-shock setting.

$$\frac{dp(t)}{dt} = -\lambda_3 p(t)(1-p(t)), \tag{10}$$

which is solved by

$$p(t) = \frac{\lambda_2}{\lambda_1 e^{\lambda_3(t-t_0)} + \lambda_2}. \tag{11}$$

2.3 A Simple Case

Before presenting the details of our analysis in subsequent sections, we consider the simplest setting to illustrate the intuition behind our results. Specifically, for the cash flow specification in (1), we set $\alpha = \sigma = 0$, and assume that there is only one shock, which can be either permanent or temporary. In this case, the cash flow process is rather simple. At all times prior to the arrival of the shock, the cash flow stream is fixed at X_0. Upon the arrival of the shock it jumps up to $X_0(1+\varphi)$. With a permanent shock the cash flow remains at the higher level $X_0(1+\varphi)$ forever after, but with a temporary shock the cash flow eventually reverts to the level X_0 at some (random) point in the future.

Let us begin by calculating some simple values. If the firm knows for sure that the shock is permanent, then the present value of the project's cash flows is $\frac{X_0(1+\varphi)}{r}$, which is simply the present value of the perpetual cash flow $X_0(1+\varphi)$. Conversely, if the firm knows for sure that the shock is temporary, then the present value of the project's cash flows is $\frac{X_0(1+\varphi+\lambda_3/r)}{r+\lambda_3}$, which is the present value of receiving a flow of $X_0(1+\varphi)$ until the shock is reversed and X_0 thereafter. Therefore, the present value of the project's underlying cash flows upon investing at the belief level p, defined by (3), is equal to

$$S(p) = (1-p)\frac{X_0(1+\varphi)}{r} + p\frac{X_0(1+\varphi+\lambda_3/r)}{r+\lambda_3}. \tag{12}$$

To ensure that a non-trivial solution exists for the problem, we make the assumption that X_0 satisfies

$$\frac{rI}{1+\varphi} < X_0 < \frac{r+\lambda_3\pi_0}{1+\varphi+\frac{\lambda_3}{r}\pi_0} I. \tag{13}$$

Economically, the lower bound means that if the shock is known to be permanent, the net present value of investing into the project is positive, $\frac{X_0(1+\varphi)}{r} - I > 0$, or else there would never be any investment in this Bayesian setting. The upper bound, as we show later in this section, is equivalent to an assumption that learning has positive value. In particular, this assumption implies that if the shock is known to be temporary, then it is not optimal to invest: $\frac{X_0(1+\varphi+\lambda_3/r)}{r+\lambda_3} - I < 0$.

Let $G(p)$ denote the value of the option to invest, while a shock persists, and where p is the current value of the belief process. By Itô's lemma for jump processes (see Shreve (2004)), the dynamics of $G(p)$ under $(\mathcal{F}_t)$ is given by

$$dG(p) = -\lambda_3 p(1-p) G'(p)\, dt + (H(X_0) - G(p))\, dN(t), \tag{14}$$

where $H(X_0)$ is the value of the option to invest after the shock reverts. In (14), the intensity of $N(t)$ is equal to $k(t)\lambda_3$, where $k(t)$ is the indicator of an outstanding temporary shock at time t, known under $(\mathcal{F}_t)$. Because the firm believes that the shock is temporary with probability p, $\mathbb{E}[dN(t)|\mathcal{G}_t] = p\lambda_3 dt$. Therefore, the expectation of the differential with respect to the information set of the firm is equal to

$$\mathbb{E}\left[\frac{dG(p)}{dt}|\mathcal{G}_t\right] = -\lambda_3 p(1-p)G'(p) + p\lambda_3(H(X_0) - G(p)). \tag{15}$$

In equilibrium, it must be the case that $\mathbb{E}[dG(p)/dt|\mathcal{G}_t] = rG(p)$. Therefore, $G(p)$ must satisfy the equilibrium differential equation:

$$(r + p\lambda_3)G(p) = -\lambda_3 p(1-p)G'(p) + p\lambda_3 H(X_0), \tag{16}$$

By (13), the option will not be exercised after the reversion, so $H(X_0) = 0$ and

$$(r + p\lambda_3)G(p) = -\lambda_3 p(1-p)G'(p). \tag{17}$$

The general solution to (17) is:

$$G(p) = C\cdot(1-p)\left(\frac{1}{p} - 1\right)^{\frac{r}{\lambda_3}}, \tag{18}$$

where C is a constant to be determined by the appropriate boundary conditions.

The option will be exercised when the conditional probability of the shock being temporary decreases to a lower threshold. Intuitively, it is optimal for the firm to invest only when it becomes sufficiently sure that the project will yield high cash flows for a long period of time. Let $\bar{p}$ denote the trigger at which the option is exercised. The exercise trigger $\bar{p}$ and constant C are jointly determined by the following boundary conditions:

$$G(\bar{p}) = (1-\bar{p})\frac{X_0(1+\varphi)}{r} + \bar{p}\frac{X_0(1+\varphi+\lambda_3/r)}{r+\lambda_3} - I, \tag{19}$$

$$G_p(\bar{p}) = -\frac{X_0(1+\varphi)}{r} + \frac{X_0(1+\varphi+\lambda_3/r)}{r+\lambda_3}. \tag{20}$$

The first equation is the value-matching condition. It reflects the fact that upon exercise, the value of the project is equal to its net present value, $S(\bar{p}) - I$. The second equation is the smooth-pasting condition.[12] It ensures that the trigger $\bar{p}$ maximizes the option value.

Combining (18) with (19) and (20) yields the optimal investment threshold $\bar{p}$ and the constant C:

$$\bar{p} = \frac{X_0(1+\varphi) - rI}{\lambda_3\left(I - \frac{X_0}{r}\right)}, \tag{21}$$

[12] This condition is also known as the high-contact condition (see Krylov (1980) and Dumas (1991) for a discussion).

$$C=\frac{1}{1-\bar{p}}\left(\frac{1}{\bar{p}}-1\right)^{-\frac{r}{\lambda_3}}\left[(1-\bar{p})\,\frac{X_0(1+\varphi)}{r}+\bar{p}\frac{X_0(1+\varphi+\lambda_3/r)}{r+\lambda_3}-I\right]. \tag{22}$$

Now we can see that the upper bound on X_0 from (13) is equivalent to assuming that $\bar{p} < \pi_0$. This restriction seems entirely reasonable since it ensures that there is some benefit to learning. Combined with the lower bound on X_0, it guarantees that $\bar{p} \in (0, \pi_0)$. Also, note that in this simple case, the option will never be exercised prior to the arrival of the shock.[13] This is because the value of exercising prior to the arrival of the shock is dominated by the value of waiting until the shock arrives and then exercising immediately, as shown in the appendix (which itself is dominated by waiting an additional period of time in order to learn).

This simple case highlights the key notion of Bayesian learning in a real options context. Perhaps the best way to gain intuition on the optimal investment policy is to rewrite the expression for the optimal trigger $\bar{p}$ outlined in Eq. (21) as:

$$X_0\,(1+\varphi)=\bar{p}\lambda_3\left(I-\frac{X_0}{r}\right)+rI. \tag{23}$$

The intuition behind expression (23) is the trade-off between investing now versus investing a moment later if the past shock persists, where the value of waiting is explicitly an "option to learn." If the firm invests now, it gets the benefit of the cash flow $X_0\,(1+\varphi)$ over the next instant. This is the term on the left-hand side of the equal sign. If the firm waits a moment and invests if the past shock persists, it faces a small chance of the shock reversing, in which case the expected value gained by not investing is equal to $\bar{p}\lambda_3\left(I-\frac{X_0}{r}\right)$. Importantly, by waiting that additional moment, it gains the opportunity to forgo investment should the past shock prove to be temporary. The second term on the right hand side of the equal sign, rI, is the savings from delaying the investment cost by an instant. At the optimal Bayesian trigger, $\bar{p}$, these two sides are exactly equal, and the firm is indifferent between investing now and waiting a moment to learn.

Fig. 1 plots simulated sample paths of the cash flow process, $X\,(t)$, and the corresponding firm's belief process, $p\,(t)$. Before the arrival of the shock, investment is suboptimal since the project does not generate enough cash flows. When the shock arrives, the cash flow process jumps from X_0 to $X_0\,(1+\varphi)$, and the net present value of the project becomes positive. Nevertheless, the firm finds investment suboptimal because of the valuable option to learn more about the nature of the past shock. As time goes by and the shock does not revert back, the firm updates its beliefs downwards. When the firm becomes sufficiently confident that the past shock is permanent, it invests. In the example in Fig. 1 this happens more than 1.5 years after the arrival of the shock.

[13] This result will not hold in several generalized versions of the model that follow.

3 Model with Bayesian and Brownian Uncertainties

In the simple case in the previous section, the cash flow process $X(t)$ was a pure jump process. Since most traditional real options models are based on Brownian uncertainty, we generalize the simple case by allowing $X(t)$ to follow a combined jump and geometric Brownian motion process. In this case the firm faces two fundamentally different types of uncertainty. First, as in traditional models, the firm is uncertain about future shocks. This is captured by the cash flow process being subject to the Brownian innovations and unknown timing of the shock before the shock arrives. This gives rise to the option to wait for realizations of future shocks usually studied in the real options literature. Second, the firm is also uncertain about past shocks, meaning that the firm is unable to perfectly identify whether the past shock is permanent or temporary. This gives rise to the option to learn more about the nature of past shocks. In order to highlight the interaction between these uncertainties, we assume only a single shock. Section 5 extends the model to the general case of multiple shocks.

Since there is only one shock which can be either permanent or temporary, we solve the model by backward induction. First, we consider the optimal timing of investment conditional on a temporary shock having reversed. In this case, there is neither uncertainty about past shocks nor the possibility of new jumps, so the problem is the standard real options problem studied in the literature. Second, we move one step back and consider the situation after a shock arrives when the firm is uncertain about its nature. Finally, we consider the situation before a shock arrives.

3.1 Optimal Investment after a Shock Reverses

First, consider the situation after a temporary shock reverses. Then, there is no uncertainty about past shocks, so the only underlying state variable is $X(t)$, which follows

$$dX(t) = \alpha X(t)dt + \sigma X(t)dB(t). \tag{24}$$

Let $H(X)$ denote the value of the investment option after the shock reverses, where X is the current value of the cash flow process. Using standard arguments (e.g., Dixit and Pindyck (1994)), in the range before investment, $H(X)$ must solve the following differential equation:

$$rH = \alpha X H_X + \frac{1}{2}\sigma^2 X^2 H_{XX}. \tag{25}$$

Eq. (25) must be solved subject to the following boundary conditions:

$$H(X^*) = \frac{X^*}{r-\alpha} - I, \tag{26}$$

$$H'(X^*) = \frac{1}{r-\alpha}, \tag{27}$$

where X^* is the investment threshold. The first boundary condition is the value-matching condition, which states that at the exercise time the value of the option is equal to the net present value of the project. The second boundary condition is the smooth-pasting or high-contact condition, which guarantees that the exercise strategy is chosen optimally. The last boundary condition is $H(0) = 0$, reflecting the fact that $X(t) = 0$ is the absorbing barrier for the cash flow process.[14]

Solving (25) subject to (26) and (27) yields the investment threshold X^*:

$$X^* = \frac{\beta}{\beta - 1}(r - \alpha) I, \tag{28}$$

where β is the positive root of the fundamental quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1)+\alpha\beta-r = 0$:

$$\beta = \frac{1}{\sigma^2}\left[-\left(\alpha - \frac{\sigma^2}{2}\right) + \sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] > 1. \tag{29}$$

The corresponding value of the investment option $H(X)$ is then given by

$$H(X) = \begin{cases} \left(\frac{X}{X^*}\right)^\beta \left(\frac{X^*}{r-\alpha} - I\right), & \text{if } X < X^*, \\ \frac{X}{r-\alpha} - I, & \text{otherwise.} \end{cases} \tag{30}$$

3.2 Optimal Investment while a Shock Persists

Consider the situation when there is an outstanding shock that can be either permanent or temporary. In this case, analogous to (1), the evolution of $X(t)$ under $(\mathcal{F}_t)$ is given by

$$dX(t) = \alpha X(t)\,dt + \sigma X(t)\,dB(t) - \frac{\varphi X(t)}{1+\varphi} dN(t), \tag{31}$$

where the intensity of $N(t)$ is equal to $k(t)\lambda_3$, where $k(t)$ is the indicator of an outstanding temporary shock at time t, known under $(\mathcal{F}_t)$ but unknown by the firm.

We begin by calculating some simple values. If the firm knows for sure that the shock is permanent, then the present value of the project's cash flows is $\frac{1}{r-\alpha}X(t)$. If the firm knows for sure that the shock is temporary, then the present value of the project's cash flows is $\frac{1+\frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r-\alpha+\lambda_3}X(t)$. Therefore, the expected present value of the project's underlying cash flows, defined by (3), is equal to

$$S(X(t), p(t)) = \left[(1-p(t))\frac{1}{r-\alpha} + p(t)\frac{1+\frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r-\alpha+\lambda_3}\right] X(t). \tag{32}$$

[14] This boundary condition applies to all of the valuation equations. However, to avoid repetition, we do not list it in the future valuation equations.

Let $G(X,p)$ denote the value of the option to invest while the shock persists, where X and p are the current values of the cash flow and the belief processes, respectively. By Itô's lemma for two-dimensional processes with jumps (see Theorem 11.5.4 in Shreve (2004)), in the range of (X,p) at which the option is not exercised, the dynamics of $G(X,p)$ under $(\mathcal{F}_t)$ satisfies

$$\begin{aligned} dG = & \left(\alpha X G_X + \tfrac{\sigma^2}{2} X^2 G_{XX} - \lambda_3 p(1-p) G_p\right) dt \\ & + \sigma X G_X dB(t) + \left(H\left(\tfrac{X}{1+\varphi}\right) - G\right) dN(t), \end{aligned} \tag{33}$$

where $H(X)$ is the value of the option when no more jumps can occur, given by (30). Because the firm believes that the shock is temporary with probability p, $\mathbb{E}[dN(t)|\mathcal{G}_t] = p\lambda_3 dt$. Hence, the instantaneous conditional expected change in $G(X,p)$ is equal to

$$\mathbb{E}\left[\frac{dG}{dt}|\mathcal{G}_t\right] = \alpha X G_X + \frac{\sigma^2}{2} X^2 G_{XX} - \lambda_3 p(1-p) G_p + p\lambda_3 \left(H\left(\frac{X}{1+\varphi}\right) - G\right). \tag{34}$$

Thus, analogous to Eq. (16), in the range of (X,p) at which the option is not exercised, $G(X,p)$ must satisfy the equilibrium partial differential equation:

$$(r + p\lambda_3) G = \alpha X G_X + \frac{\sigma^2}{2} X^2 G_{XX} - \lambda_3 p(1-p) G_p + p\lambda_3 H\left(\frac{X}{1+\varphi}\right). \tag{35}$$

Exercise will be triggered when cash flows rise to an upper trigger, which is itself a function of the firm's beliefs. Let $\bar{X}(p)$ denote the exercise trigger function. Conjecture that $\bar{X}(p) < (1+\varphi) X^*$ for all $p \in (0, \pi_0]$.[15] Eq. (35) is solved subject to the following value-matching and smooth-pasting conditions:

$$G\left(\bar{X}(p), p\right) = \left[(1-p)\frac{1}{r-\alpha} + p\frac{1 + \frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r - \alpha + \lambda_3}\right] \bar{X}(p) - I, \tag{36}$$

$$G_X(\bar{X}(p), p) = \left[(1-p)\frac{1}{r-\alpha} + p\frac{1 + \frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r - \alpha + \lambda_3}\right], \tag{37}$$

$$G_p(\bar{X}(p), p) = \left[-\frac{1}{r-\alpha} + \frac{1 + \frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r - \alpha + \lambda_3}\right] \bar{X}(p). \tag{38}$$

The value-matching condition (36) captures the fact that at the time of investment the value of the investment option equals the expected payoff from immediate investment $S\left(\bar{X}(p), p\right) - I$, while the smooth-pasting conditions (37) - (38) guarantee that the trigger function is chosen optimally. In the appendix we use the results in Peskir and Shiryaev (2006) to show that the boundary $\bar{X}(p)$ indeed satisfies the smooth-pasting conditions (37) - (38).

[15]This conjecture is verified in the appendix.

Evaluating (35) at $\bar{X}(p)$, plugging in the boundary conditions (36) - (38) and simplifying, we get the following expression for the optimal trigger function $\bar{X}(p)$:[16]

$$\begin{aligned}\bar{X}(p) = p\lambda_3 \left[\left(\frac{\bar{X}(p)}{(1+\varphi)X^*}\right)^{\beta}\left(\frac{X^*}{r-\alpha} - I\right) - \left(\frac{\bar{X}(p)}{(1+\varphi)(r-\alpha)} - I\right)\right] \\ +rI + \frac{\sigma^2}{2}\bar{X}(p)^2 G_{XX}\left(\bar{X}(p), p\right).\end{aligned} \tag{39}$$

In the appendix we demonstrate that there indeed exists a solution $\bar{X}(p)$ to the fixed-point problem in Eq. (23) for all $p \in (0, \pi_0)$.

Eq. (39) provides intuition on the additive forces of the option to wait and the option to learn. If the firm chooses to invest now over waiting an instant, it receives the benefits of the current cash flow, which is the term on the left side of (39). If the firm waits a moment, it faces a small chance of the shock reverting, in which case the benefits of not investing are equal to $\left(\frac{\bar{X}(p)}{(1+\varphi)X^*}\right)^{\beta}\left(\frac{X^*}{r-\alpha} - I\right) - \left(\frac{\bar{X}(p)}{(1+\varphi)(r-\alpha)} - I\right)$. Thus, the first term on the right-hand side of (39) corresponds to the value of waiting to learn more about the nature of the past shock. The second term on the right-hand side of (39), rI, is simply the savings from delaying the investment cost by an instant. Finally, the third term is a new convexity term, $\frac{\sigma^2}{2}\bar{X}(p)^2 G_{XX}\left(\bar{X}(p), p\right)$, which represents the traditional option to wait for future realizations of Brownian uncertainty.[17] In the sense that the right-hand side of (39) is the sum of the three components, Brownian uncertainty is additive to Bayesian uncertainty. In other words, the addition of Brownian uncertainty to the model increases the investment trigger by a further component due to the option value of waiting for the evolution of Brownian uncertainty over future cash flows.

The trigger function $\bar{X}(p)$ is increasing in p. For any two values of p at exercise, the higher the value of p, the greater the risk that immediate exercise could be followed by a cash flow reversal. Thus in order to induce exercise at higher levels of p, the immediate payoff must be higher. Given this monotonicity, we can invert the function and express the exercise trigger by the function $\bar{p}(X)$. In this formulation, the firm's investment strategy can be characterized in the following way. At any time t, given the current value of the cash flow process $X(t)$, the firm compares its beliefs $p(t)$ with the boundary level $\bar{p}(X(t))$ and invests at the first instant when $p(t)$ becomes lower than $\bar{p}(X(t))$.

The quantitative effects of the addition of Brownian uncertainty to the model are illustrated in Fig. 2, which shows the investment trigger function $\bar{X}(p)$ for different values

[16] In general, $\bar{X}(p)$ and $G(X,p)$ can be computed numerically. In the appendix we also derive the closed-form solution for the special case of $\sigma = 0$.

[17] In order to ensure optimality of the exercise trigger, $\bar{X}(p)$, $G(X,p)$ must be convex at $\bar{X}(p)$. To see this, note that for a given p, $G(X,p) > h(p)X - I$ for all $X < \bar{X}(p)$, where $h(p) = \left[(1-p)\frac{1}{r-\alpha} + p\frac{1+\frac{\lambda_3}{(r-\alpha)(1+\phi)}}{r-\alpha+\lambda_3}\right]$. From the value-matching condition, at $\bar{X}(p)$, $G(\bar{X}(p),p) = h(p)\bar{X}(p) - I$, and from the smooth-pasting condition $G_X(\bar{X}(p),p) = h(p)$. Thus, at $\bar{X}(p)$, it must be the case that $G_{XX}(\bar{X}(p),p) > 0$.

of σ.[18] Importantly, while σ is constant, p is falling over time, so that the Bayesian effect is dynamic and declining over time as learning accumulates. Both Bayesian and Brownian uncertainties lead to a significant increase in the investment trigger. If there were no Brownian or Bayesian uncertainty and all shocks were permanent, the trigger would equal 1. Now, consider the case in which we add only Bayesian uncertainty, corresponding to the bottom curve in which $\sigma = 0$. Consider the case of $p = 2/3$, equaling the value of π_0 for our parameter specification. Here we find that the impact of pure Bayesian uncertainty increases the investment threshold by 12.6%, which is labeled as the "Bayesian effect" in Fig. 2. Now, consider the additional impact of Brownian uncertainty. The middle and top curves correspond to the cases of $\sigma = 0.05$ and $\sigma = 0.10$, respectively. The addition of Brownian uncertainty leads to an *additional* increase in the threshold by 5.6% for the case of $\sigma = .05$ and by 20% for $\sigma = .10$, which are labeled as the "Brownian effects" in Fig. 2.[19] As discussed above, the addition of Brownian uncertainty does not undo the effect that Bayesian learning has on the optimal exercise rule. Indeed, the shape of the trigger function $\bar{X}(p)$ does not change much as the Brownian volatility parameter σ increases. For any σ, $\bar{X}(p)$ is an increasing and concave function of p.

3.3 Optimal Investment prior to the Arrival of a Shock

To complete the solution, consider the value of the investment option before the shock arrives. In this case, the only underlying uncertainty concerns the future path of $X(t)$. Specifically, $X(t)$ evolves according to

$$dX(t) = \alpha X(t)\,dt + \sigma X(t)\,dB(t) + \varphi X(t)\left(dM_1(t) + dM_2(t)\right), \tag{40}$$

where $M_1(t)$ and $M_2(t)$ are arrival processes of permanent and temporary shocks, respectively.

Denote the option value by $F(X)$, where X is the current value of the state variable. Prior to the investment, $F(X)$ solves[20]

$$(r + \lambda_1 + \lambda_2) F(X) = \alpha X F'(X) + \frac{1}{2}\sigma^2 X^2 F''(X) + (\lambda_1 + \lambda_2) G(X(1+\varphi), \pi_0), \tag{41}$$

where $G(X, p)$ is the value of the investment option while the shock persists. Because the option has only one state variable, the optimal trigger will be a fixed value, $\hat{X}$.

Since in this region, Bayesian learning plays no role, we relegate the derivation of $\hat{X}$ to the appendix. The optimal investment trigger prior to the arrival of the shock is

[18] To compute the trigger functions we used a variation of the least-squares method developed by Longstaff and Schwartz (2003). The procedure is outlined in the appendix.

[19] Note that this implies that the total instantaneous volatility of the cash flow process is higher than σ, since it also includes volatility due to the jumps.

[20] As before, Eq. (40) describes the dynamics of $X(t)$ under $(\mathcal{F}_t)$. However, in terms of valuation, it is immaterial, because $\mathbb{E}\left[dM_1(t) + dM_2(t)\,|\mathcal{F}_t\right] = \mathbb{E}\left[dM_1(t) + dM_2(t)\,|\mathcal{G}_t\right] = (\lambda_1 + \lambda_2)\,dt$

$$\hat{X} = \frac{\gamma_1}{\gamma_1 - 1} \frac{(r - \alpha + \lambda_1 + \lambda_2)\, rI}{r + \lambda_1 + \lambda_2} + \frac{\gamma_1 - \gamma_2}{\gamma_1 - 1} A_2 \hat{X}^{\gamma_2}, \tag{42}$$

where A_2 is defined in Eq. (88) in the appendix, and where γ_1 and γ_2 are the positive and negative roots of the fundamental quadratic equation $\frac{1}{2}\sigma^2\gamma(\gamma - 1) + \alpha\gamma = r + \lambda_1 + \lambda_2$. The corresponding value of the investment option $F(X)$ is shown in the appendix.

We can now fully summarize the optimal investment strategy in the following proposition:

Proposition 1. *The optimal investment strategy for the model with both Bayesian and Brownian uncertainty is:*

1. *If the shock does not occur until $X(t)$ reaches $\hat{X}$, then it is optimal for the firm to invest when $X(t) = \hat{X}$;*

2. *If the shock occurs before $X(t)$ reaches $\hat{X}$, and at the time of the shock $X(t)$ is above $\bar{X}(\pi_0)/(1+\varphi)$, then it is optimal for the firm to invest immediately after the shock;*

3. *If the shock occurs before $X(t)$ reaches $\hat{X}$, and at the time of the shock $X(t)$ is below $\bar{X}(\pi_0)/(1+\varphi)$, then it is optimal for the firm to invest at the first time when $X(t) = \bar{X}(p(t))$;*

4. *If the shock occurs before $X(t)$ reaches $\hat{X}$, at the time of the shock $X(t)$ is below $\bar{X}(\pi_0)/(1+\varphi)$, and it reverts back before $X(t)$ reaches $\bar{X}(p(t))$, then it is optimal to invest when $X(t) = X^*$.*

3.4 Discussion

Proposition 1 demonstrates that there are four different scenarios for the firm's investment timing. Under the first scenario, the cash flow process $X(t)$ increases up to $\hat{X}$ before the jump occurs. There is no learning prior to the arrival of the shock, so the investment trigger is constant. One simulated sample path that satisfies this scenario is shown in the upper left corner of Fig. 3.

Under the other three possible scenarios, the shock occurs before the cash flow process reaches $\hat{X}$, so the firm does not invest prior to the arrival of the shock. After the shock arrives, the investment trigger is a function of the firm's beliefs about the past shock given by (39). Thus, as time passes and the shock persists, the firm learns more about the nature of the past shock. When the firm observes that the shock does not revert back, it lowers its assessment of the probability that the past shock was temporary. Since immediate investment is more attractive when the past shock is permanent, the investment trigger $\bar{X}(p(t))$ decreases over time.

Under the second scenario, the value of the cash flow process immediately after the shock overshoots $\bar{X}(\pi_0)$. In this case, the firm invests immediately after the shock arrives. A sample path satisfying this scenario is shown in the upper right corner of Fig. 3.

Under the third scenario, the cash flow process $X(t)$ reaches the investment trigger $\bar{X}(p(t))$ following a period of learning, but before the shock reverts. A simulated sample path illustrating this scenario is shown in the lower left corner of Fig. 3. This graph illustrates several interesting properties of investment in our Bayesian setting such as the dependence on the timing of past shocks and the sluggish response of investment to shocks. We discuss these and other implications in more details in the next section.

Finally, under the fourth scenario, the past temporary shock reverts back before the firm invests. If this happens, the problem becomes standard. After the reversal, the investment trigger is constant at the level X^*. The firm invests at the first time when the cash flow process reaches X^*. A simulated sample path that describes the fourth scenario is shown in the lower right corner of Fig. 3.

4 Model Implications

In this section we analyze some important implications of the model.

4.1 Bayesian Uncertainty and the Valuable Option to Learn

In traditional real options models focusing on uncertainty about future shocks only, investment takes place when cash flows rise to a constant trigger. This is due to the fact that uncertainty is entirely driven by the Brownian component, which is constant over time. This trigger embodies the notion of the traditional option to wait as investment is triggered when the real option is sufficiently in the money.

As we add the Bayesian uncertainty over past shocks, the investment trigger is no longer constant, but is a function of the firm's beliefs about the nature of past shocks. More specifically, the trigger is high soon after the arrival of a shock, but then goes down as time goes by. Thus, in the Bayesian framework, uncertainty is no longer constant, but declines over time as the firm becomes progressively more confident in a past shock's permanence. Hence, in addition to the standard option to wait, we now also have a valuable option to learn. We see in Eq. (39) that the investment trigger exceeds the Marshallian trigger of rI due to two forces: the option to learn (waiting until a moment later and seeing if a past shock reverses) and the option to wait (and benefit from cash flows rising over the next moment). As demonstrated in Fig. 2, these forces are additive and appear to be of similar magnitudes. Ignoring either option would lead to a similar investment timing error.

An important implication of Bayesian uncertainty over past shocks is the sluggishness in responding to positive cash flow shocks. In traditional real options models, in which uncertainty is about future shocks only, investment responds immediately to past cash flow shocks. In contrast, uncertainty about past shocks rationalizes sluggish response of investment to shocks. Specifically, the firm might find it optimal to wait an additional period of time and see if the past positive shock is indeed permanent. Given this, the firm is now less likely to invest soon after the arrival of a positive shock, where the option to learn is the most valuable. Then, as time goes by, uncertainty is reduced, and the firm is willing to invest at a lower level of the cash flow process.[21]

The slow response of investment to cash flow shocks generated by the option to learn is consistent with the slow response of economic variables such as investment, labor demand and prices to shocks (e.g., Caballero and Engel (2004)). Another stylized fact which is consistent with Bayesian learning about past shocks is that the response of investment to shocks is time-dependent. In particular, aggregate investment is shown to be more responsive to shocks if the period preceding expansion is longer (Caballero et al. (1995), Bachmann et al. (2008)). Our argument appears consistent with this evidence since firms are likely to believe that positive shocks following the period of preceding expansion are more likely to be permanent, so the option to learn is less valuable in this case. The result that uncertainty about the identity of the shock can lead to sluggish response of investment to shocks is very general and can hold in different environments with shocks of different types. For example, Moore and Schaller (2002) show in simulations that uncertainty about the persistence of interest rate shocks can lead to the sluggishness of investment in the context of the neoclassical q-theory.

The magnitude of the slow response of investment to cash flow shocks depends on two key parameters: the size of the shock φ and the speed of learning λ_3. Papers that calibrate models of investment decisions usually assume that temporary shocks are relatively persistent. For example, Thomas (2002) and Gourio and Kashyap (2007) assume that a productivity shock follows an AR(1) process with coefficient 0.9225. In the context of our model this would mean that learning about the identity of a shock would be rather slow. This suggests that incorporating uncertainty about whether shocks are permanent or temporary into the models of aggregate investment can have important quantitative effects.

4.2 Investment in the Face of Stable or Declining Cash Flows

Another interesting property of our Bayesian model is that investment may occur in the face of stable or even decreasing cash flows. In standard real options models, investment will be triggered when shocks increase the cash flow process up to a constant upper threshold. Because of this, the firm can rationally exercise the investment option only at the times of increasing cash flows. In contrast, when the traditional real options model

[21] See Bloom (2009) for the evidence that such uncertainty shocks as the 1973 oil price shock have large aggregate effects due to more "cautious" behavior by economic agents.

is extended to include Bayesian uncertainty over the nature of past shocks, the firm may invest at the times of stable or even declining cash flows simply because it becomes sufficiently confident in the permanence of the past shocks.

As an illustration of this property, consider a particular sample path of cash flows and the investment response presented in Fig. 4. We see that investment is made at a time of steady cash flows. At the moment of investment, it was the reduction in Bayesian uncertainty rather than the increase in current cash flows that triggered investment. This is made possible by the monotonically decreasing trigger function, $\bar{X}(p(t))$.

Empirically, we certainly see examples of firms (and industries) investing in markets where cash flows are stable, or even declining. As detailed in Grenadier (1996), during the late 1970's and early 1980's several U.S. cities saw explosive growth in office building development in the face of rapidly increasing office vacancy rates. Specifically, consider the cases of the Denver and Houston office markets. Over the thirty-year period from 1960 through 1990, over half of all office construction was completed in a four-year interval: 1982-1985. Since office space takes an estimated average length of time between the initiation and completion of construction of 2.5 years, this investment was likely initiated over the period from 1979-1983. Throughout this period, office vacancies in these two cities were around 30%, considerably above previous levels. Notably, these two cities (as well as most of the cities experiencing unprecedented office growth during this period) were oil-patch cities where developers likely concluded (incorrectly) that high oil prices in the late 1970's and early 1980's would last indefinitely, as discussed in the quote provided in the introduction. In other words, our model explains this period of rapid construction in times of declining cash flows by saying that the firms became very sure that high oil prices, and hence, high rents in these oil cities would last forever.

Fig. 4 also makes evident that accounting for uncertainty about past shocks may lead to a failure of the "record-setting news principle." The record-setting news principle is that investment occurs only at instants in which the value of the cash flow process is the highest in its whole history. In Fig. 4 we see that investment occurs at a cash flow level that is more than 5% lower than its previous maximum. While the record-setting news principle holds for a large class of real options models (Boyarchenko (2004))[22], the addition of Bayesian uncertainty over the nature of past shocks can overturn this result. Thus, even when current cash flows are not at their all-time maximum, the decline in uncertainty about past shocks can trigger investment.

[22]The record-setting news principle, a term credited to Maxwell Stinchcombe, is usually formulated in terms of asset values, rather than cash flows. Because the model is defined in cash flows, we focus on the cash flow analogue of the record-setting news principle. The two definitions are equivalent if the cash flow process is the only state variable and the asset value is a monotone function of the current value of the cash flow process.

4.3 Investment Strategy and the Maturity Structure of the Project's Cash Flows

In the standard real options literature, there is a simple equivalence between options that pay off in a stream of cash flows and those that pay off with an identical lump sum value. Thus, the timing of cash flows, per se, does not matter for the firm's investment strategy.[23] Ultimately, at exercise the firm chooses a trigger such that the value of future cash flows at exercise provides the needed option premium to induce investment. For example, suppose that the project's cash flows are driven solely by Brownian uncertainty about future shocks, and the maturity structure of these cash flows is relatively short. Suppose that for this option on a short-maturity project the firm chooses to exercise when the value of the future cash flows exceeds the investment cost by 50%, a premium consistent with the value of the option to wait. Then, it is also true that the optimal exercise policy for an option on a project with a long-term cash flow maturity structure but the same other characteristics will be one where investment is triggered when the value of future cash flows exceeds the investment cost by 50%. For example, Chapter 5 in Dixit and Pindyck (1994) considers the optimal exercise rule for options that pay off with a lump sum value of $V(t)$. Then, in Chapter 6, they perform an analogous analysis for options that pay off with a perpetuity cash flow of $P(t)$, with identical present value to the lump sum value $V(t)$. They show that the optimal exercise rules are identical.

However, in the context of valuations that are driven by uncertainty of past shocks, the timing of cash flows can be quite important in determining the exercise strategy. The greater is the "front-loadedness" of the project's cash flows, the less important is the assessment of the relative likelihood that a shock is temporary or permanent. As a result, the "option to learn" is less valuable, so the firm invests at a lower trigger. Thus, while the firm might invest in a short-maturity project when the value of the future cash flows exceeds the investment cost by 50%, the firm might only invest in a longer-maturity project if the value of the future cash flows exceeds the investment cost by 100%. This is because the option to learn is fundamentally linked to the cash flow structure of the project.

In this section we consider a simple parametrization of the front-loadedness of the option payoff, ranging from payoffs that are equivalent to a one-time lump sum payment to payoffs that are equivalent to perpetual cash flows. Consider the simplest case studied in Section 2.3, with one alteration. Assume now that if an option is exercised at time τ, it provides a stream of payments $\left(1+\frac{w}{r}\right)e^{-w(t-\tau)}X(t)$, $t \geq \tau$. Parameter $w \in [0, +\infty)$ captures the degree of front-loadedness of the project's cash flows. Projects with low

[23] When we refer to the standard real options literature, we refer to models in which there is an initial lump-sum cost followed by a single lump-sum cash flow or a sequence of positive cash flows, and in which investment is irreversible. However, in a broader class of real options, in which costs could be paid over time and partial reversibility is possible, the timing of the project's cash flows could matter for investment.

values of w are relatively back-loaded: much of their cash flows are generated long after the exercise time. High values of w mean that the project is relatively front-loaded, with most cash flows coming close to the exercise time. The particular parametrization was chosen so as to make the present value of cash flows from the immediate exercise of the project in the no-shock case independent of w: $\int_{\tau}^{\infty} X(\tau)\left(1+\frac{w}{r}\right) e^{-w(t-\tau)} e^{-r(t-\tau)} dt = \frac{X(\tau)}{r}$. Of course, other reasonable parameterizations are possible.

This specification of cash flows captures two cases widely used in the real options literature. First, when $w = 0$, the model reduces to the one studied in Section 2.3. In this case, the project pays a perpetual flow of $X(t)$ upon exercise. Second, if $w \to \infty$, payments from the project converge to a one time lumpy payment of $\frac{X(\tau)}{r}$ at the time of exercise τ.

Similar to (13), to ensure that the project has a potentially positive net present value and that there is positive value to learning we make the assumption that X_0 satisfies

$$\frac{rI}{1+\varphi} < X_0 < \frac{r+\lambda_3\pi_0}{1+\varphi+\frac{\lambda_3}{r}\pi_0+\frac{\lambda_3}{r}\pi_0\frac{w\varphi}{(r+\lambda_3+w)}} I. \tag{43}$$

Compared to (13), (43) puts the same lower bound and a more restrictive upper bound. As previously, these bounds guarantee that the solution to the investment timing problem is non-trivial.

As in Section 2.3, let the value of the option while the shock persists be denoted by $G(p)$. Over the range of p at which the option is not exercised, the standard argument implies

$$(r + p\lambda_3)\, G(p) = -G'(p)\lambda_3 p(1-p). \tag{44}$$

This equation has the general solution

$$G(p) = C_w (1-p)\left(\frac{1}{p}-1\right)^{\frac{r}{\lambda_3}}, \tag{45}$$

where C_w is a constant. Note that this general solution coincides with the one obtained in Section 2.3, Eq. (18). However, because of the more general cash flow timing assumption, the boundary conditions are now different.

Because the payoff from the project if the current shock is temporary is affected by the parameter w, the value-matching condition at the exercise trigger $\bar{p}_w$ is now

$$G(\bar{p}_w) = (1-\bar{p}_w)\frac{X_0(1+\varphi)}{r} + \bar{p}_w \frac{X_0\left(1+\varphi+\frac{\lambda_3}{r}+\frac{w(1+\varphi)}{r}\right)}{r+\lambda_3+w} - I. \tag{46}$$

Note that although the present value of the firm's cash flows in case of the permanent jump does not depend on w, in case of the temporary jump it depends on w positively.

Intuitively, a more front-loaded project allows the firm to capture more of the temporary high cash flows than a more back-loaded project.

As in Section 2.3, the exercise trigger is chosen to maximize the value of the option (or equivalently, to satisfy the smooth-pasting condition), giving the resulting optimal trigger value:

$$\bar{p}_w = \frac{X_0 (1+\varphi) - rI}{\lambda_3 \left(I - \frac{X_0}{r} - \frac{X_0 w \varphi}{r(r+\lambda_3+w)} \right)}. \tag{47}$$

Note, when $w = 0$, $\bar{p}_w = \bar{p}$ which equals the trigger we obtained in Section 2.3.

As in Section 2.3, given that there is value to learning, it is straightforward to show that the option will never be exercised prior to the arrival of the shock. Thus, the optimal investment rule is indeed for the firm to invest at the first moment that the firm's belief $p(t)$ falls to the trigger $\bar{p}_w$, and never if the trigger is not reached.

Consider how the parameter of front-loadedness affects the trigger value:

$$\frac{\partial \bar{p}_w}{\partial w} = \frac{r\varphi(r+\lambda_3)}{\lambda_3} \frac{X_0(1+\varphi) - Ir}{\left[X_0 \left(r+\lambda_3 + w(1+\varphi)\right) - Ir\left(r+\lambda_3+w\right)\right]^2} X_0 > 0. \tag{48}$$

We therefore find that the greater the front-loadedness, the lower the exercise trigger.

Now, consider how the parameter of front-loadedness affects the value at exercise. The option premium at the time of exercise is

$$\begin{aligned} \frac{G(\bar{p}_w)}{I} &= \frac{1}{I}\left(1 - \frac{X(1+\varphi)-rI}{\lambda_3\left(I - \frac{X}{r} - \frac{Xw\varphi}{r(r+\lambda_3+w)}\right)}\right)\frac{X(1+\varphi)}{r} \\ &+ \frac{1}{I}\frac{X(1+\varphi)-rI}{\lambda_3\left(I-\frac{X}{r}-\frac{Xw\varphi}{r(r+\lambda_3+w)}\right)}\frac{X\left(1+\varphi+\frac{\lambda_3}{r}+\frac{w(1+\varphi)}{r}\right)}{r+\lambda_3+w} - 1. \end{aligned} \tag{49}$$

Differentiating with respect to w gives us

$$\frac{\partial \left(\frac{G(\bar{p}_w)}{I}\right)}{\partial w} = -\frac{X_0\varphi \left(X_0 (1+\varphi) - rI\right)^2}{I\left(\left(X_0 - rI\right)\left(r+\lambda_3+w\right) + X_0 \varphi w\right)^2} < 0. \tag{50}$$

Thus, the greater the front-loadedness of the project, the lower the option premium necessary to induce exercise. In other words, the firm invests at a lower net present value for projects with lower value of learning.

Intuitively, if the project's cash flows are very front-loaded, then knowing the identity of the shock is not very important for the firm. In this case the firm gets a large part of the project cash flows very soon after the exercise date. Because of that, the learning option is not very valuable, so the firm invests at a lower premium. On the other hand, if the project's cash flows are very back-loaded, then it is very important for the firm to be sure that past shocks are permanent. As a result, the learning option is very valuable and the firm invests at a higher premium.

5 A Model with an Unlimited Number of Shocks

In this section, we analyze our most general version of the model, in which there is an unlimited number of shocks as well as a geometric Brownian motion component.[24] In this case, the evolution of $X(t)$ under $(\mathcal{F}_t)$ is given by (1), where the probability of observing reversal over a next instant depends on the number of outstanding temporary shocks. Given our assumptions, under the information set of the firm, at each time t the state can be described by a pair $(X(t), p(t))$, where $X(t)$ is the value of the cash flow process and $p(t) = (p_0(t)\ p_1(t)\ p_2(t)\ \ ...)'$ is the infinitely dimensional vector of the firm's beliefs. The initial state is $(X(0), p(0)) = (X(0), \mathbf{0})$, and at any time t, $p_i(t) = 0$ for all $i > n(t)$, where $n(t)$ is the total number of outstanding shocks, $n(t) = M_1(t) + M_2(t) - N(t)$. The Bayesian learning process $p(t)$ is derived in Section 2.2.

As before, we begin by calculating the present value of the project's underlying cash flows, defined by (3). In the appendix we show that it is equal to

$$S(X(t), p(t)) = a_0 X(t) + \sum_{i=1}^{\infty} p_i(t)(a_i - a_0) X(t), \tag{51}$$

where constants $a_0, a_1, ...$ are given by (105). Intuitively, if the firm knew for sure that there are exactly k outstanding temporary shocks, then the present value of the project's cash flows would be linear in $X(t)$ and equal to $a_k X(t)$ for some constant a_k. Taking the expectation with respect to the beliefs of the firm yields (51).

Let $G(X, p)$ denote the value of the option to invest, where X and p are the current values of the cash flow and the belief processes, respectively. The optimal investment decision can be described by a trigger function $\bar{X}(p)$. In the appendix we provide the details of the determination of $\bar{X}(p)$ and $G(X, p)$, which is done using the same argument as in Section 3.2. In particular, we show that $\bar{X}(p)$ can be expressed as

$$\begin{aligned} \bar{X}(p) = \left(\lambda_3 \textstyle\sum_{k=1}^{\infty} p_k k\right) \left[G\left(\tfrac{\bar{X}(p)}{1+\varphi}, \tilde{p}(p)\right) + I - S\left(\tfrac{\bar{X}(p)}{1+\varphi}, \tilde{p}(p)\right)\right] \\ + rI + \tfrac{\sigma^2}{2} \bar{X}(p)^2 G_{XX}\left(\bar{X}(p), p\right), \end{aligned} \tag{52}$$

where $\tilde{p}(p)$ is the updated after the reversion vector of beliefs, defined by (9).

Given the similarity between (52) and (39), it becomes clear how the multi-shock case generalizes the single-shock case. As before, the intuition behind the equation for the optimal trigger is the trade-off between investing now versus investing a moment later if all of the past shocks persist, where the value of waiting is explicitly an "option to learn." If the firm invests now, it gets the benefit of the cash flow $\bar{X}(p)$ over the next instant. This is the term on the left-hand side of the equal sign. If the firm waits a moment and

[24] The authors have also solved the model for any finite number of potential shocks. The results are very similar to those presented for the case of a countably infinite number of shocks, but with some additional notational burdens.

invests only if all of the past shocks persist, it faces a small chance of one of the shocks reversing, in which case the expected value gained by waiting is equal to the first term on the right-hand side. Specifically, $G\left(\frac{\bar{X}(p)}{1+\varphi}, \tilde{p}(p)\right) + I - S\left(\frac{\bar{X}(p)}{1+\varphi}, \tilde{p}(p)\right)$ is the value gained by not investing should an existing shock reverse over the next instant, while $\lambda_3 \sum_{i=1}^{\infty} p_i i$ is the likelihood of this occurring. The second term on the right-hand side of the equal sign, rI, is the savings from delaying the investment cost by an instant. The third term is the convexity term, which represents the value of waiting due to Brownian uncertainty. At the optimal trigger, $\bar{X}(p)$, these two sides are exactly equal, and the firm is indifferent between investing now and waiting a moment.

We can now summarize the solution to the optimal investment timing problem in this multi-shock setting.

Proposition 2. *The optimal investment strategy for the model outlined in this section is to invest when $X(t)$ exceeds $\bar{X}(p(t))$ for the first time.*

To provide intuition for the multi-shock extension, Fig. 5 plots a simulated path of the cash flow process $X(t)$ along with the corresponding investment trigger $\bar{X}(p(t))$. Due to computational complexity, for this numerical solution we consider a pure-jump process by setting $\alpha = \sigma = 0$. In this example, each shock increases the value of the cash flows by 5% ($\varphi = 0.05$). A new permanent shock occurs, on average, every two years ($\lambda_1 = 0.5$). A new temporary shock occurs, on average, every year ($\lambda_2 = 1$) and takes six months to revert ($\lambda_3 = 2$). The dynamics of the cash flow process is very straightforward. In the 5-year period there are four arrivals of new shocks, corresponding to upward jumps in $X(t)$. Two out of four shocks revert, corresponding to downward jumps in $X(t)$. The dynamics of the investment boundary is more interesting. Prior to the arrival of the first shock, there is no uncertainty about past shocks. Since there is no Bayesian updating at this time, the trigger is constant at $\bar{X}(\mathbf{0})$. After the first shock arrives, the trigger jumps up to $\bar{X}(\pi_0, 0, ...)$ as the firm becomes unsure if the outstanding shock is permanent or temporary. As time goes by and the cash flow process does not revert back, the first shock is more likely to be permanent. As a result, the firm's beliefs become more optimistic, and the investment trigger decreases. This intuition underlies the whole dynamics of the trigger in Fig. 5. An arrival of a new shock leads to an upward jump in the trigger due to an increase in uncertainty. When the cash flow process is stable, the trigger goes down as the firm updates its beliefs about the past shocks. When an outstanding shock reverses, the firm learns for sure that one of the outstanding shocks is temporary. Hence, reversal leads to a downward jump in the trigger. The investment occurs when $X(t)$ exceeds the investment trigger $\bar{X}(p(t))$ for the first time. In the example in Fig. 5, this happens at $t = 2.7$ when the firm becomes sufficiently sure that the outstanding shock is permanent.

By allowing for multiple shocks, the model of this section illuminates the nature of the Bayesian investment problem beyond that of the simpler one-shock model. Most notably, the exercise trigger, $\bar{X}(p(t))$, is a multi-dimensional function, which demonstrates its dependence on all persisting past shocks. Thus, as illustrated in Fig. 5, the investment trigger jumps by a discrete amount at all moments when either a new jump occurs or an existing shock reverses. In between such jumps, the trigger declines as Bayesian uncertainty is reduced.

6 Conclusion

This paper studies the optimal timing of investment in the presence of uncertainty about both future and past shocks. We augment the standard Brownian uncertainty driving traditional real options models with additional Bayesian uncertainty over distinguishing between the temporary and permanent nature of past cash flow shocks. As a result, the evolving uncertainty is no longer constant and is driven by Bayesian updating, or learning. We show that this gives rise to two real options. In addition to the "waiting" option as in traditional literature, the firm now has a "learning" option: the option to learn more about the nature of past shocks. We solve for the optimal investment rule and show that the implied investment behavior differs significantly from that predicted by standard real options models. In particular, investment often responds sluggishly to cash flow shocks, investment may occur at a time of stable or even decreasing cash flows, and the optimal investment timing critically depends on the maturity structure of the project's cash flows.

Several further extensions of the model would prove interesting. First, the model could be extended to a multiple firm industry equilibrium in the manner of Grenadier (2002). Of particular interest would be the fact that firms need to update their beliefs about not only their own past shocks, but also those of their competitors. Thus, while competition is known to erode the option to wait, the benefits from waiting to learn could mitigate such erosion. Second, the structure of the jump component could be made richer by permitting more than just either permanent or temporary shocks. For example, shocks could have a greater variety of (unobserved) types, such as n types with reversion intensities $\lambda_1^{reversion}, \lambda_2^{reversion}, ..., \lambda_n^{reversion}$.

References

[1] Bachmann, Ruediger, Ricardo J. Caballero, and Eduardo M.R.A. Engel (2008). Aggregate implications of lumpy investment: new evidence and a DSGE model, Working Paper.

[2] Bergemann, Dirk, and Ulrich Hege (1998). Venture capital financing, moral hazard, and learning, *Journal of Banking and Finance* 22, 703-735.

[3] Bloom, Nicholas (2009). The impact of uncertainty shocks, *Econometrica* 77, 623-685.

[4] Bolton, Patrick, and Christopher Harris (1999). Strategic experimentation, *Econometrica* 67, 349-374.

[5] Boyarchenko, Svetlana (2004). Irreversible decisions and record-setting news principle, *American Economic Review* 94, 557-568.

[6] Brennan, Michael J., and Eduardo S. Schwartz (1985). Evaluating natural resource investments, *Journal of Business* 58, 135-157.

[7] Caballero, Ricardo J., and Eduardo M.R.A. Engel (2004). Adjustment is much slower than you think, Working Paper.

[8] Caballero, Ricardo J., Eduardo M.R.A. Engel, and John C. Haltiwanger (1995). Plant-level adjustment and aggregate investment dynamics, *Brookings Papers on Economic Activity*, No. 2, pp. 1-54.

[9] Décamps, Jean-Paul, Thomas Mariotti, and Stephane Villeneuve (2005). Investment timing under incomplete information, *Mathematics of Operations Research* 30, 472-500.

[10] Dixit, Avinash K., and Robert S. Pindyck (1994). *Investment under Uncertainty*, Princeton University Press, Princeton, NJ.

[11] Dumas, Bernard (1991). Super contact and related optimality conditions, *Journal of Economic Dynamics and Control* 15, 675-685.

[12] Gorbenko, Alexander S., and Ilya A. Strebulaev (2009). Temporary vs permanent shocks: explaining corporate financial policies, Working Paper.

[13] Gourio, Francois, and Anil K. Kashyap (2007). Investment spikes: new facts and a general equilibrium exploration, *Journal of Monetary Economics*, 54, 1-22.

[14] Grenadier, Steven R. (1996). The strategic exercise of options: development cascades and overbuilding in real estate markets, *Journal of Finance* 56, 1653-1679.

[15] Grenadier, Steven R. (2002). Option exercise games: an application to the equilibrium investment strategies of firms, *Review of Financial Studies* 15, 691-721.

[16] Grenadier, Steven R., and Neng Wang (2005). Investment timing, agency, and information, *Journal of Financial Economics* 75, 493-533.

[17] Grenadier, Steven R., and Neng Wang (2007). Investment under uncertainty and time-inconsistent preferences, *Journal of Financial Economics* 84, 2-39.

[18] Jarrow, Robert A., and Fan Yu (2001). Counterparty risk and the pricing of defaultable securities, *Journal of Finance* 56, 1765-1799.

[19] Keller, Godfrey, and Sven Rady (1999). Optimal experimentation in a changing environment, *Review of Economic Studies* 66, 475-507.

[20] Keller, Godfrey, Sven Rady, and Martin Cripps (2005). Strategic experimentation with exponential bandits, *Econometrica* 73, 39-68.

[21] Klein, Manuel (2007). Irreversible investment under incomplete information, Working Paper.

[22] Klein, Manuel (2009). Comment on "Investment timing under incomplete information," *Mathematics of Operations Research* 35, 249-254.

[23] Krylov, Nicolai V. (1980). *Controlled Diffusion Processes*, Springer, Berlin.

[24] Lambrecht, Bart M., and William Perraudin (2003). Real options and preemption under incomplete information, *Journal of Economic Dynamics and Control* 27, 619-643.

[25] Longstaff, Francis A., and Eduardo S. Schwartz (2003). Valuing American options by simulation: a simple least-squares approach, *Review of Financial Studies* 14, 113-147.

[26] Majd, Saman, and Robert S. Pindyck (1987). Time to build, option value, and investment decisions, *Journal of Financial Economics* 18, 7-27.

[27] Malueg, David A., and Shunichi O. Tsutsui (1997). Dynamic R&D Competition with learning, *Rand Journal of Economics* 28, 751-772.

[28] McDonald, Robert, and David Siegel (1986). The value of waiting to invest, *Quarterly Journal of Economics* 101, 707-728.

[29] Miao, Jianjun, and Neng Wang (2007). Experimentation under uninsurable idiosyncratic risk: an application to entrepreneurial survival, Working Paper.

[30] Moore, Bartholomew, and Huntley Schaller (2002). Persistent and transitory shocks, learning, and investment dynamics, *Journal of Money, Credit, and Banking* 34, 650-677.

[31] Moscarini, Giuseppe, and Lones Smith (2001). The optimal level of experimentation, *Econometrica* 69, 1629-1644.

[32] Nishimura, Kiyohiko G., and Hiroyuki Ozaki (2007). Irreversible investment and Knightian uncertainty, *Journal of Economic Theory* 136, 668-694.

[33] Novy-Marx, Robert (2007). An equilibrium model of investment under uncertainty, *Review of Financial Studies* 20, 1461-1502.

[34] Peskir, Goran, and Albert Shiryaev (2006). *Optimal Stopping and Free-Boundary Problems*, Lectures in Mathematics, ETH Zurich, Birkhauser.

[35] Protter, Philip E. (2004). *Stochastic Integration and Differential Equations*, 2nd ed., Springer-Verlag, New York, NY.

[36] Shreve, Steven E. (2004). *Stochastic Calculus for Finance II: Continuous-Time Models,* Springer-Verlag, New York, NY.

[37] Thomas, Julia K. (2002). Is lumpy investment relevant for the business cycle, *Journal of Political Economy*, 110, 508-534.

[38] Titman, Sheridan (1985). Urban land prices under uncertainty, *American Economic Review* 75, 505-514.

[39] Williams, Joseph T. (1991). Real estate development as an option, *Journal of Real Estate Finance and Economics* 4, 191-208.

Appendix

Proofs and Derivations

Proof that in the simple case of Section 2.3, the option will never be exercised prior to the arrival of the shock.

First, we will demonstrate that the strategy of exercising prior to the arrival of the shock is dominated by the strategy of investing immediately upon the arrival of the shock. If the firm invests prior to the arrival of the shock, it pays the cost I, and obtains the cash flow of X_0 until the arrival of the shock. Upon the arrival of the shock, it obtains the additional cash flow of φX_0 until the time (if ever) that the shock reverses. If the firm instead waits until the shock arrives and then invests immediately, it gets no cash flow until the shock arrives, pays the cost I when the shock arrives, and obtains the cash flow of φX_0 until the time (if ever) that the shock reverses.

Let NB denote the net benefit of investing prior to the shock versus waiting until the shock arrives. Let τ denote the random arrival time of the shock. Thus we can write NB as:

$$NB = \mathbb{E}\left[\int_0^{\tau} X_0 e^{-rt} dt\right] + \left(\mathbb{E}\left[e^{-r\tau}\right] I - I\right) = \left(\frac{X_0}{r} - I\right)\left(1 - \mathbb{E}\left[e^{-r\tau}\right]\right). \qquad (53)$$

where NB is thus equal to the value of the cash flow X_0 until the arrival of the shock (the first term), plus the cost of paying I immediately rather than at the time of the shock's arrival (the second term).

The upper bound on X_0 in (13) implies

$$X_0 - rI < \frac{r + \lambda_3\pi_0 - r - r\varphi - \lambda_3\pi_0}{1 + \varphi + \frac{\lambda_3}{r}\pi_0} I = -\frac{r\varphi}{1 + \varphi + \frac{\lambda_3}{r}\pi_0} I < 0. \qquad (54)$$

Thus $NB < 0$.

We have proven that it is always better to wait until the shock arrives and invest immediately at that time then investing prior to the shock. In addition, once the shock arrives, it is always optimal to wait further until $p(t)$ falls to the threshold $\bar{p}$ before exercising. Thus, investing prior to the arrival of the shock is dominated by the strategy of waiting until the shock arrives and then waiting until $p(t)$ falls to the threshold $\bar{p}$ before exercising.

Smooth-pasting conditions in Eqs. (37) and (38).

Let $\mathfrak{M}$ denote the set of $(\mathcal{G}_t)$-stopping times and $\tau \in \mathfrak{M}$ be an element of this set. By definition, the value of the option $G(X(t), p(t))$ satisfies

$$\begin{aligned} G(X(t), p(t)) = \sup\nolimits_{\tau\in\mathfrak{M}} \mathbb{E}\Big[\textstyle\int_t^{\tau} e^{-r(s-t)}\lambda_3 p(s)\, e^{-\int_t^s \lambda_3 p(u)du} H\left(\frac{X(s)}{1+\varphi}\right) ds \\ + e^{-\int_t^{\tau}\lambda_3 p(s)ds} e^{-r(\tau-t)} \left(S(X(\tau), p(\tau)) - I\right) |\mathcal{G}_t\Big]. \end{aligned} \qquad (55)$$

The right-hand side of (55) consists of two terms. The first term corresponds to the payoff if the shock reverts before the firm invests. The second term corresponds to the payoff if the firm invests before the shock reverses. Let

$$D\left(X\left(t\right),p\left(t\right)\right)\equiv\mathbb{E}\left[\int_{t}^{+\infty}e^{-r(s-t)}\lambda_{3}p\left(s\right)e^{-\int_{t}^{s}\lambda_{3}p(u)du}H\left(\frac{X\left(s\right)}{1+\varphi}\right)ds|\mathcal{G}_{t}\right],\tag{56}$$

$$d\left(p\left(t\right)\right)\equiv\int_{t_{0}}^{t}\left(r+\lambda_{3}p\left(s\right)\right)ds,\tag{57}$$

where t_0 is the time when the shock arrives. Then, for any $X\left(t\right)>0$ and $p\left(t\right)\in\left(0,\pi_{0}\right]$, we can rewrite (55) as

$$\begin{aligned}&e^{-d(p(t))}\left(G\left(X\left(t\right),p\left(t\right)\right)-D\left(X\left(t\right),p\left(t\right)\right)\right)\\&=\sup\nolimits_{\tau\in\mathfrak{M}}\mathbb{E}\left[e^{-d(p(\tau))}\left(S\left(X\left(\tau\right),p\left(\tau\right)\right)-I-D\left(X\left(\tau\right),p\left(\tau\right)\right)\right)|\mathcal{G}_{t}\right]\end{aligned}\tag{58}$$

Because $e^{-d(p(\tau))}\left(S\left(X\left(\tau\right),p\left(\tau\right)\right)-I-D\left(X\left(\tau\right),p\left(\tau\right)\right)\right)$ is C^1 everywhere and the dependence of $\left(X\left(t\right),p\left(t\right)\right)$ on any initial point (X,p) is explicit and smooth, the problem satisfies the smooth-fit principle (Peskir and Shiryaev (2006)).[25] Therefore, at all points $\left(\bar{X}\left(p\right),p\right)$ the derivatives of $e^{-d(p)}\left(G\left(X,p\right)-D\left(X,p\right)\right)$ and $e^{-d(p)}\left(S\left(X,p\right)-I-D\left(X,p\right)\right)$ with respect to X and p must be the same. Taking the two derivatives, we obtain

$$G_{X}\left(\bar{X}\left(p\right),p\right)=S_{X}\left(\bar{X}\left(p\right),p\right),\tag{59}$$

$$G_{p}\left(\bar{X}\left(p\right),p\right)=S_{p}\left(\bar{X}\left(p\right),p\right).\tag{60}$$

Proof of the existence of a solution to the fixed-point problem in Eq. (39). Note that a solution $\left(G(X,p),\bar{X}(p)\right)$ to Eqs. (35) - (38) will satisfy (39). From the above we know that if $\bar{X}\left(p\right)$ exists, then it satisfies (35) - (38). We thus need to demonstrate the existence of a boundary between an exercise region and a continuation region for any $p\in\left(0,\pi_{0}\right]$.

Consider state (X,p), and suppose that the firm exogenously learns the type of the outstanding shock in a moment. Then, the value of the investment option just before the firm learns the type of the shock is equal to $\left(1-p\right)G\left(X,0\right)+pG\left(X,1\right)$. Because the option value is convex in p,

$$\left(1-p\right)G\left(X,0\right)+pG\left(X,1\right)\geq G\left(X,p\right).\tag{61}$$

This can be shown by contradiction. If (61) did not hold, there would exist a stopping strategy τ such that the firm gets more in expectation from the uninformed situation

[25] On p. 152 Peskir and Shiryaev (2006) prove this result for one-dimensional problems, but state that it also extends higher dimensions (see p. 150). See also p. 144, where they state that the smooth-fit principle holds in multiple dimensions if the state process after starting at the boundary of the stopping region enters the interior of the stopping region immediately.

compared to the case when it learns the type of the shock in a moment. However, for any sample path $\{X(t), T_3\}$, where T_3 is the time at which the shock reverts ($T_3 \equiv \infty$, if the shock is permanent), the firm can also exercise the option according to τ in the informed case, where it learns the type of the shock in a moment. This is because if τ is adapted to the information set, in which the firm learns the type of the shock when it reverts, then it is also adapted to the information set, in which the firm learns the type of the shock immediately.

Now, consider $G(X,0)$ and $G(X,1)$. There is no learning when $p \in \{0,1\}$, so the option pricing problems are standard. The exercise boundary for $p = 0$ is given by $\bar{X}(0) = X^*$. To solve for $\bar{X}(1)$, note that in the range $p = 1$, $X < (1+\varphi) X^*$, Eq. (35) is solved by

$$G(X,1) = \tilde{C} X^{\gamma} + \left(\frac{X}{(1+\varphi) X^*}\right)^{\beta} \left(\frac{X^*}{r-\alpha} - I\right), \tag{62}$$

where

$$\gamma = \frac{1}{\sigma^2}\left[-\left(\alpha - \frac{\sigma^2}{2}\right) + \sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2(r+\lambda_3)\sigma^2}\right] > \beta > 1. \tag{63}$$

The unknown constant $\tilde{C}$ and the exercise boundary $\bar{X}(1)$ are given by the boundary conditions (36) - (37). Simplifying, we obtain an implicit solution for $\bar{X}(1)$:

$$\frac{\gamma-1}{\gamma}\frac{1+\frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r-\alpha+\lambda_3}\bar{X}(1) = I + \frac{\gamma-\beta}{\gamma}\left(\frac{\bar{X}(1)}{(1+\varphi) X^*}\right)^{\beta}\left(\frac{X^*}{r-\alpha} - I\right). \tag{64}$$

Define

$$Q(x) = I + \frac{\gamma-\beta}{\gamma}\left(\frac{x}{(1+\varphi) X^*}\right)^{\beta}\left(\frac{X^*}{r-\alpha} - I\right) - \frac{\gamma-1}{\gamma}\left(\frac{1+\frac{\lambda_3}{(r-\alpha)(1+\varphi)}}{r-\alpha+\lambda_3}x\right). \tag{65}$$

We have

$$\begin{gathered} Q(0) = I > 0, \\ Q((1+\varphi) X^*) = -\tfrac{\gamma-1}{\gamma}\tfrac{\varphi}{r-\alpha+\lambda_3}X^* < 0, \\ Q''(x) = \beta(\beta-1)\tfrac{\gamma-\beta}{\gamma}\left(\tfrac{1}{(1+\varphi)X^*}\right)^{\beta}\left(\tfrac{X^*}{r-\alpha} - I\right)x^{\beta-1} > 0. \end{gathered} \tag{66}$$

Hence, there exists a unique point between 0 and $(1+\varphi) X^*$, at which $Q(x) = 0$. It satisfies the sufficient conditions (Dixit and Pindyck (1994)), so it is equal to $\bar{X}(1)$.

Because both $\bar{X}(0)$ and $\bar{X}(1)$ are below $(1+\varphi) X^*$, for any $X > (1+\varphi) X^*$, $G(X,0) = \frac{1}{r-\alpha}X - I$ and $G(X,1) = \frac{1+\frac{\lambda_3}{(1+\varphi)(r-\alpha)}}{r+\lambda_3-\alpha}X - I$. Thus, for any $X > (1+\varphi) X^*$ and $p \in (0, \pi_0]$,

$$(1-p) G(X,0) + pG(X,1) = S(X,p) - I. \tag{67}$$

By definition of $G(X,p)$,

$$G(X,p) \geq S(X,p) - I. \tag{68}$$

Combining Eqs. (61), (67), and (68), we find that for any $p \in (0, \pi_0]$ and any $X > (1+\varphi) X^*$, $G(X,p) = S(X,p) - I$, and thus it is always optimal to exercise the investment option. Also, for any $p \in (0, \pi_0]$ and any $X < (r-\alpha) I$, it is always optimal not to exercise the option, as the payoff from the exercise is negative in this range. Therefore, for any $p \in (0, \pi_0]$, there is a boundary between exercise and continuation regions.

Proof of Section 3.2 conjecture that $\bar{X}(p) < (1+\varphi)X^*$.

From the proof of the existence of a boundary $\bar{X}(p)$, it follows that it is always optimal to exercise the option whenever $X > (1+\varphi) X^*$. Therefore, for any $p \in (0, \pi_0]$, $\bar{X}(p)$ must be below $(1+\varphi) X^*$.

Closed form solutions for the model of Section 3.2 when $\alpha > 0$ and $\sigma = 0$.

While the exercise trigger $\bar{p}(X)$ is characterized by (39), it is not solvable in closed-form, since the value function $G(X,p)$ itself is not available in closed-form. However, for the special case in which $\sigma = 0$, $\alpha \geq 0$, the closed-form solution for the trigger is

$$\bar{p}(X)|_{\sigma=0} = \frac{X - rI}{\lambda_3 \left[I + \left(\frac{X}{rI(1+\varphi)} \right)^{\frac{r}{\alpha}} \frac{\alpha I}{r-\alpha} - \frac{X}{(1+\varphi)(r-\alpha)} \right]}. \tag{69}$$

The corresponding value of the investment option equals

$$G(X,p)|_{\sigma=0} = p \frac{\alpha I}{r-\alpha} \left(\frac{X}{(1+\varphi) rI} \right)^{\frac{r}{\alpha}} + (1-p) X^{\frac{r}{\alpha}} \Gamma \left(X \left(\frac{1}{p} - 1 \right)^{-\frac{\alpha}{\lambda_3}} \right), \tag{70}$$

where

$$\begin{aligned} \Gamma(y) = {} & \left(\frac{\lambda_1}{\lambda_2} \right)^{\frac{\alpha}{\lambda_3}} y \left(\frac{\lambda_1}{r-\alpha} e^{(\alpha - r) t^* \left(\left(\frac{\lambda_1}{\lambda_2} \right)^{\frac{\alpha}{\lambda_3}} y \right)} + \lambda_2 \frac{1 + \varphi + \frac{\lambda_3}{r-\alpha}}{(r+\lambda_3-\alpha)(1+\varphi)} e^{(\alpha - r - \lambda_3) t^* \left(\left(\frac{\lambda_1}{\lambda_2} \right)^{\frac{\alpha}{\lambda_3}} y \right)} \right) \\ & - \left(\frac{y}{(1+\varphi) rI} \right)^{\frac{r}{\alpha}} \frac{\alpha I}{r-\alpha} \left(\frac{\lambda_1}{\lambda_2} \right)^{\frac{r}{\lambda_3}} \lambda_2 e^{-\lambda_3 t^* \left(\left(\frac{\lambda_1}{\lambda_2} \right)^{\frac{\alpha}{\lambda_3}} y \right)}, \end{aligned} \tag{71}$$

where $t^*(z)$ is a function defined implicitly by

$$\frac{\lambda_2 \lambda_3}{\lambda_1 e^{\lambda_3 t^*} + \lambda_2} = \frac{z e^{\alpha t^*} - rI}{\left(\frac{z}{(1+\varphi) rI} \right)^{\frac{r}{\alpha}} \frac{\alpha I e^{rt^*}}{r-\alpha} + I - \frac{z e^{\alpha t^*}}{(r-\alpha)(1+\varphi)}}. \tag{72}$$

Notice that when $\alpha = 0$, investment does not occur when the jump reverts. Therefore, in this case, the trigger (69) coincides with (21)[26].

Derivation of the investment trigger $\hat{X}$ and the investment option value $F(X)$ for the model with Brownian uncertainty in Section 3.3.

[26]Note that since in (21) X_0 denotes the level of $X(t)$ before the positive jump occurred, we need to use $\bar{p}(X_0(1+\varphi))$ to ensure equivalence.

In the range before investment, $F(X)$ solves

$$(r+\lambda_1+\lambda_2)\,F(X)=\alpha XF'(X)+\frac{1}{2}\sigma^2X^2F''(X)+(\lambda_1+\lambda_2)\,G((1+\varphi)\,X,\pi_0)\,, \quad (73)$$

where $G(X,p)$ is the value of the investment option while the shock persists. If the investment option is exercised prior to the arrival of a shock, the firm receives:

$$S(X)-I=\frac{X}{r-\alpha}+\frac{\varphi X}{r-\alpha+\lambda_1+\lambda_2}\left(\frac{\lambda_1}{r-\alpha}+\frac{\lambda_2}{r-\alpha+\lambda_3}\right)-I. \quad (74)$$

The first term of (74) is the discounted cash flows that the firm gets if the jump never occurs, while the last term is the investment cost that the firm needs to incur to launch the project. The two other terms correspond to the additional cash flows the firm gets from the shocks. If the shock is permanent, the firm gets additional expected cash flows of $\frac{\varphi Xe^{\alpha\tau}}{(r-\alpha)}$ at the time of the shock τ. If the shock turns out to be temporary, the additional expected cash flows at the time of the shock τ are $\frac{\varphi Xe^{\alpha\tau}}{(r-\alpha+\lambda_3)}$. Integrating over τ yields the second and third terms of (74).

Let $\hat{X}$ denote the optimal investment trigger before the arrival of the shock. Conjecture that it is strictly optimal to wait when $X(t)$ is below $\bar{X}(\pi_0)/(1+\varphi)$, that is, $\hat{X}\geq\bar{X}(\pi_0)/(1+\varphi)$. This is quite intuitive, in that it implies that if it is not optimal to invest at an instant prior to the shock, then it would not be optimal to invest if the jump occurs and immediately reverts. The conjecture is verified below. Then, the investment option value $F(X)$ can be divided into two parts $F_L(X)$ and $F_H(X)$, corresponding to the lower and the higher regions, respectively. By Itô's lemma, $F_L(X)$ and $F_H(X)$ satisfy the following differential equations:

- in the region $X<\bar{X}(\pi_0)/(1+\varphi)$,

$$(r+\lambda_1+\lambda_2)\,F_L(X)=\alpha XF_L'(X)+\frac{1}{2}\sigma^2X^2F_L''(X)+(\lambda_1+\lambda_2)\,G((1+\varphi)\,X,\pi_0)\,; \quad (75)$$

- in the region $\bar{X}(\pi_0)/(1+\varphi)<X<\hat{X}$,

$$\begin{aligned}(r+\lambda_1+\lambda_2)\,F_H(X)&=\alpha XF_H'(X)+\tfrac{1}{2}\sigma^2X^2F_H''(X)\\&+\left(\lambda_1\tfrac{1+\varphi}{r-\alpha}+\lambda_2\tfrac{1+\varphi+\frac{\lambda_3}{r-\alpha}}{r-\alpha+\lambda_3}\right)X-(\lambda_1+\lambda_2)\,I.\end{aligned} \quad (76)$$

Eqs. (75) and (76) differ due to the implied investment behavior at the moment of the arrival of a shock. In the lower region the arrival of a shock does not induce immediate investment, while in the higher region it does.

Eqs. (75) and (76) are solved subject to the following boundary conditions:

$$F_H\left(\hat{X}\right) = \frac{\hat{X}}{r-\alpha} + \frac{\varphi\hat{X}}{r-\alpha+\lambda_1+\lambda_2}\left(\frac{\lambda_1}{r-\alpha} + \frac{\lambda_2}{r-\alpha+\lambda_3}\right) - I, \tag{77}$$

$$F'_H\left(\hat{X}\right) = \frac{1}{r-\alpha} + \frac{\varphi}{r-\alpha+\lambda_1+\lambda_2}\left(\frac{\lambda_1}{r-\alpha} + \frac{\lambda_2}{r-\alpha+\lambda_3}\right), \tag{78}$$

$$\lim_{X\uparrow\bar{X}(\pi_0)/(1+\varphi)} F_L(X) = \lim_{X\downarrow\bar{X}(\pi_0)/(1+\varphi)} F_H(X), \tag{79}$$

$$\lim_{X\uparrow\bar{X}(\pi_0)/(1+\varphi)} F'_L(X) = \lim_{X\downarrow\bar{X}(\pi_0)/(1+\varphi)} F'_H(X), \tag{80}$$

$$\lim_{X\to 0} F_L(X) = 0. \tag{81}$$

As before, the value-matching condition (77) imposes equality at the exercise point between the value of the option and the net present value of the project's cash flows, while the smooth-pasting condition (78) ensures that the exercise point is chosen optimally. Conditions (79) and (80) guarantee that the value of the investment option is continuous and smooth. Finally, (81) is a boundary condition that reflects the fact that $X = 0$ is the absorbing barrier for the cash flow process.

The general solutions to Eq. (75) and (76) are given by

$$F_L(X) = C_1X^{\gamma_1} + C_2X^{\gamma_2} + \tfrac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)}\left(\Gamma_2(X) - \Gamma_1(X)\right), \tag{82}$$

$$F_H(X) = A_1X^{\gamma_1} + A_2X^{\gamma_2} + \tfrac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)}X - \tfrac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2}I, \tag{83}$$

where

$$\Gamma_i(X) = X^{\gamma_i}\int_0^X \frac{G((1+\varphi)x,\pi_0)}{x^{\gamma_i+1}}dx,\ i\in\{1,2\}. \tag{84}$$

We have five boundary conditions (77)-(81) to determine the four unknown constants (A_1, A_2, C_1 and C_2) and the investment trigger $\hat{X}$. Consider condition (81). It implies

$$\lim_{X\to 0} C_2X^{\gamma_2} + \frac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)}\left(\lim_{X\to 0}\Gamma_2(X) - \lim_{X\to 0}\Gamma_1(X)\right) = 0. \tag{85}$$

Because $\gamma_1 > 0$, $\lim_{X\to 0}\Gamma_1(X) = 0$. Also, by l'Hôpital's rule,

$$\begin{aligned} \lim\nolimits_{X\to 0}\Gamma_2(X) &= \lim\nolimits_{X\to 0} X^{\gamma_2}\textstyle\int_0^X \frac{G((1+\varphi)x,\pi_0)}{x^{\gamma_2+1}}dx \\ = \lim\nolimits_{X\to 0}\tfrac{G((1+\varphi)X,\pi_0)X^{-\gamma_2-1}}{-\gamma_2X^{-\gamma_2-1}} &= -\tfrac{1}{\gamma_2}\lim\nolimits_{X\to 0} G((1+\varphi)X,\pi_0) = 0. \end{aligned} \tag{86}$$

Therefore, condition (81) implies $C_2 = 0$. The other four boundary conditions give the following system of equations:

$$\begin{aligned}
A_1 \hat{X}^{\gamma_1} + A_2 \hat{X}^{\gamma_2} &= \tfrac{\hat{X}}{r-\alpha+\lambda_1+\lambda_2} - \tfrac{rI}{r+\lambda_1+\lambda_2} \\
\gamma_1 A_1 \hat{X}^{\gamma_1} + \gamma_2 A_2 \hat{X}^{\gamma_2} &= \tfrac{\hat{X}}{r-\alpha+\lambda_1+\lambda_2} \\
A_1 \left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)^{\gamma_1} + A_2 \left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)^{\gamma_2} &+ \tfrac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)} \tfrac{\bar{X}(\pi_0)}{1+\varphi} - \tfrac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2} I \\
&= C_1 \left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)^{\gamma_1} + \tfrac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)} \left(\Gamma_2\left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right) - \Gamma_1\left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)\right) \\
\gamma_1 A_1 \left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)^{\gamma_1} + \gamma_2 A_2 \left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)^{\gamma_2} &+ \tfrac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)} \tfrac{\bar{X}(\pi_0)}{1+\varphi} \\
&= \gamma_1 C_1 \left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)^{\gamma_1} + \tfrac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)} \left(\gamma_2 \Gamma_2\left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right) - \gamma_1 \Gamma_1\left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)\right).
\end{aligned} \tag{87}$$

Combining the last two equations, we get

$$\begin{aligned}
A_2 = \left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)^{-\gamma_2} &\Big[\tfrac{1-\gamma_1}{\gamma_1-\gamma_2} \tfrac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)} \tfrac{\bar{X}(\pi_0)}{1+\varphi} \\
&+ \tfrac{\gamma_1}{\gamma_1-\gamma_2} \tfrac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2} I + \tfrac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)} \Gamma_2\left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)\Big].
\end{aligned} \tag{88}$$

Combining the first two equations, we get

$$\hat{X} = \frac{\gamma_1}{\gamma_1 - 1} \frac{(r-\alpha+\lambda_1+\lambda_2)\, rI}{r+\lambda_1+\lambda_2} + \frac{\gamma_1-\gamma_2}{\gamma_1-1} A_2 \hat{X}^{\gamma_2}. \tag{89}$$

The corresponding value of the investment opportunity is

$$F(X) = \begin{cases} C_1 X^{\gamma_1} + \tfrac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)} \left(\Gamma_2(X) - \Gamma_1(X)\right), & X \le \tfrac{\bar{X}(\pi_0)}{1+\varphi} \\ A_1 X^{\gamma_1} + A_2 X^{\gamma_2} + \tfrac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)} X - \tfrac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2} I, & \tfrac{\bar{X}(\pi_0)}{1+\varphi} \le X \le \hat{X} \\ \tfrac{X}{r-\alpha} + \tfrac{\varphi X}{r-\alpha+\lambda_1+\lambda_2}\left(\tfrac{\lambda_1}{r-\alpha} + \tfrac{\lambda_2}{r-\alpha+\lambda_3}\right) - I, & X \ge \hat{X}, \end{cases} \tag{90}$$

where A_2 is given by (88), and A_1 and C_1 satisfy

$$A_1 = \hat{X}^{-\gamma_1}\left(\frac{\hat{X}}{r-\alpha+\lambda_1+\lambda_2} - \frac{rI}{r+\lambda_1+\lambda_2} - A_2 \hat{X}^{\gamma_2}\right), \tag{91}$$

$$\begin{aligned}
C_1 = A_1 + A_2 \left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)^{\gamma_2-\gamma_1} + \left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)^{-\gamma_1} &\Big(\tfrac{(\lambda_1+\lambda_2)((1+\varphi)(r-\alpha)+\lambda_3)+\lambda_1\lambda_3\varphi}{(r-\alpha+\lambda_1+\lambda_2)(r-\alpha)(r-\alpha+\lambda_3)} \tfrac{\bar{X}(\pi_0)}{1+\varphi} \\
- \tfrac{\lambda_1+\lambda_2}{r+\lambda_1+\lambda_2} I + \tfrac{2(\lambda_1+\lambda_2)}{\sigma^2(\gamma_1-\gamma_2)} &\left(\Gamma_1\left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right) - \Gamma_2\left(\tfrac{\bar{X}(\pi_0)}{1+\varphi}\right)\right)\Big).
\end{aligned} \tag{92}$$

We now confirm the conjecture that $\hat{X} \ge \bar{X}(\pi_0)/(1+\varphi)$. By contradiction, suppose that it is optimal to invest at some trigger X' below $\bar{X}(\pi_0)/(1+\varphi)$. Let us modify the problem in the following way. Suppose that the upward jump occurs immediately, that is, $\lambda_1 + \lambda_2 = +\infty$ with π_0 being unchanged. For each sample path, the project in the modified problem yields the same cash flows as the project in the original problem with the difference that the range of extra cash flows generated by the upward shock occurs earlier. Hence, investment in the modified problem occurs earlier than in the original problem. In particular, since it was optimal to invest at X' in the original problem, it is optimal at X' in the modified problem. Since in the modified problem the jump occurs immediately, the value of the investment option is $G\left(X\left(1+\varphi\right),\pi_0\right)$. However, from the previous subsection we know that it is strictly optimal to wait for all $X\left(1+\varphi\right)<\bar{X}\left(\pi_0\right)$. Therefore, it cannot be optimal to invest at any X'. This implies that indeed, it is strictly optimal to wait for any X below $\bar{X}\left(\pi_0\right)/\left(1+\varphi\right)$. Therefore, the optimal investment policy is characterized by the critical value (42) at which it is optimal to invest.

Derivation of the investment trigger for the case of an unlimited number of shocks in Section 5.

Let $Y\left(X\left(t\right),p\left(t\right)\right)$ be a continuously differentiable function, where $X\left(t\right)$ is the cash flow process and $p\left(t\right)$ is the belief process, and let $Y\left(t\right)=Y\left(X\left(t\right),p\left(t\right)\right)$. Applying Itô's lemma for semimartingales (see Theorem 33 in Protter (2004)), we get the dynamics of $Y\left(t\right)$ under $\left(\mathcal{F}_t\right)$:

$$\begin{aligned} dY = &\left(\alpha XY_X+\tfrac{1}{2}\sigma^2X^2Y_{XX}-\sum_{k=1}^{\infty}\tfrac{\partial Y}{\partial p_k}\lambda_3p_k\left(k-\textstyle\sum_{i=1}^{\infty}p_ii\right)\right)dt+\sigma XY_xdB\left(t\right) \\ &+\left[Y\left(X(1+\varphi),\hat{p}\left(p\right)\right)-Y\right]\left(dM_1\left(t\right)+dM_2\left(t\right)\right)+\left[Y\left(\tfrac{X}{1+\varphi},\tilde{p}\left(p\right)\right)-Y\right]dN\left(t\right), \end{aligned} \tag{93}$$

where X, p_k, and Y denote $X\left(t\right)$, $p_k\left(t\right)$, and $Y\left(t\right)$, respectively, and $\hat{p}\left(p\right)$ and $\tilde{p}\left(p\right)$ are the updated vectors of beliefs defined by (8) - (9). By the law of iterated expectations,

$$\mathbb{E}\left[dN\left(t\right)|\mathcal{G}_t\right]=\mathbb{E}\left[\mathbb{E}\left[dN\left(t\right)|\mathcal{F}_t\right]\mathcal{G}_t\right]=\mathbb{E}\left[k\left(t\right)\lambda_3dt|\mathcal{G}_t\right]=\sum_{k=1}^{\infty}\lambda_3kp_kdt. \tag{94}$$

Therefore, the instantaneous conditional expected change in $Y\left(X,p\right)$ is equal to

$$\begin{aligned} \mathbb{E}\left[\tfrac{dY(X,p)}{dt}|\mathcal{G}_t\right] = &\ \alpha XY_X+\tfrac{1}{2}\sigma^2X^2Y_{XX}-\sum_{k=1}^{\infty}\tfrac{\partial Y}{\partial p_k}\lambda_3p_k\left(k-\textstyle\sum_{i=1}^{\infty}p_ii\right) \\ &+\left[Y\left(X(1+\varphi),\hat{p}\left(p\right)\right)-Y\right]\left(\lambda_1+\lambda_2\right)+\left[Y\left(\tfrac{X}{1+\varphi},\tilde{p}\left(p\right)\right)-Y\right]\sum_{k=1}^{\infty}\lambda_3kp_k. \end{aligned} \tag{95}$$

If $Y\left(X,p\right)$ denotes the value of a contingent claim that pays a continuous cash flow of $y\left(X,p\right)$ and the discount rate is equal to r, then it must be the case that

$\mathbb{E}\left[\frac{dY(X,p)+y(X,p)dt}{dt}|\mathcal{G}_t\right]=rY\left(X,p\right)$. Hence,

$$rY=\alpha XY_X+\tfrac{1}{2}\sigma^2X^2Y_{XX}-\sum_{k=1}^{\infty}\tfrac{\partial Y}{\partial p_k}\lambda_3 p_k\left(k-\textstyle\sum_{i=1}^{\infty}p_i i\right)$$
$$+\left[Y\left(X(1+\varphi),\hat{p}\left(p\right)\right)-Y\right]\left(\lambda_1+\lambda_2\right)+\left[Y\left(\tfrac{X}{1+\varphi},\tilde{p}\left(p\right)\right)-Y\right]\sum_{k=1}^{\infty}\lambda_3 kp_k+y\left(X,p\right). \tag{96}$$

Simplifying:

$$\left(r+\lambda_1+\lambda_2+\lambda_3\sum_{k=1}^{\infty}p_k k\right)Y=\alpha XY_X+\tfrac{1}{2}\sigma^2X^2Y_{XX}-\lambda_3\sum_{k=1}^{\infty}\tfrac{\partial Y}{\partial p_k}p_k\left(k-\textstyle\sum_{i=1}^{\infty}p_i i\right)$$
$$+\left(\lambda_1+\lambda_2\right)Y\left(X(1+\varphi),\hat{p}(p)\right)+\left(\lambda_3\sum_{k=1}^{\infty}p_k k\right)Y\left(\tfrac{X}{1+\varphi},\tilde{p}(p)\right)+y(X,p). \tag{97}$$

Let $S\left(X,p\right)$ denote the value of the underlying project. It is the expected discounted value of cash flows that the firm gets if it immediately exercises the investment option. $S(X,p)$ is thus a special case of $Y(X,p)$, with $y(X,p)=X$. Thus $S\left(X,p\right)$ must satisfy:

$$\left(r+\lambda_1+\lambda_2+\lambda_3\sum_{k=1}^{\infty}p_k k\right)S\left(X,p\right)=\alpha XS_X+\tfrac{1}{2}\sigma^2X^2S_{XX}-\lambda_3\sum_{k=1}^{\infty}\tfrac{\partial S}{\partial p_k}p_k\left(k-\textstyle\sum_{i=1}^{\infty}p_i i\right)$$
$$+\left(\lambda_1+\lambda_2\right)S\left(X(1+\varphi),\hat{p}(p)\right)+\left(\lambda_3\sum_{k=1}^{\infty}p_k k\right)S\left(\tfrac{X}{1+\varphi},\tilde{p}(p)\right)+X. \tag{98}$$

The solution can be written as

$$S\left(X,p\right)=a_0X+\sum_{i=1}^{\infty}p_i\left(a_i-a_0\right)X, \tag{99}$$

where constants $a_0, a_1, ...$ are defined in Eq. (105) later in the appendix.

Let $G\left(X,p\right)$ denote the value of the investment option. Again, $G(X,p)$ is thus a special case of $Y(X,p)$, with $y(X,p)=0$. Thus $G\left(X,p\right)$ must satisfy:

$$\left(r+\lambda_1+\lambda_2+\lambda_3\sum_{k=1}^{\infty}p_k k\right)G\left(X,p\right)=\alpha XG_X+\frac{1}{2}\sigma^2X^2G_{XX}-\lambda_3\sum_{k=1}^{\infty}\frac{\partial G}{\partial p_k}p_k\left(k-\sum_{i=1}^{\infty}p_i i\right) \tag{100}$$
$$+\left(\lambda_1+\lambda_2\right)G\left(X(1+\varphi),\hat{p}(p)\right)+\left(\lambda_3\sum_{k=1}^{\infty}p_k k\right)G\left(\frac{X}{1+\varphi},\tilde{p}(p)\right).$$

The optimal investment decision can be described by a trigger function $\bar{X}(p)$. Eq. (100) is solved subject to the following value-matching and smooth-pasting conditions:

$$\begin{gathered} G\left(\bar{X}(p), p\right) = S\left(\bar{X}(p), p\right) - I, \\ G_X\left(\bar{X}(p), p\right) = S_X\left(\bar{X}(p), p\right), \\ \left(\lambda_3 \sum_{k=1}^{\infty} \left(\frac{\partial G\left(\bar{X}(p), p\right)}{\partial p_k} - \frac{\partial S\left(\bar{X}(p), p\right)}{\partial p_k}\right)\right) p_k \left(k - \sum_{i=1}^{\infty} p_i i\right) = 0. \end{gathered} \tag{101}$$

The first equation is the value-matching condition. The second and third equations are the smooth-pasting conditions (with respect to X and t, respectively).

Combining (98), (100) and (101) gives us:[27]

$$\begin{gathered} \bar{X}(p) - rI = (\lambda_1 + \lambda_2)\left[G\left(\bar{X}(p)(1+\varphi), \hat{p}(p)\right) + I - S\left(\bar{X}(p)(1+\varphi), \hat{p}(p)\right)\right] \\ + \left(\lambda_3 \sum_{k=1}^{\infty} p_k k\right)\left[G\left(\frac{\bar{X}(p)}{1+\varphi}, \tilde{p}(p)\right) + I - S\left(\frac{\bar{X}(p)}{1+\varphi}, \tilde{p}(p)\right)\right] + \frac{\sigma^2}{2}\bar{X}(p)^2 G_{XX}\left(\bar{X}(p), p\right). \end{gathered} \tag{102}$$

When (X, p) is very close to the trigger $\bar{X}(p)$, the arrival of a new positive shock will result in immediate investment. The intuition is simple. If the firm is indifferent between investing and waiting today, an upward jump in X will certainly induce the firm to invest immediately. This implies:

$$G\left(\bar{X}(p)(1+\varphi), \hat{p}(p)\right) = S\left(\bar{X}(p)(1+\varphi), \hat{p}(p)\right) - I. \tag{103}$$

Thus, the first term on the right-hand side of (102) is zero, so (102) is equivalent to (52).

Numerical Procedures

Numerical procedure for computing $\bar{X}(p)$ in Section 3.

To compute the trigger functions we use a variation of the least-squares method developed by Longstaff and Schwartz (2003). Note that when $p = 0$, the model becomes standard, so $\bar{X}(0) = X^*$. Also, note that $p(T) \to 0$ as $T \to \infty$, where T is the time that passes after the arrival of the shock. Because of that, we can approximate $\bar{X}(p(T))$ for some large T by $\bar{X}(0)$. After that, take some small Δ, and compute $p(T-\Delta)$ from (11). Then, use the least squares method of Longstaff and Schwartz (2003) to estimate the second derivative of the conditional expected payoff from waiting at time $T-\Delta$ until time T. Then, use this estimate and (39) to compute $\bar{X}(p(T-\Delta))$. We repeat this N times for a sufficiently large N such that $p(T-N\Delta) > \pi_0$. More specifically, at any step n:

1. Use $\bar{X}(p(T-k\Delta))$, $k = 0, 1, ..., n-1$ and the least squares method to estimate the second derivative of the conditional expected payoff from waiting at time $T - n\Delta$;

[27] Note that since $S(X,p)$ is linear in X, $S_{XX} = 0$.

2. Use this estimate as input in (39) to compute $\bar{X}(p(T-n\Delta))$.

Numerical procedure for computing $\bar{X}(p)$, $p=(p_1,p_2,0,0,...)'$ in Section 5. First, we compute $S(X,p)$. Plugging (99) into (98), we get

$$\begin{array}{c}(r+\lambda_1+\lambda_2+\lambda_3\sum_i ip_i)\,a_0+(r+\lambda_1+\lambda_2)\sum_i(a_i-a_0)\,p_i=\\=(\lambda_1+\lambda_2)(1+\varphi)\left(p_0a_1+(1-p_0)\,a_0+\sum_i\left[p_0\,(a_{i+1}-a_1)+(1-p_0)\,(a_i-a_0)\right]p_i\right)\\+\frac{\lambda_3}{1+\varphi}\sum_i a_{i-1}ip_i-\lambda_3\sum_i(a_i-a_0)\,ip_i+1.\end{array}\tag{104}$$

This equation must hold for any p. This happens if and only if coefficients with 1, p_1,p_2,... on the left-hand and right-hand sides are equal. Matching the coefficients, we get

$$a_k=\begin{cases}\frac{(1+\varphi)\lambda_2}{r+\lambda_2-\varphi\lambda_1}a_1+\frac{1}{r+\lambda_2-\varphi\lambda_1}\text{ for }k=0,\\ \frac{1}{r+\lambda_2-\varphi\lambda_1+\lambda_3k}+\frac{\lambda_3k}{(1+\varphi)(r+\lambda_2-\varphi\lambda_1+\lambda_3k)}a_{k-1}+\frac{(1+\varphi)\lambda_2}{r+\lambda_2-\varphi\lambda_1+\lambda_3k}a_{k+1}\text{ for }k=1,2,...\end{cases}\tag{105}$$

Hence, coefficients a_0,a_1,... are defined as solutions to this recurrence relation subject to the boundary condition $\lim_{k\to\infty}a_k=0$. This system of equations is solved numerically.

Second, we compute $\bar{X}(p)$ for $p=(p_1,0,...,0,...)$. In this case, $\bar{X}(p)$ if found as a solution to

$$\bar{X}(p)=\lambda_3p_1\left[G\left(\frac{\bar{X}(p)}{1+\varphi},0\right)+I-S\left(\frac{\bar{X}(p)}{1+\varphi},0\right)\right]+rI,$$
$$G\left(\frac{\bar{X}(p)}{1+\varphi},0\right)=\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+r}G\left(\bar{X}(p),(\pi_0,0,...,0,...)'\right).$$

If p_1 is above π_0, then

$$G\left(\bar{X}(p),(\pi_0,0,...,0,...)'\right)=S\left(\bar{X}(p),(\pi_0,0,...,0,...)'\right)-I.$$

If p_1 is below π_0, then $G\left(\bar{X}(p),(\pi_0,0,...,0,...)'\right)$ is computed by simulations. In this case, investment occurs at the first instant when an upward jump arrives or $p(t)$ reduces from π_0 to p.

Finally, we use these results to approximate $\bar{X}(p)$ for $p=(p_1,p_2,0,...,0,...)$ using the following procedure. First, we take $\bar{X}(p)$ to be equal to $\bar{X}(\tilde{p})$, where $\tilde{p}=(\sum_{i=1}^{\infty}p_i,0,...,0,...)$. Using this threshold, we simulate the option values $G(X,p)$ for $p=(p_1,...,p_{M-1},0,...,0,...)$. Then, we use the simulated option values as input in (52) to re-evaluate $\bar{X}(p)$ for $p=(p_1,...,p_M,0,...,0,...)$. After that, we take $\bar{X}(p)$ to be equal to $\bar{X}(\tilde{p})$, where $\tilde{p}=(p_1,...,p_{M-1},\sum_{i=M}^{\infty}p_i,0,...,0,...)$. Using this threshold, we again simulate the option values $G(X,p)$ for $p=(p_1,...,p_{M-1},0,...,0,...)$. Then, we use the simulated option values as input in (52) to re-evaluate $\bar{X}(p)$ for $p=(p_1,...,p_M,0,...,0,...)$. This is repeated until convergence. If $M\to\infty$, this procedure should converge to the true threshold $\bar{X}(p)$. Since the procedure is computationally intensive, we are able to use only the maximum of $M=5$. We use the resulting $\bar{X}(p)$ for $p=(p_1,p_2,0,...,0,...)$ to plot the simulated sample paths in Section 5.

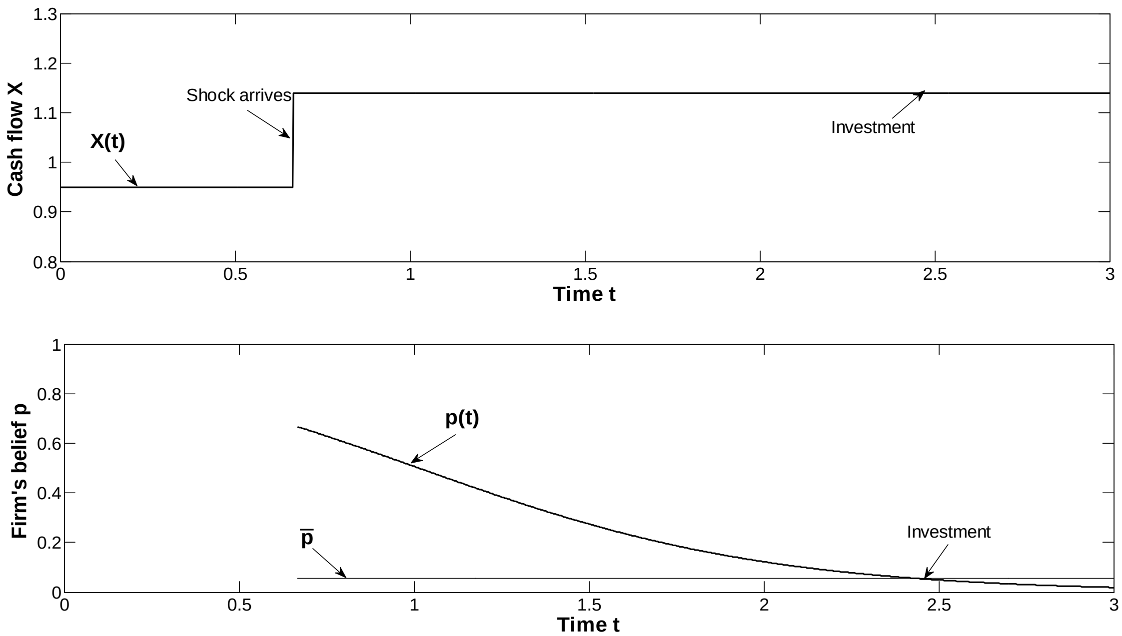

Figure 1. Simulation of the firm's investment strategy. The top graph shows the simulated sample path of the cash flow process, X(t), for the simple model of Section 2.3. An upward step corresponds to the arrival of a shock. For this sample path, the shock arrives at time 0.67 and persists through year 3, implying that over time the firm becomes more confident that the shock is permanent. The bottom graph shows the dynamics of the firm's belief process as well as the investment threshold $\bar{p}$. The investment occurs when the firm's belief process falls to $\bar{p}$. At this point, the benefits of learning and the costs of deferring the receipt of the cash flows are precisely offset. The parameter values are r=0.04, φ=0.2, λ_1=0.5, λ_2=1, λ_3=2, I=25, and $X_0 = 0.95$.

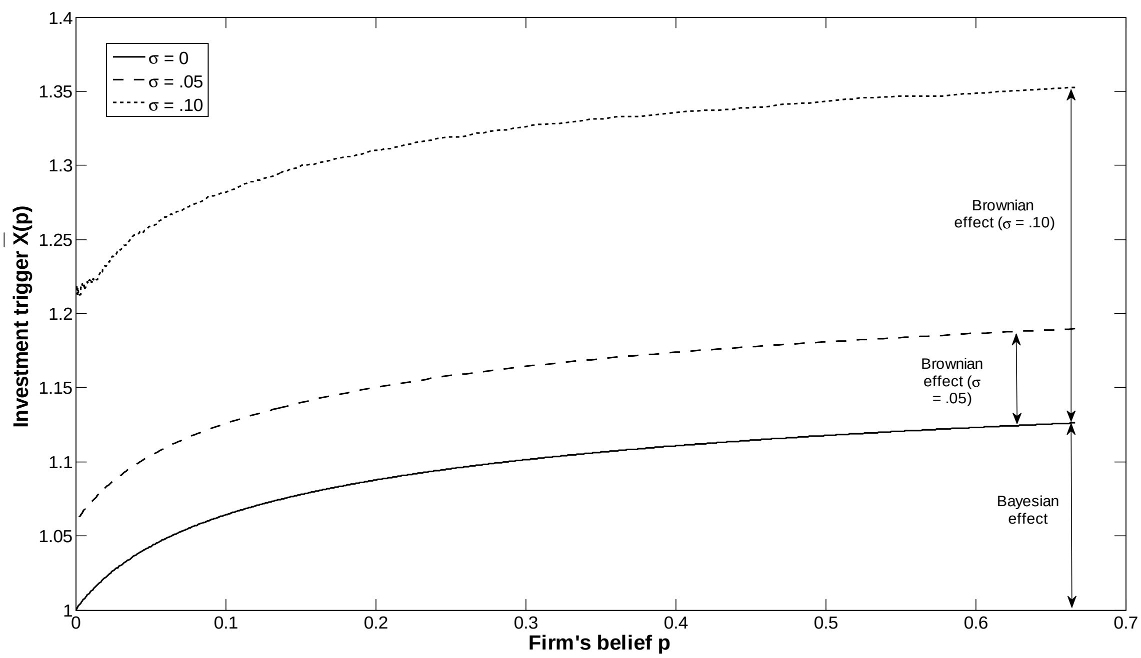

Figure 2. Investment trigger functions for different values of the Brownian volatility parameter. The graph plots the investment trigger function $\bar{X}(p)$ for different values of the Brownian volatility parameter σ. The bottom curve corresponds to the case of pure Bayesian uncertainty (σ=0). The middle and the top curves correspond to the cases of both Bayesian and Brownian uncertainties (σ=0.05 and σ=0.10, respectively). As a result, the change of the trigger along each curve is due to the impact of Bayesian uncertainty, while the upward shift of the whole trigger function is due to the impact of Brownian uncertainty. The parameter values are r=0.04, α=0.02, φ=0.2, λ_1=0.5, λ_2=1, λ_3=2, and I=25.

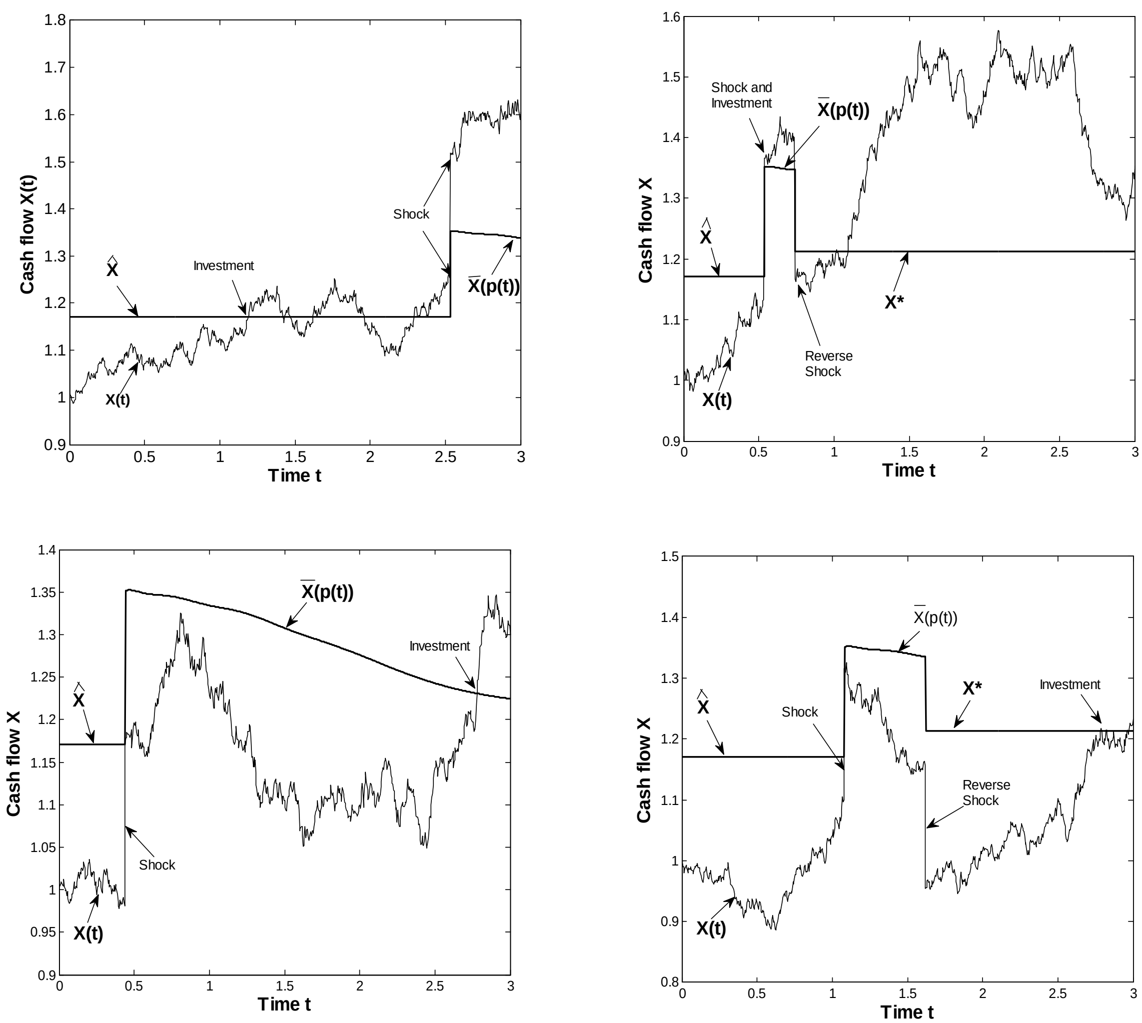

Figure 3. Simulations of the firm's investment strategies. The figure shows the simulated paths of the cash flow process, X(t), (plotted in thin lines) and the corresponding investment triggers (plotted in bold lines) for four different scenarios specified in Proposition 1. The optimal exercise strategy is to invest at the first time when X(t) reaches the investment trigger for the first time. The parameter values are r=0.04, α=0.02, σ=0.10, φ=0.2, λ_1=0.5, λ_2=1, λ_3=2, I=25, and X(0)=1.

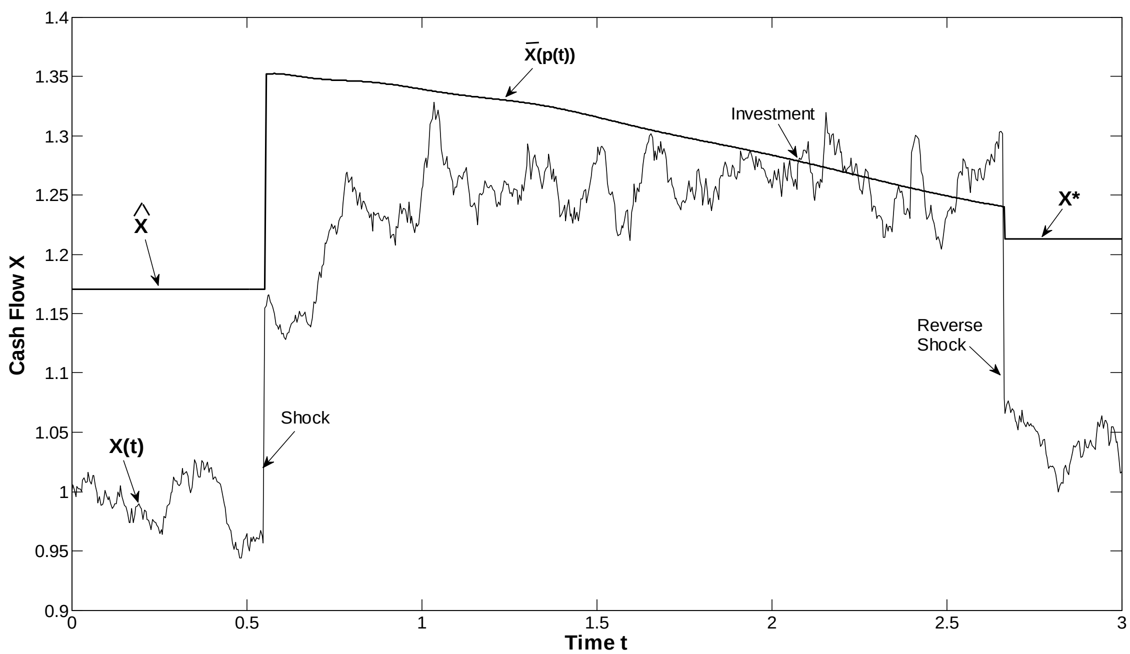

Figure 4. Investment in the Face of Stable Cash Flows: Illustration. The figure shows the simulated path of the cash flow process, X(t), (plotted in the thin line) and the corresponding investment trigger (plotted in the bold line). The optimal exercise strategy is to invest at the first time when X(t) reaches the investment trigger for the first time. The parameter values are r=0.04, α=0.02, σ=0.10, φ=0.2, λ_1=0.5, λ_2=1, λ_3=2, I=25, and X(0)=1.

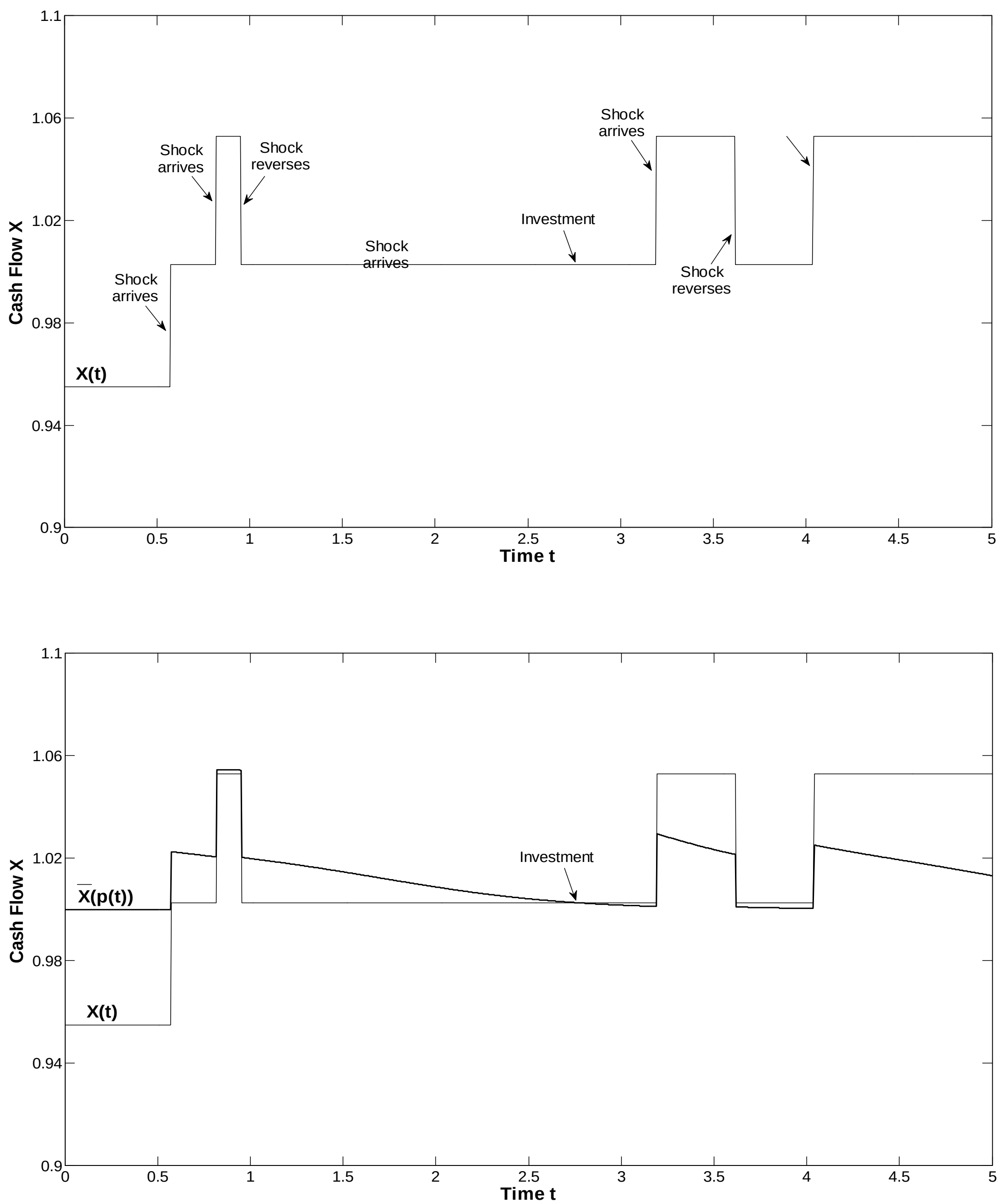

Figure 5. Simulation of the firm's investment strategy. The top graph shows the simulated sample path of the cash flow process, X(t), for the model of Section 5. Each of the four upward steps corresponds to the arrival of a new shock. Similarly, each of the two downward steps corresponds to the reversal of an outstanding shock. The bottom graph adds the investment trigger $\bar{X}$(p(t)) to the simulated sample path from the top graph. The optimal investment strategy is to invest at the first time when X(t) reaches the investment trigger. The parameter values are r=0.04, α=0, σ=0, φ=0.05, λ_1=0.5, λ_2=1, λ_3=2, I=25, and X(0)=0.955.