Optical Network Synthesis Using Birefringent Crystals.* I. Synthesis of Lossless Networks of Equal-Length Crystals

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A procedure for the synthesis of birefringent networks having arbitrarily prescribed transfer functions is presented. The basic network configuration consists of n identical cascaded birefringent crystals between an input and an output polarizer. The crystals are cut with their optic axes perpendicular to their length. The variables determined by the synthesis procedure are the angles of the optic axes of the crystals and the angle of the output polarizer. Any transfer function which is periodic with frequency and whose corresponding impulse response is real and causal can, in theory, be realized. A network of n crystals allows the approximation of a desired function by $(n+1)$ terms of a Fourier exponential series. Bandwidths of less than 1 Å appear possible.

I. INTRODUCTION

THE advent of the laser has made possible various types of optical systems. This has produced a need for optical elements or networks whose transfer functions can be arbitrarily prescribed as a function of frequency. In a manner analogous to that used at radio frequencies, such optical networks could be utilized as discriminators and ratio detectors, equalizers and compensators, frequency selective hybrids, and delay networks, to name just a few. Of particular importance is the possibility of realizing very narrow-band filters having prescribed transmission characteristics.

The purpose of this paper is to present a basic network configuration and synthesis procedure whereby optical networks having arbitrary transfer functions can be constructed using a set of cascaded birefringent crystals. Although synthesis procedures exist for other types of optical devices,[1–4] the very narrow bandwidths and tunability of birefringent devices make them particularly attractive for the above-mentioned applications. The type of network to be considered is shown in Fig. 1. In simplest form, it consists of a number of identical birefringent crystals placed between two polarizers. Although Fig. 1 pictures a network containing four stages (four birefringent crystals), any number can be used. In principle, either uniaxial or biaxial[5] crystals may be employed, but for simplicity we will assume uniaxial crystals are used. Each crystal is cut with its optic axis perpendicular to its length and with end faces which are flat and parallel. The S's and F's in Fig. 1 denote the crystals' "slow" and "fast" axes, respectively. If a negative crystal is used, the fast axis will be the optic axis, while for a positive crystal the slow axis will be the optic axis. The variables to be determined by the synthesis procedure are the angles to which the crystals are rotated, the angle of the output polarizer, and the length L of the crystals used. In the following sections, we will show that by properly choosing these variables, it is possible, in theory, to synthesize any desired transfer function, subject only to the restrictions that it be periodic with frequency and that it satisfy the usual requirements imposed by the necessity for a real and causal impulse response. The basic periodicity of the network response is determined by the type and length of birefringent crystals used. For example, if calcite crystals 1 cm in length are used, the basic period of

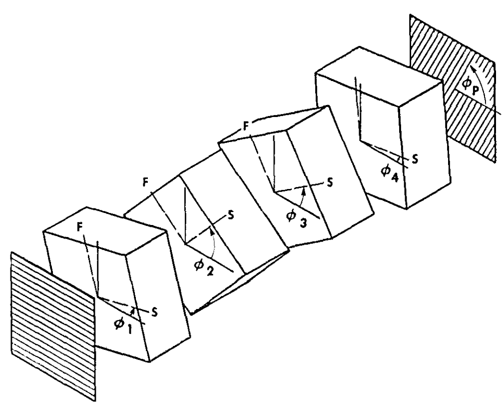

Fig. 1. Basic configuration of optical network (four stages). Polarizers are shown shaded.

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[1] H. Pohlack, Jenaer Jahrbuch, **1962**, p. 181 (in German).
[2] L. Young, J. Opt. Soc. Am. **51**, 967 (1961).
[3] J. S. Seeley, Proc. Phys. Soc. (London) **78**, 998 (1961).
[4] R. J. Pegis, J. Opt. Soc. Am. **51**, 1255 (1961).
[5] If biaxial crystals are used, crystals in the monoclinic and triclinic systems will probably not be satisfactory since the directions of their principal axes are dependent upon temperature and wavelength.

the response will be about 175 Gc (about 2 Å in the red).

An important modification of the basic configuration of Fig. 1 is the addition of a variable optical compensator[6] before or after each birefringent crystal. The compensators allow one to tune the network transfer function without distortion over its basic period and, in addition, compensate for slightly incorrect crystal lengths.

The optical network described here is a lossless or nondissipative network in that it does not contain any internal polarizers; if the final polarizer is nonabsorbing, e.g., a Rochon prism, then all of the optical energy incident on the first birefringent crystal is, in principle, available at the network output. It is planned to consider the synthesis of dissipative birefringent networks, i.e., networks containing internal polarizers, in a following paper.

A central idea of this paper is the consideration of the impulse response of a system of birefringent crystals. This approach was used by Mertz[7] to analyze the Solc birefringent filter and was independently suggested as an approach to the synthesis problem by Harris.[8] It is first presented and then used to obtain an exact synthesis procedure. The question of tunability is considered and an example given.

II. HISTORY OF BIREFRINGENT DEVICES

Before proceeding further, it is appropriate to note that two birefringent filters having particular transfer functions have been proposed considerably earlier. The first of these was proposed in 1933 by the French astronomer, Lyot,[9] who suggested a birefringent filter consisting of alternating polarizers and birefringent crystals. The length of each crystal is twice that of the preceding crystal. A four-stage Lyot filter is shown in Fig. 2. The transfer function of the Lyot filter has the form $\sin x/x$, repeated at periodic intervals. More recently, Solc[10] proposed two types of birefringent filters, termed fan and folded filters. Figure 3 shows a four-stage Solc fan filter. These filters have the same structural form as our basic network. In the Solc filters, however, the relative rotation angle between each successive crystal is related in a simple manner to the number of birefringent crystals employed. In contrast, the relative angles of the crystals in our network are determined by the chcice of optical transfer function—which may be arbitrary. Complete discussions of both the Lyot and Solc filters have been given by Evans.[11,12]

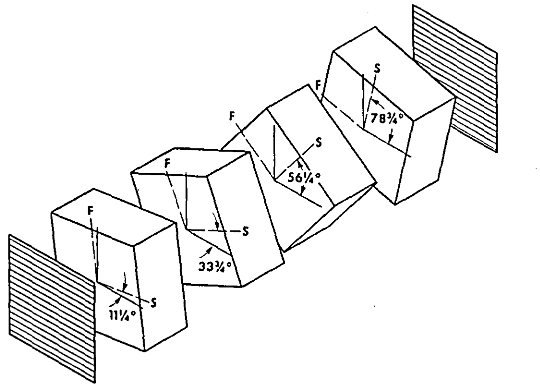

FIG. 3. Four-stage Solc fan filter.

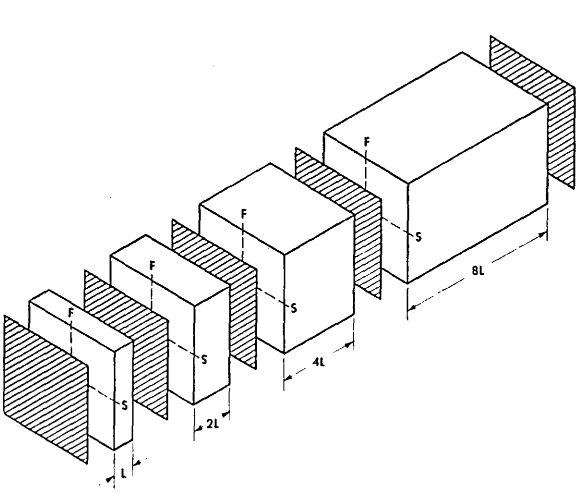

FIG. 2. Four-stage Lyot filter. Polarizers are shown shaded.

Numerous Lyot and Solc filters have been built and operated.[11,13–18] These filters are used primarily in astronomy where their very narrow bandwidths are utilized to observe sclar prominences. Recently, Steel *et al.*[17] have constructed a Lyot filter with a bandwidth of $\frac{1}{8}$ Å in the red. By using the synthesis techniques proposed in this paper, it should be possible to attain similar bandwidths with prescribed transmission characteristics.

III. GENERAL CONSIDERATIONS

A. Impulse Response of a Series of Birefringent Crystals

Analysis by means of impulse response is a concept that is familiar to electrical engineers.[19] If an impulse, i.e., a Dirac delta function in time is applied to a

[6] H. G. Jerrard, J. Opt. Soc. Am. 38, 35 (1948).
[7] L. Mertz, J. Opt. Soc. Am. 50 (June 1960) (advertisement facing p. xii).
[8] S. E. Harris and E. O. Ammann, Proc. IEEE 52, 411 (1964).
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[13] Y. Öhman, Nature 141, 157 (1938); Nature 141, 291 (1938); Pop. Astron. Tidskrift, No. 1–2, 11, 27 (1938).
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[16] B. H. Billings, J. Opt. Soc. Am. 37, 738 (1947).
[17] W. H. Steel, R. N. Smartt, and R. G. Giovanelli, Australian J. Phys. 14, 201 (1961).
[18] J. W. Evans, Appl. Opt. 2, 193 (1963).
[19] J. A. Aseltine, *Transform Method in Linear System Analysis* (McGraw-Hill Book Company, Inc., New York, 1958).

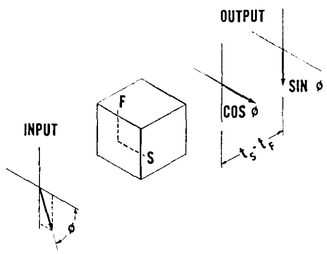

FIG. 4. Impulse response of a single birefringent crystal.

linear network, the Fourier transform of the impulse response of the network is the frequency domain transfer function of the network.

We first consider the impulse response of the single birefringent crystal of Fig. 4. The crystal is cut with its optic axis perpendicular to its length and with end faces flat and parallel. A linearly polarized impulse of optical electric field is assumed to be normally incident on the crystal. Since the incoming signal is normally incident, double refraction will not occur. The impulse will divide into orthogonally polarized ordinary and extraordinary impulses whose amplitudes are dependent on the polarization of the incident impulse with respect to the principal axes of the birefringent crystal. These impulses travel with different velocities, therefore emerging at different times. The difference in the times at which they emerge from the crystal is given by

$$t_S - t_F = L\Delta\eta / c, \tag{1}$$

where $\Delta\eta$ is the difference between the extraordinary and ordinary indices of refraction of the crystal, L is the crystal length, and c is the velocity of light in a vacuum.

We assume here that $\Delta\eta$ is a constant independent of frequency. This is not the actual situation, however, for $\Delta\eta$ will be a function of frequency, at least to some degree. The birefringence of calcite, for example, varies approximately 11% between 4000 and 8000 Å. The effect of the dispersion of $\Delta\eta$ has been ignored in this paper for two reasons. First, to include its effect would greatly complicate the synthesis procedure and obscure the basic ideas. Second, the effects of dispersion upon the resulting transfer function will generally be small, particularly if the synthesized network has a small bandwidth. Existing analyses of the Lyot and Solc filters have also neglected dispersion; yet experimental results have agreed quite well with theory.

Thus, the impulse response of a single birefringent crystal is two orthogonally polarized impulses whose amplitudes depend upon ϕ, the angle between the principal axes of the crystal and the incident optical polarization. If ϕ is equal to zero, all of the light will emerge at time t_S; if ϕ is equal to 45°, the light will emerge as two equal impulses at times t_F and t_S.

We next consider the impulse response of several cascaded birefringent crystals having arbitrary lengths and orientations, as shown schematically in Fig. 5. This figure contains information about the time of emission of the impulses, but none about their polarizations. First consider the case of two crystals. The output of the first crystal is, in general, two orthogonally polarized impulses. Each of these impulses is incident on the second crystal and produces two more impulses. Thus, in general, the impulse response of two cascaded birefringent crystals is four impulses, two of which are polarized along the fast axis and two along the slow axis of the second crystal. With more crystals this process continues, giving us the result that the impulse response of n birefringent crystals having arbitrary lengths and orientations is a set of 2^n impulses. The magnitudes and polarizations of these impulses are determined by the crystal angles, while their relative times of emergence from the crystal are determined by the birefringence and lengths of the crystals used. Thus we reach the important conclusion that the impulse response of a series of birefringent crystals is a train of impulses of finite duration. In contrast, the impulse responses of Fabry–Perot and multilayer dielectric-film filters consist of infinite trains of impulses.

Now suppose that all of the n crystals are chosen to be identical, i.e., the same material and equal lengths. The output will now consist of only $(n+1)$ rather than 2^n impulses. Furthermore, the emerging impulses will be equally spaced in time. The reason that fewer impulses emerge when the crystals are chosen of equal length is seen by examining the two-crystal case. For two crystals of equal length, the impulse which travels along the fast axis of the first and the slow axis of the second will emerge at the same time as the impulse which travels along the slow axis of the first and the fast axis of the second. These two combine, and the output, therefore, consists of three rather than four impulses.

Thus we are led to the network configuration of Fig. 1. The basic idea of the synthesis procedure is to utilize the relative angles of n birefringent crystals and one output polarizer to control the amplitudes of $(n+1)$ equally spaced output impulses. The first step of the procedure is to specify the desired impulse amplitudes at the output of the final polarizer. These amplitudes may be selected arbitrarily as is seen in the following section. We then use a systematic procedure to arrive at angles for the network elements so this final set of impulses is obtained from a single impulse incident on the first crystal of the network. This is equivalent to saying, of course, that the desired set of impulses is the impulse response of the network.

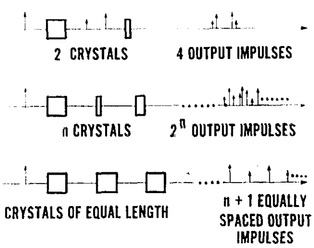

FIG. 5. Impulse response of several birefringent crystals.

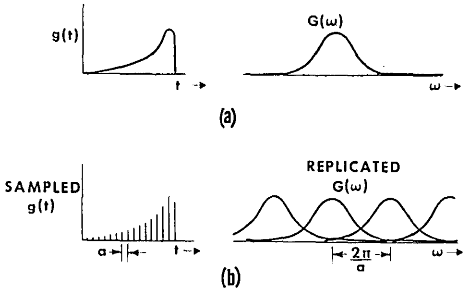

FIG. 6. Impulse responses and corresponding transfer functions for a network whose impulse response is (a) $g(t)$, and (b) $g(t)$ sampled.

B. Specifying the Desired Response

Let us now consider what types of responses we can realize and how we specify them. As in lumped-element circuit theory, a convenient approach is to first choose an ideal response and then approximate this to the necessary degree.

We should note that the frequency transfer function of the optical network must be periodic. This can be seen readily from Fourier theory or sampling theory. Suppose that a network has an impulse response $g(t)$ and a corresponding transfer function $G(\omega)$, where both $g(t)$ and $G(\omega)$ are continuous and aperiodic as shown in Fig. 6(a). Next, suppose another network has an impulse response which is $g(t)$ sampled at a uniform rate of $1/a$ samples/sec. This is the case for a network consisting of a set of birefringent crystals, each of whose length is such that $t_S - t_F$ of Eq. (1) equals a seconds. This network will have a periodic transfer function like that shown in Fig. 6(b), which is the original $G(\omega)$ replicated with a period of $2\pi/a$ rad/sec.[20] Figure 7 shows the transfer function periodicity that can be obtained using readily available lengths of some common crystals.

Assume that a desired periodic transfer function $G(\omega)$ has been chosen. The next step is to find a satisfactory approximation to $G(\omega)$ which can be realized using the optical network of Fig. 1. The approximation is made by an exponential series containing a finite number of terms.

$$C(\omega) = C_0 + C_1 e^{-ia\omega} + C_2 e^{-i2a\omega} + \cdots + C_n e^{-ina\omega} = \sum_{k=0}^{n} C_k e^{-ika\omega}. \tag{2}$$

The impulse response corresponding to Eq. (2) is found by taking the inverse Fourier transform, giving

$$C(t) = C_0\delta(t) + C_1\delta(t-a) + C_2\delta(t-2a) + \cdots + C_n\delta(t-na) \tag{3}$$
$$= \sum_{k=0}^{n} C_k\delta(t-ka).$$

[20] E. A. Guillemin, *Theory of Linear Physical Systems* (John Wiley & Sons, Inc., New York, 1963), p. 430.

Thus it is clear why an exponential series is used to approximate the desired transfer function. The exponential series has a Fourier transform consisting of uniformly spaced impulses, and this is the form of the impulse response of our optical network. If there are $n+1$ terms in $C(\omega)$ [as there are in Eq. (2)], an n-stage optical network is required.

There are various methods available for finding the C_i of Eq. (3) from a given $G(\omega)$. One obvious possibility is to choose the C_i to be the Fourier coefficients of the series. However, if the desired $G(\omega)$ contains discontinuities, some other approximation such as a Cesàro approximation may well be more desirable. Such topics have been treated in detail elsewhere,[21] so we will not discuss this problem further.

It is likely that $|G(\omega)|^2$ or arg $G(\omega)$ will sometimes be given instead of $G(\omega)$. It will then be necessary to approximate $|G(\omega)|^2$ or arg $G(\omega)$ in a suitable manner and calculate $C(\omega)$ from this.

Two points should be noted concerning the approximating functions $C(\omega)$ and $C(t)$. First, since the impulse response of a physical network must be real, the real and imaginary parts of $C(\omega)$ must be even and odd functions of frequency, respectively. This means that all C_i must be real. Second, it is *not* necessary that $C(\omega)$ and $C(t)$ be causal. While it is true, of course, that the impulse response of a network must be zero for $t<0$, we are free to shift our time scale to a new origin when writing $C(\omega)$ and $C(t)$ if this will be more convenient. Thus, in writing Eqs. (2) and (3), we have neglected most of the uniform time delay associated with the network, i.e., the time delay accumulated by passage of the signal through each crystal, in the space between crystals, and in transit to the point of detection. We have chosen our new time origin to be the time at which the first output impulse occurs. For this choice of origin $C(\omega)$ is causal, but equally well, we could have chosen a time origin which results in a noncausal $C(\omega)$. As far as the synthesis procedure is concerned, the important point is that only the relative positions in time of the various impulses are important. In this paper, we will always

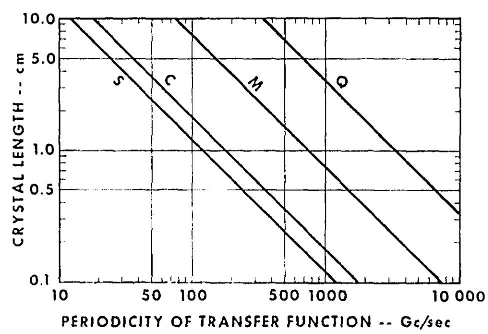

FIG. 7. Periodicity of network response for several types of birefringent crystals. Q: quartz, $\Delta n = 0.009$; M: mica, $\Delta n = 0.04$; C: calcite, $\Delta n = 0.17$; S: sodium nitrate, $\Delta n = 0.24$.

[21] Ref. 20, p. 408.

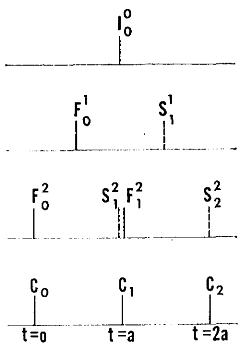

FIG. 8. Summary of impulse notation: Impulse pyramid for a two-stage network. Top: input; next to top: output from first crystal; next to bottom: output from second crystal; bottom: output from polarizer. Solid strokes: polarized along fast axis of crystal. Broken strokes: polarized along slow axis of crystal.

choose our time origin to be synonymous with the occurrence of the first impulse of the train.

The number of birefringent crystals that are necessary to synthesize a desired function will depend on the nature of the function and on the closeness of the approximation desired. Many applications of the synthesis procedure to problems of optical communications will require functions which do not possess discontinuities and whose width is equal to their basic periodicity. (One such function is the triangular waveform of Fig. 11 which might be used to convert a frequency-modulated light signal to an amplitude-modulated light signal.) For functions of this type, the first five or six terms of an exponential series (and, therefore, four or five birefringent crystals) will generally yield a satisfactory approximation.

For narrow-band filter applications, it is necessary to synthesize transfer functions whose basic periodicity is considerably wider than their width. An estimate of the number of crystals necessary for this case may be obtained from sampling considerations and can be written

$$\text{Number of crystals necessary} \approx q\,\frac{\text{periodicity}}{\text{bandwidth}}, \tag{4}$$

where q is an integer which generally will be between 2 and 7. This statement can be understood by noting that the length of the time-domain impulse response is approximately related by the reciprocal width property of Fourier transforms to the bandwidth of the transfer function, and may be written as q/bandwidth, where q is the aforementioned integer. By the length of the impulse response, we mean the time between the first and the last impulses which have significant amplitude. The number of necessary impulses is then the length of the impulse train divided by the spacing between impulses, plus one. Since the spacing between impulses is the reciprocal of the periodicity, and since the number of necessary birefringent crystals is one less than the necessary number of impulses, Eq. (4) follows. The integer q will depend on the function chosen, the degree of approximation desired, and on the definition of bandwidth. As an example, $q=2$ if the desired function is $\sin x/x$ and bandwidth is defined as the number of cycles between its first zeros.

IV. SYNTHESIS PROCEDURE

The object of the synthesis procedure is to find the n birefringent crystal angles and the output polarizer angle which give the desired transfer function $C(\omega)$. The C_i of Eqs. (2) and (3) can have any value, provided that each is real.

A. Notation

The notation and conventions used in the synthesis procedure are discussed here. We refer repeatedly to Fig. 1 which pictures the basic optical network.

Rather than dealing with the ϕ's of Fig. 1, it is more convenient to solve for the relative angles (additional angles of rotation measured from the preceding component) of the crystals and output polarizer. Therefore, we define

$$\begin{aligned} \theta_1 &= \phi_1, \\ \theta_2 &= \phi_2 - \phi_1, \\ &\vdots \\ \theta_n &= \phi_n - \phi_{n-1}, \\ \theta_p &= \phi_p - \phi_n. \end{aligned} \tag{5}$$

The magnitudes of the impulses composing the impulse train emitted from the network are denoted by the C_i of Eqs. (2) and (3). It is also necessary to describe quantitatively the impulse trains which occur between the various stages within the network. In describing them, we must convey information about the polarization of the impulse train, as well as about the magnitudes of the individual impulses. For although we know that $C(t)$ is polarized parallel to the transmission axis of the output polarizer, the impulse train which leaves one of the birefringent crystals on its way toward the output has components polarized parallel to both the S and F axes of that crystal. This points up a fundamental difference between the synthesis procedure described here and conventional synthesis procedures in other fields. Namely, we must be concerned with not only the time variation of the signal, but also with its polarization as it passes through the network.

We illustrate the impulse notation with the aid of the "impulse pyramid" of Fig. 8. Suppose a single impulse (polarized parallel to the transmission axis of the input polarizer) is incident upon a network consisting of two birefringent crystals plus an output polarizer. The resulting output from the second birefringent crystal contains components polarized in both the S and F directions of that crystal.

$$\mathrm{F}^2(t) = \mathrm{F}_0{}^2\delta(t) + \mathrm{F}_1{}^2\delta(t-a), \tag{6a}$$

$$\mathrm{S}^2(t) = \mathrm{S}_1{}^2\delta(t-a) + \mathrm{S}_2{}^2\delta(t-2a). \tag{6b}$$

S denotes that an impulse is emitted polarized parallel to the slow axis of the crystal, while F denotes polarization parallel to the fast axis. In Fig. 8, slow-axis and fast-axis polarizations are denoted by dotted and solid lines, respectively. The superscript 2 means that we are

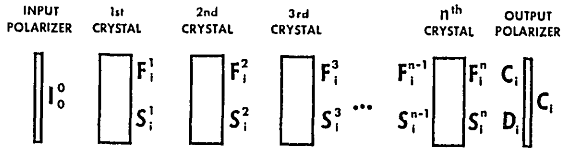

FIG. 9. n-stage network. Compare with two-stage network in Fig. 8.

dealing with the output from the second crystal of the network. The subscripts give the time of occurrence of the impulses. The first impulse, emitted at $t=0$, has the subscript 0; the next two impulses, emitted at $t=a$, have the subscript 1; and so on. Notice, in particular, that S_0^2 and F_2^2 are zero.

Since the impulses are evenly spaced in time, it is not necessary henceforth to write the delta functions when describing an impulse train. All the information of Eqs. (6) is given when F_0^2, F_1^2, S_1^2, and S_2^2 are stated.

As noted earlier, the desired transfer function and corresponding set of impulses are denoted by $C(\omega)$ and C_i, respectively. There is also an orthogonally polarized component which is stopped by the output polarizer. This signal and its corresponding set of impulses is denoted by $D(\omega)$ and D_i. Finally, the area of the impulse incident on the first crystal of the network is denoted by I_0^0. The notation is further summarized in Fig. 9.

B. Procedure

At the outset, two points should be stressed. First, it is assumed that the birefringent crystals of the network are lossless. This means that at all points between the input and output polarizers, energy must be conserved. Energy conservation places certain important restrictions on the F_i and S_i which are derived and listed in Appendix B. Secondly, it should be noted that $F_i^i=S_0^i=0$. This is just a statement of the fact that the first and last impulses out of the ith crystal must have propagated along its fast and slow axes, respectively.

We begin by assuming that $C(\omega)$ and, therefore, the desired C_i of Eqs. (2) and (3) have been chosen. We must next find the orthogonal signal, i.e., the signal $D(\omega)$ that is stopped by the output polarizer. By conservation of energy, we have

$$D(\omega)D^*(\omega)=(I_0^0)^2-C(\omega)C^*(\omega). \quad (7)$$

The left side of this equation must be non-negative for all frequencies and, therefore, for the equation to be valid, $(I_0^0)^2$ must be chosen greater than the maximum value of $C(\omega)C^*(\omega)$. As long as $(I_0^0)^2$ exceeds this value, its choice is arbitrary. However, it will generally be desirable to choose $(I_0^0)^2$ equal to the maximum value of $C(\omega)C^*(\omega)$, since this insures 100% transmission at the frequency at which this maximum occurs. Appendix A shows one method for calculating $D(\omega)$ from $D(\omega)D^*(\omega)$. It is also shown in Appendix A that as long as $(I_0^0)^2$ is chosen sufficiently large, at least one real set of D_i can always be found. Once $(I_0^0)^2$ has been chosen, $D(\omega)$ is calculated and written in the form

$$D(\omega)=D_0+D_1e^{-ia\omega}+D_2e^{-i2a\omega}+\cdots+D_ne^{-ina\omega}. \quad (8)$$

The corresponding orthogonal impulse response is then

$$D(t)=D_0+D_1\delta(t-a)+D_2\delta(t-2a)+\cdots +D_n\delta(t-na). \quad (9)$$

With the C_i and D_i specified, we now have a complete description of the input to the output polarizer. This, of course, is also the output from the last (nth) crystal. It is convenient here to transform this output into the principal axis system of the final crystal. With the help of Fig. 10(a), we have

$$\begin{bmatrix} F_i^n \\ S_i^n \end{bmatrix}=\begin{bmatrix} \sin\theta_p & -\cos\theta_p \\ \cos\theta_p & \sin\theta_p \end{bmatrix}\begin{bmatrix} C_i \\ D_i \end{bmatrix}, \quad (10)$$

where θ_p is the relative angle of the output polarizer.

As mentioned earlier, a requirement is that

$$F_n^n=S_0^n=0. \quad (11)$$

Using Eq. (10), we see that Eq. (11) will be satisfied if

$$\tan\theta_p=D_n/C_n \quad (12a)$$

and

$$\tan\theta_p=-C_0/D_0. \quad (12b)$$

In order for Eqs. (12a) and (12b) to be satisfied simultaneously, it must be true that $C_0C_n+D_0D_n=0$. But we know this is satisfied from conservation of energy, since it is Eq. (B13) of Appendix B.

Thus by using either Eq. (12a) or (12b), the angle of the final polarizer is determined. Then, substituting this calculated value of θ_p into Eq. (10), we obtain F_i^n and S_i^n, the outputs along the fast and slow axes of the last crystal. We now must find the rotation angles of the n crystals.

To accomplish this, we first find expressions relating the input and output of each crystal. This is a matter of taking projections along S and F axes of the crystals. With the help of Figs. 10(b) and 10(c), we find that

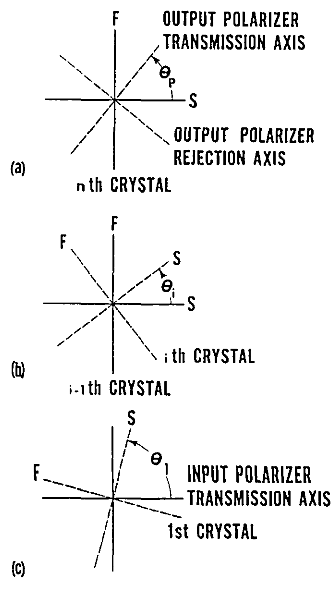

FIG. 10. Angle conventions used in the synthesis procedure: (a) output polarizer; (b) relative crystal angles; (c) input polarizer.

First Crystal

$$\begin{bmatrix} F_0^1 \\ S_1^1 \end{bmatrix} = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \end{bmatrix} [I_0^0], \tag{13a}$$

Second Crystal

$$\begin{bmatrix} F_0^2 \\ F_1^2 \\ S_1^2 \\ S_2^2 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & 0 \\ 0 & -\sin\theta_2 \\ \sin\theta_2 & 0 \\ 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} F_0^1 \\ S_1^1 \end{bmatrix}, \tag{13b}$$

Third Crystal

$$\begin{bmatrix} F_0^3 \\ F_1^3 \\ F_2^3 \\ S_1^3 \\ S_2^3 \\ S_3^3 \end{bmatrix} = \begin{bmatrix} \cos\theta_3 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & -\sin\theta_3 \\ \sin\theta_3 & 0 & 0 & 0 \\ 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 0 & \cos\theta_3 \end{bmatrix} \begin{bmatrix} F_0^2 \\ F_1^2 \\ S_1^2 \\ S_2^2 \end{bmatrix}. \tag{13c}$$

From the pattern established, we can write for the ith crystal

ith Crystal

$$\begin{bmatrix} F_0^i \\ F_1^i \\ F_{2i} \\ \cdots \\ \cdots \\ \cdots \\ F_{i-3}^i \\ F_{i-2}^i \\ F_{i-1}^i \\ S_1^i \\ S_2^i \\ S_3^i \\ \cdots \\ \cdots \\ \cdots \\ S_{i-2}^i \\ S_{i-1}^i \\ S_i^i \end{bmatrix} = \begin{bmatrix} \cos\theta_i & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cos\theta_i & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cos\theta_i & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\sin\theta_i & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\sin\theta_i & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -\sin\theta_i \\ \sin\theta_i & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \sin\theta_i & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \sin\theta_i & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cos\theta_i & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cos\theta_i & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cos\theta_i \end{bmatrix} \begin{bmatrix} F_0^{i-1} \\ F_1^{i-1} \\ F_2^{i-1} \\ \cdots \\ \cdots \\ \cdots \\ F_{i-3}^{i-1} \\ F_{i-2}^{i-1} \\ S_1^{i-1} \\ S_2^{i-1} \\ S_3^{i-1} \\ S_4^{i-1} \\ \cdots \\ \cdots \\ \cdots \\ S_{i-2}^{i-1} \\ S_{i-1}^{i-1} \end{bmatrix}. \tag{13d}$$

Our procedure is to start with the output from the last crystal. From these F_i^n and S_i^n, we calculate the crystal angle and the input to the crystal (the F_i^{n-1} and S_i^{n-1}). Since the input to the nth stage is the output from the $(n-1)$th stage, we can repeat the entire process for the $(n-1)$th crystal. Thus we work our way back through the entire network alternately finding crystal angles and crystal inputs.

The calculation of the angles and inputs is accomplished as follows: Consider, for example, Eq. (13c) which relates the input and output of the third crystal. We know the output (the F_i^3 and S_i^3) and wish to find θ_3 and the input (the F_i^2 and S_i^2). In the language of linear equation theory, the problem may be restated as, "Does the system of nonhomogeneous equations (13c) have a solution?"

A set of nonhomogeneous equations has a solution if and only if the rank of the matrix of the coefficients is equal to the rank of the augmented matrix.[22] For Eqs. (13c), this means that a solution exists if the rank of the coefficient matrix

$$\begin{bmatrix} \cos\theta_3 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & -\sin\theta_3 \\ \sin\theta_3 & 0 & 0 & 0 \\ 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 0 & \cos\theta_3 \end{bmatrix}$$

equals the rank of the augmented matrix

$$\begin{bmatrix} \cos\theta_3 & 0 & 0 & 0 & F_0^3 \\ 0 & \cos\theta_3 & -\sin\theta_3 & 0 & F_1^3 \\ 0 & 0 & 0 & -\sin\theta_3 & F_2^3 \\ \sin\theta_3 & 0 & 0 & 0 & S_1^3 \\ 0 & \sin\theta_3 & \cos\theta_3 & 0 & S_2^3 \\ 0 & 0 & 0 & \cos\theta_3 & S_3^3 \end{bmatrix}.$$

Since the rank of the coefficient matrix is 4, the rank of the augmented matrix must also be 4 for a solution to exist. Several procedures exist[22] for determining the rank of a matrix. Applying one of these, we find the rank of the augmented matrix to be 4 if

$$\tan\theta_3 = -(F_2^3/S_3^3) \tag{14a}$$

and

$$F_0^3F_2^3 + S_1^3S_3^3 = 0. \tag{14b}$$

[22] D. C. Murdoch, *Linear Algebra for Undergraduates* (John Wiley & Sons, Inc., New York, 1947), p. 50–51.

TABLE I. Related sets of D_i and their corresponding θ_i.

	Solutions for D_i 1st set	2nd set	3rd set	4th set		Corresponding crystal and polarizer angles 1st set	2nd set	3rd set	4th set
D_0	Δ_0	$-\Delta_0$	Δ_n	$-\Delta_n$	θ_1	Θ_1	$-\Theta_1$	Θ_p	$-\Theta_p$
D_1	Δ_1	$-\Delta_1$	Δ_{n-1}	$-\Delta_{n-1}$	θ_2	Θ_2	$-\Theta_2$	Θ_n	$-\Theta_n$
D_2	Δ_2	$-\Delta_2$	Δ_{n-2}	$-\Delta_{n-2}$	θ_3	Θ_3	$-\Theta_3$	Θ_{n-1}	$-\Theta_{n-1}$
D_3	Δ_3	$-\Delta_3$	Δ_{n-3}	$-\Delta_{n-3}$	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	θ_{n-1}	Θ_{n-1}	$-\Theta_{n-1}$	Θ_3	$-\Theta_3$
D_{n-1}	Δ_{n-1}	$-\Delta_{n-1}$	Δ_1	$-\Delta_1$	θ_n	Θ_n	$-\Theta_n$	Θ_2	$-\Theta_2$
D_n	Δ_n	$-\Delta_n$	Δ_0	$-\Delta_0$	θ_p	Θ_p	$-\Theta_p$	Θ_1	$-\Theta_1$

The first equation gives the angle of the crystal. Using this angle in Eq. (13c), we can now calculate the input (F_0^2, F_1^2, S_1^2, and S_2^2). The calculation is an easy one, involving for any stage no worse than the solution of two simultaneous equations. Appendix C shows a systematic method of performing this calculation. Equation (14b) is seen by comparison with Eq. (B9) to be simply a restatement of the fact that the F_i^3 and S_i^3 must satisfy conservation of energy. This requirement is automatically satisfied by the F_i and S_i of *all* stages since $D(\omega)$ was calculated using conservation of energy.

In a similar manner, the conditions for existence of solutions to Eqs. (13a), (13b), and (13d) result in the equations:

First Crystal

$$\tan\theta_1 = -(F_0^1/S_1^1), \tag{15a}$$

$$(F_0^1)^2 + (S_1^1)^2 = (I_0^0)^2, \tag{15b}$$

Second Crystal

$$\tan\theta_2 = -(F_1^2/S_2^2), \tag{16a}$$

$$F_0^2F_1^2 + S_1^2S_2^2 = 0, \tag{16b}$$

*i*th Crystal

$$\tan\theta_i = -(F_{i-1}^i/S_i^i), \tag{17a}$$

$$F_0^iF_{i-1}^i + S_1^iS_i^i = 0. \tag{17b}$$

The crystal angles are given by Eqs. (15a), (16a), and (17a), while Eqs. (15b), (16b), and (17b) are statements of conservation of energy. We now have all the information necessary for performing the synthesis. The entire procedure is summarized below.

C. Summary of Synthesis Procedure

(1) Choose the desired output response $C(\omega)$ and write it in the form of Eq. (2). The C_i must be real.

(2) Calculate the crystal length L from $L = ac/\Delta\eta$. The quantity a is obtained by comparing the $C(\omega)$ written in step (1) to $C(\omega)$ as given by Eq. 2. A rough estimate of L can be obtained from Fig. 7.

(3) Choose a value for I_0^0; the choice is arbitrary so long as $(I_0^0)^2$ exceeds the maximum value of $C(\omega)C^*(\omega)$. It will often be advantageous to make $(I_0^0)^2$ equal to the maximum value of $C(\omega)C^*(\omega)$.

(4) Calculate $D(\omega)D^*(\omega)$ from Eq. (7). Solve for $D(\omega)$ from $D(\omega)D^*(\omega)$ using the method of Appendix A (or some equivalent method). The D_i must be real.

(5) Calculate the output polarizer angle θ_p from Eq. (12a).

(6) Calculate the F_i^n and S_i^n from Eq. (10).

(7) Calculate the crystal angle θ_n of the last stage using Eq. (17a). From Eqs. (C1) and (C2) calculate the input to the last stage (which is the output from the preceding stage).

(8) Repeat the procedure of (7) on each succeeding stage until all crystal angles have been found.

D. Number of Possible Networks

It has been stated that at least one real set of D_i can always be found. It would perhaps be more correct to amend this to read that at least four real sets can always be found, for the calculated sets of D_i can always be placed conveniently into groups of four. The relations between the D_i of these four sets and between the corresponding θ_i are shown in Table I. We see that these four sets give four network configurations which are related. For example, the optical network corresponding to the second set is the "mirror image" of the network obtained from the first set. It can be obtained from it simply by rotating each crystal and the output polarizer a negative, instead of positive, angle.

In addition, it is of interest to note that the network of the third set is precisely the same network that is obtained by turning the first set network end for end. This means that the output of a network will be the same, regardless of which end is used as the input end. Finally, the network resulting from the fourth set is the mirror image of that network obtained from the third set. Therefore, these four sets of D_i do not really give four different networks, but rather one network and three variations.

It is stated in Appendix A that a desired transfer function can be realized by $2^{(n-\frac{1}{2}m+1)}$ different networks, where m is the number of complex roots of Eq. (A8). If we consider that the networks of Table I represent only one, rather than four, networks, the statement should read $2^{(n-\frac{1}{2}m-1)}$ networks.

V. EXAMPLE

A sample calculation will now be performed to illustrate the synthesis procedure of Sec. IV. Suppose that the ideal transfer function $G(\omega)$ which we wish to approximate is the triangular wave of Fig. 11. A network having such a transfer function might be used as a linear discriminator to accomplish the conversion of frequency-modulated light to amplitude-modulated light.[23] As shown in Fig. 11, $G(\omega)$ is real and has a basic period of $2\pi/a$ rad/sec. We must first approximate $G(\omega)$ by a finite exponential series. A series containing six terms will be used. The number of terms is arbitrary, but in the case of the triangular wave six terms give a satisfactory approximation. For this example the Fourier approximation is used to find the series coefficients, although there are other approximations which could have been used.

The exponential Fourier series approximated to the triangular wave is

$$K(\omega)=4/\pi^2[(1/25)e^{+i5a\omega}+(1/9)e^{+i3a\omega}+e^{ia\omega}+e^{-ia\omega}+(1/9)e^{-i3a\omega}+(1/25)e^{-i5a\omega}], \quad (18)$$

which is plotted in Fig. 11. Note that $K(\omega)$ is the transfer function of a noncausal network. It is often more convenient to first calculate the approximating function in a noncausal form such as Eq. (18), and then make the function causal. We can make $K(\omega)$ causal by multiplying it by $e^{-i5a\omega}$, which gives

$$C(\omega)=e^{-i5a\omega}K(\omega)=4/\pi^2[1/25+(1/9)e^{-i2a\omega}+e^{-i4a\omega}+e^{-i6a\omega}+(1/9)e^{-i8a\omega}+(1/25)e^{-i10a\omega}]. \quad (19)$$

Multiplication by $e^{-i5a\omega}$ is equivalent to introducing a pure time delay in the time domain. Thus the network impulse response and transfer function are essentially unchanged by this operation.

Since alternate harmonics in Eqs. (18) and (19) are zero, we may let $2a\omega=b\omega$. Using this in Eq. (19), we obtain the final form for $C(\omega)$

$$C(\omega)=0.01621+0.04503e^{-ib\omega}+0.40528e^{-i2b\omega}+0.40528e^{-i3b\omega}+0.04503e^{-i4b\omega}+0.01621e^{-i5b\omega}. \quad (20)$$

We must now calculate $D(\omega)$. From Eq. (7) we have $|D(\omega)|^2=D(\omega)D^*(\omega)=(I_0^0)^2-C(\omega)C^*(\omega)$,

$$=(I_0^0)^2-0.33309-0.40443\cos b\omega-0.09928\cos 2b\omega-0.03034\cos 3b\omega-0.00292\cos 4b\omega-0.000526\cos 5b\omega. \quad (21)$$

The area I_0^0 of the input impulse must now be chosen in order to obtain $|D(\omega)|^2$. It may have any real value as long as $(I_0^0)^2$ is larger than the maximum value of $C(\omega)C^*(\omega)$. From Fig. 11 we see that the maximum value of $C(\omega)C^*(\omega)$ occurs at $\omega=0$ and has a value of

[23] S. E. Harris, Appl. Phys. Letters 2, 47 (1963).

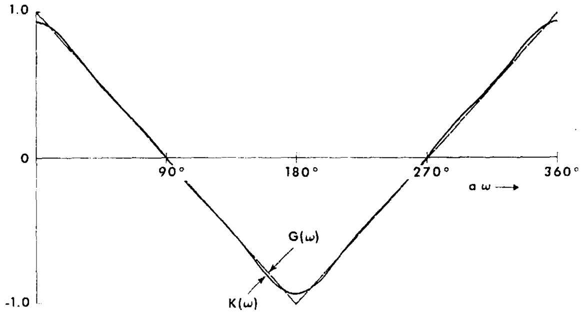

FIG. 11. Ideal and approximating transfer functions of example. Ideal transfer function is shown by dotted line and approximating transfer function by solid line.

0.87059. Let us choose $I_0^0=1$. Equation (21) then becomes

$$|D(\omega)|^2=0.66691-0.40443\cos b\omega-0.09928\cos 2b\omega-0.03034\cos 3b\omega-0.00292\cos 4b\omega-0.000526\cos 5b\omega. \quad (22)$$

We will now use the method outlined in Appendix A to calculate $D(\omega)$. We first form Eq. (A5)

$$-0.00263x^5-0.00146x^4-0.01517x^3-0.04964x^2-0.20222x+0.66691-0.20222x^{-1}-0.04964x^{-2}-0.01517x^{-3}-0.00146x^{-4}-0.000263x^{-5}=0. \quad (23)$$

We next wish to put Eq. (23) in the form of (A6). To determine the B_i, we equate similar coefficients in (A5) and (A6) which gives

$$\begin{aligned}
B_5&=A_5 &&=-0.00263,\\
B_4&=A_4 &&=-0.00146,\\
B_3&=A_3-5A_5 &&=-0.01385,\\
B_2&=A_2-4A_4 &&=-0.04380,\\
B_1&=A_1+5A_5-3A_3 &&=-0.15803,\\
B_0&=A_0+2A_4-2A_2 &&=+0.76327.
\end{aligned}$$

Substituting these into (A6) and letting $(x+x^{-1})=y$, we have

$$-0.000263y^5-0.00146y^4-0.01385y^3-0.04380y^2-0.15803y+0.76327=0. \quad (24)$$

The roots of (24) are

$$y_1=-4.07379+i3.93269,\quad y_2=-4.07379-i3.93269,$$
$$y_3=0.18957+i6.39532,\quad y_4=0.18957-i6.39532,$$
$$y_5=2.21289. \quad (25)$$

From Eq. (A7) the corresponding x_i are found to be

$$x_1=-3.95066+i4.05920,\quad x_2=-3.95066-i4.05920,$$
$$x_3=0.18525+i6.54791,\quad x_4=0.18525-i6.54791,$$
$$x_5=1.57997; \quad (26)$$
$$x_1^{-1}=-0.12313-i0.12652,\quad x_2^{-1}=-0.12313+i0.12652,$$
$$x_3^{-1}=0.00432-i0.15260,\quad x_4^{-1}=0.00432+i0.15260,$$
$$x_5^{-1}=0.63293.$$

TABLE II. The 16 real sets of D_i.

Set	D_0	D_1	D_2	D_3	D_4	D_5
$x_1x_2x_3x_4x_5$	−0.75607	0.29887	0.07414	0.02091	0.00207	0.00035
$x_1x_2x_3x_4x_5$	0.75607	−0.29887	−0.07414	−0.02901	−0.00207	−0.00035
$x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}$	−0.00035	−0.00207	−0.02091	−0.07414	−0.29887	0.75607
$x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}$	0.00035	0.00207	0.02091	0.07414	0.29887	−0.75607
$x_1x_2x_3^{-1}x_4^{-1}x_5$	−0.01762	0.01334	−0.75640	0.29188	0.09415	0.01491
$x_1x_2x_3^{-1}x_4^{-1}x_5$	0.01762	−0.01334	0.75640	−0.29188	−0.09415	−0.01491
$x_1^{-1}x_2^{-1}x_3x_4x_5^{-1}$	−0.01491	−0.09415	−0.29188	0.75640	−0.01334	0.01762
$x_1^{-1}x_2^{-1}x_3x_4x_5^{-1}$	0.01491	0.09415	0.29188	−0.75640	0.01334	−0.01762
$x_1x_2x_3^{-1}x_4^{-1}x_5^{-1}$	−0.01115	0.01901	−0.48006	0.63730	0.17107	0.02356
$x_1x_2x_3^{-1}x_4^{-1}x_5^{-1}$	0.01115	−0.01901	0.48006	−0.63730	−0.17107	−0.02356
$x_1^{-1}x_2^{-1}x_3x_4x_5$	−0.02356	−0.17107	−0.63730	0.48006	−0.01901	0.01115
$x_1^{-1}x_2^{-1}x_3x_4x_5$	0.02356	0.17107	0.63730	−0.48006	0.01901	−0.01115
$x_1x_2x_3x_4x_5^{-1}$	−0.47854	0.64236	0.15462	0.03696	0.00379	0.00055
$x_1x_2x_3x_4x_5^{-1}$	0.47854	−0.64236	−0.15462	−0.03696	−0.00379	−0.00055
$x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5$	−0.00055	−0.00379	−0.03696	−0.15462	−0.64236	0.47854
$x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5$	0.00055	0.00379	0.03696	0.15462	0.64236	−0.47854

Since there are four complex roots to Eq. (24), there will be $2^{(n-\frac{1}{2}m+1)}=16$ real sets of D_i which can be obtained by multiplying the factors $(x-x_i)$ together in various ways. Eight of these sets are simply the negatives of the other eight. Consider the set that is found by constructing the polynomial

$$(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5).$$

Performing the indicated multiplication, we obtain

$$x^5+5.95085x^4+60.16845x^3+213.29090x^2 +859.85121x-2175.20862.$$

As stated by Eq. (A11), a set of D_i is proportional to the coefficients of this polynomial.

$$\begin{aligned} D_0&=-2175.20862q, & D_1&=859.85121q,\\ D_2&=213.29090q, & D_3&=60.16845q, \\ D_4&=5.95085q, & D_5&=q. \end{aligned} \tag{27}$$

The value of q is different for each set of D_i. For the above set, q is found from (A12) to be

$$q=\pm 3.47586\times 10^{-4}.$$

Substituting this value back into Eq. (27), we find that one set of D_i is

$$\begin{aligned} D_0&=-0.75607, & D_1&=0.29887,\\ D_2&=0.07414, & D_3&=0.02091,\\ D_4&=0.002068, & D_5&=0.000348. \end{aligned}$$

All 16 real sets of D_i are shown in Table II. We now go through, in detail, the synthesis procedure for the first set.

We first calculate the output polarizer angle from Eq. (12a). Doing so, we obtain

$$\tan\theta_p=D_5/C_5=0.02144,$$

which gives

$$\theta_p=1°14'.$$

Several equations provide checks on the numerical computations and should be used during the synthesis. For example, we should also calculate θ_p from (12b) to verify that (12a) and (12b) do indeed give the same result. These checks are available at various points in the synthesis procedure and will be pointed out when appropriate.

Equation (10) is now used to calculate the F_i^5 and S_i^5, giving

$$\begin{pmatrix} F_0^5\\ F_1^5\\ F_2^5\\ F_3^5\\ F_4^5 \end{pmatrix}=\begin{pmatrix} 0.75625\\ -0.29784\\ -0.06543\\ -0.01222\\ -0.00110 \end{pmatrix}. \qquad \begin{pmatrix} S_1^5\\ S_2^5\\ S_3^5\\ S_4^5\\ S_5^5 \end{pmatrix}=\begin{pmatrix} 0.05143\\ 0.40678\\ 0.40564\\ 0.04507\\ 0.01621 \end{pmatrix}.$$

We are now able to calculate θ_5, the angle of the last birefringent crystal. Using (17a) we find

$$\tan\theta_5=-(F_4^5/S_5^5)=0.06799,$$

which gives

$$\theta_5=3°53'.$$

As a check, we should see that Eq. (17b) is satisfied.

The input impulses to the fifth crystal are calculated next from Eqs. (C1) and (C2).

$$\begin{pmatrix} F_0^4\\ F_1^4\\ F_2^4\\ F_3^4\\ F_4^4 \end{pmatrix}=\frac{1}{\{(F_4^5)^2+(S_5^5)^2\}^{\frac{1}{2}}}\begin{pmatrix} F_0^5 & S_1^5\\ F_1^5 & S_2^5\\ F_2^5 & S_3^5\\ F_3^5 & S_4^5\\ F_4^5 & S_5^5 \end{pmatrix}\begin{bmatrix} S_5^5\\ -F_4^5 \end{bmatrix}=\begin{pmatrix} 0.75799\\ -0.26955\\ -0.03777\\ -0.00913\\ 0 \end{pmatrix},$$

TABLE III. Summary of results of example.

Set	θ_1	θ_2	θ_3	θ_4	θ_5	θ_p
$x_1x_2x_3x_4x_5$	$-88°46'$	$3°53'$	$29°21'$	$29°21'$	$3°53'$	$1°14'$
$x_1x_2x_3x_4x_5$	$88°46'$	$-3°53'$	$-29°21'$	$-29°21'$	$-3°53'$	$-1°14'$
$x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}$	$-1°14'$	$-3°53'$	$-29°21'$	$-29°21'$	$-3°53'$	$88°46'$
$x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5^{-1}$	$1°14'$	$3°53'$	$29°21'$	$29°21'$	$3°53'$	$-88°46'$
$x_1x_2x_3^{-1}x_4^{-1}x_5$	$-47°23'$	$60°25'$	$-68°34'$	$-68°34'$	$60°25'$	$42°37'$
$x_1x_2x_3^{-1}x_4^{-1}x_5$	$47°23'$	$-60°25'$	$68°34'$	$68°34'$	$-60°25'$	$-42°37'$
$x_1^{-1}x_2^{-1}x_3x_4x_5^{-1}$	$-42°37'$	$-60°25'$	$68°34'$	$68°34'$	$-60°25'$	$47°23'$
$x_1^{-1}x_2^{-1}x_3x_4x_5^{-1}$	$42°37'$	$60°25'$	$-68°34'$	$-68°34'$	$60°25'$	$-47°23'$
$x_1x_2x_3^{-1}x_4^{-1}x_5^{-1}$	$-34°32'$	$64°28'$	$-64°24'$	$-64°24'$	$64°28'$	$55°28'$
$x_1x_2x_3^{-1}x_4^{-1}x_5^{-1}$	$34°32'$	$-64°28'$	$64°24'$	$64°24'$	$-64°28'$	$-55°28'$
$x_1^{-1}x_2^{-1}x_3x_4x_5$	$-55°28'$	$-64°28'$	$64°24'$	$64°24'$	$-64°28'$	$34°32'$
$x_1^{-1}x_2^{-1}x_3x_4x_5$	$55°28'$	$64°28'$	$-64°24'$	$-64°24'$	$64°28'$	$-34°32'$
$x_1x_2x_3x_4x_5^{-1}$	$-88°04'$	$7°56'$	$45°40'$	$45°40'$	$7°56'$	$1°56'$
$x_1x_2x_3x_4x_5^{-1}$	$88°04'$	$-7°56'$	$-45°40'$	$-45°40'$	$-7°56'$	$-1°56'$
$x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5$	$-1°56'$	$-7°56'$	$-45°40'$	$-45°40'$	$-7°56'$	$88°04'$
$x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1}x_5$	$1°56'$	$7°56'$	$45°40'$	$45°40'$	$7°56'$	$-88°04'$

$$\begin{bmatrix} S_0^4 \\ S_1^4 \\ S_2^4 \\ S_3^4 \\ S_4^4 \end{bmatrix} = \frac{1}{\{(F_4^5)^2+(S_5^5)^2\}^{\frac{1}{2}}} \begin{bmatrix} F_0^5 & S_1^5 \\ F_1^5 & S_2^5 \\ F_2^5 & S_3^5 \\ F_3^5 & S_4^5 \\ F_4^5 & S_5^5 \end{bmatrix} \begin{bmatrix} F_4^5 \\ S_5^5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.42605 \\ 0.40914 \\ 0.04579 \\ 0.01625 \end{bmatrix}.$$

We can now calculate the angle of the fourth birefringent crystal using Eq. (17a), which gives $\tan\theta_4 = -(F_3^4/S_4^4) = 0.56197$, and $\theta_4 = 29°20'$. Equation (17b) again affords a check.

The synthesis procedure may now be completed by alternately using Appendix C and Eq. (17a) until all crystal angles have been determined. The steps are given below:

$$\begin{bmatrix} F_0^3 \\ F_1^3 \\ F_2^3 \\ F_3^3 \end{bmatrix} = \frac{1}{\{(F_3^4)^2+(S_4^4)^2\}^{\frac{1}{2}}} \begin{bmatrix} F_0^4 & S_1^4 \\ F_1^4 & S_2^4 \\ F_2^4 & S_3^4 \\ F_3^4 & S_4^4 \end{bmatrix} \begin{bmatrix} S_4^4 \\ -F_3^4 \end{bmatrix} = \begin{bmatrix} 0.86952 \\ -0.03454 \\ -0.01049 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} S_0^3 \\ S_1^3 \\ S_2^3 \\ S_3^3 \end{bmatrix} = \frac{1}{\{(F_3^4)^2+(S_4^4)^2\}^{\frac{1}{2}}} \begin{bmatrix} F_0^4 & S_1^4 \\ F_1^4 & S_2^4 \\ F_2^4 & S_3^4 \\ F_3^4 & S_4^4 \end{bmatrix} \begin{bmatrix} F_3^4 \\ S_4^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.48873 \\ 0.05842 \\ 0.01864 \end{bmatrix}.$$

$$\tan\theta_3 = -(F_2^3/S_3^3) = 0.56268, \quad \theta_3 = 29°20'.$$

$$\begin{bmatrix} F_0^2 \\ F_1^2 \\ F_2^2 \end{bmatrix} = \frac{1}{\{(F_2^3)^2+(S_3^3)^2\}^{\frac{1}{2}}} \begin{bmatrix} F_0^3 & S_1^3 \\ F_1^3 & S_2^3 \\ F_2^3 & S_3^3 \end{bmatrix} \begin{bmatrix} S_3^3 \\ -F_2^3 \end{bmatrix} = \begin{bmatrix} 0.99746 \\ -0.00145 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} S_0^2 \\ S_1^2 \\ S_2^2 \end{bmatrix} = \frac{1}{\{(F_2^3)^2+(S_3^3)^2\}^{\frac{1}{2}}} \begin{bmatrix} F_0^3 & S_1^3 \\ F_1^3 & S_2^3 \\ F_2^3 & S_3^3 \end{bmatrix} \begin{bmatrix} F_2^3 \\ S_3^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.06785 \\ 0.02139 \end{bmatrix}.$$

$$\tan\theta_2 = -(F_1^2/S_2^2) = 0.06802, \quad \theta_2 = 3°53'.$$

$$\begin{bmatrix} F_0^1 \\ F_1^1 \end{bmatrix} = \frac{1}{\{(F_1^2)^2+(S_2^2)^2\}^{\frac{1}{2}}} \begin{bmatrix} F_0^2 & S_1^2 \\ F_1^2 & S_2^2 \end{bmatrix} \begin{bmatrix} S_2^2 \\ -F_1^2 \end{bmatrix} = \begin{bmatrix} 0.99977 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} S_0^1 \\ S_1^1 \end{bmatrix} = \frac{1}{\{(F_1^2)^2+(S_2^2)^2\}^{\frac{1}{2}}} \begin{bmatrix} F_0^2 & S_1^2 \\ F_1^2 & S_2^2 \end{bmatrix} \begin{bmatrix} F_1^2 \\ S_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.02144 \end{bmatrix}.$$

$$\tan\theta_1 = -(F_0^1/S_1^1) = -46.62959, \quad \theta_1 = -88°46',$$

and the synthesis is completed.

The angles calculated from all sixteen real sets of D_i are summarized in Table III. Notice that it is necessary to go through the synthesis procedure for only four of the sets of D_i. The angles for the other 12 sets can be deduced from Table I. The results of the example have been verified by applying the matrix method of Jones[24] to the resulting networks.

VI. DISCUSSION

An important modification of the basic network of Fig. 1 results from associating a variable optical compensator with each crystal of the network. Such compensation can be accomplished either internally to the crystal (for example, by thermal control) or externally[6] (for example, by using a Soleil compensator). To the extent that the transfer function is sufficiently narrow band that the compensation may be considered achromatic, the transfer function may be tuned, without distortion, over its basic period. If we associate a compensation of θ rad with each crystal of the network, the resulting transfer function becomes

$$\begin{aligned} C_{\text{tuned}}(\omega) &= \sum_{k=0}^{n} C_k e^{-ik(a\omega-\theta)} \\ &= C_{\text{untuned}}(\omega-\theta/a). \end{aligned} \tag{28}$$

The tuned $C(\omega)$ is thus equal to the original $C(\omega)$ shifted by $\theta/2\pi$ of its basic period. Since the required compensation for each crystal is identical, a simple method of tuning such as uniform temperature variation of the entire filter might be attempted. Methods of tuning birefringent filters have been considered by a number of authors.[8–17]

The synthesis procedure is based on the assumption that all crystals have the same length. At first this may seem to be a rather severe restriction, but in reality it is not, for networks containing crystals of different lengths can result from the procedure. It is possible that one or more calculated angles θ_i will be zero, and two crystals with a relative angle of zero degrees are equivalent to a single crystal of twice the length.

An exact procedure for the synthesis of birefringent networks possessing arbitrary transfer functions has been presented. Interesting problems which merit further investigation include: (1) consideration of the effects of crystal misalignment, changes in birefringence, and errors in crystal length; (2) analysis of the angular aperture of these networks and maximization of it; and (3) consideration of the effects of dispersion of $\Delta\eta$.

APPENDIX A

In this Appendix, we give a method for calculating $D(\omega)$ from $|D(\omega)|^2$. In addition, we show that at least one real set of D_i exists, provided $|D(\omega)|^2$ never becomes negative.

[24] R. C. Jones, J. Opt. Soc. Am. **31**, 488 (1941).

Suppose we are given the positive semidefinite polynomial

$$|D(\omega)|^2 = A_0 + 2A_1 \cos a\omega + \cdots + 2A_n \cos na\omega. \tag{A1}$$

Rewriting (A1) we have

$$\begin{aligned} |D(\omega)|^2 = {} & A_n e^{ina\omega} + A_{n-1} e^{i(n-1)a\omega} + \cdots + A_1 e^{ia\omega} + A_0 \\ & + A_1 e^{-ia\omega} + \cdots + A_{n-1} e^{-i(n-1)a\omega} + A_n e^{-ina\omega}. \end{aligned} \tag{A2}$$

Notice that the zeros of (A2) appear in reciprocal pairs. Equation (A2) can, therefore, be factored as

$$\begin{aligned} |D(\omega)|^2 = {} & (D_0 + D_1 e^{ia\omega} + D_2 e^{i2a\omega} + \cdots + D_n e^{ina\omega}) \\ & \times (D_0 + D_1 e^{-ia\omega} + \cdots + D_n e^{-ina\omega}). \end{aligned} \tag{A3}$$

The D_i are not unique, but rather there are 2^{n+1} possible sets of them. Since $|D(\omega)|^2$ is even and always positive, we may write it as

$$|D(\omega)|^2 = D(\omega) D^*(\omega). \tag{A4}$$

Comparing (A3) and (A4), we see that (A4) can be satisfied only if the D_i of (A3) are real. Therefore, at least one *real* set of the coefficients must exist.

The following method of obtaining the D_i is due to Pegis.[4] For more details and explanation of the procedure, the reader should refer to his paper. We begin with $|D(\omega)|^2$ as given by Eq. (A1). Form the equation

$$\begin{aligned} & A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0 + A_1 x^{-1} + \cdots \\ & \quad + A_{n-1} x^{-(n-1)} + A_n x^{-n} = 0. \end{aligned} \tag{A5}$$

Put Eq. (A5) into the form

$$B_n (x+x^{-1})^n + B_{n-1} (x+x^{-1})^{n-1} + \cdots + B_0 = 0, \tag{A6}$$

and obtain the B_i from the A_i by equating similar coefficients. Make the substitution

$$x + x^{-1} = y \tag{A7}$$

into Eq. (A6), which gives

$$B_n y^n + B_{n-1} y^{n-1} + \cdots + B_0 = 0. \tag{A8}$$

Solve for the n roots of (A8) and call them $y_1, y_2, \cdots, y_n$. Using Eq. (A7), solve for the reciprocal pairs of roots

$$\begin{matrix} x_1, & 1/x_1, \\ x_2, & 1/x_2, \\ \vdots & \vdots \\ x_n. & 1/x_n. \end{matrix} \tag{A9}$$

Next, construct all possible equations having real coefficients d_i using one root from each row of (A9); e.g., one possibility would be

$$\begin{aligned} & (x-x_1)(x-1/x_2)(x-x_3)\cdots(x-x_n) \\ & \quad = x^n + d_{n-1} x^{n-1} + \cdots + d_2 x^2 + d_1 x + d_0. \end{aligned} \tag{A10}$$

The D_i are proportional to the d_i

$$\begin{aligned} D_0&=qd_0,\\ D_1&=qd_1,\\ \vdots\ &\quad\ \vdots\\ D_n&=dq_n. \end{aligned} \tag{A11}$$

The quantity q is found from

$$q^2(d_0^2+d_1^2+\cdots+d_n^2)=A_0, \tag{A12}$$

and upon substituting this value into (A11), we obtain the D_i.

The number of real sets of D_i will depend upon the number of y_i which are complex. If m of the y_i are complex, there will be $2^{(n-\frac{1}{2}m+1)}$ real sets of D_i. Half of these, however, will just be the negative of the other half, for q can be negative or positive.

APPENDIX B

We derive here the conditions which the F_i and S_i satisfy because of conservation of energy. Assume, for convenience, that we are dealing with a four-crystal network. Since the birefringent crystals are assumed to be lossless, the energy in the fast-axis output plus the energy in the slow-axis output of the fourth crystal must equal the input energy. Mathematically, this is expressed by

$$F^4(\omega)F^{4*}(\omega)+S^4(\omega)S^{4*}(\omega)=(I_0^0)^2. \tag{B1}$$

Writing out (B1) and equating similar coefficients, we obtain

$$(F_0^4)^2+(F_1^4)^2+(F_2^4)^2+(F_3^4)^2+(S_1^4)^2+(S_2^4)^2 +(S_3^4)^2+(S_4^4)^2=(I_0^0)^2, \tag{B2}$$

$$F_0^4F_1^4+F_1^4F_2^4+F_2^4F_3^4+S_1^4S_2^4+S_2^4S_3^4+S_3^4S_4^4=0, \tag{B3}$$

$$F_0^4F_2^4+F_1^4F_3^4+S_1^4S_3^4+S_2^4S_4^4=0, \tag{B4}$$

$$F_0^4F_3^4+S_1^4S_4^4=0. \tag{B5}$$

Similarly, we can derive for the ith stage

$$(F_0^i)^2+(F_1^i)^2+\cdots+(F_{i-1}^i)^2+(S_1^i)^2+(S_2^i)^2+\cdots +(S_i^i)^2=(I_0^0)^2, \tag{B6}$$

$$F_0^iF_1^i+F_1^iF_2^i+\cdots+F_{i-2}^iF_{i-1}^i+S_1^iS_2^i +S_2^iS_3^i+\cdots+S_{i-1}^iS_i^i=0, \tag{B7}$$

$$F_0^iF_2^i+\cdots+F_{i-3}^iF_{i-1}^i+S_1^iS_3^i+\cdots+S_{i-2}^iS_i^i=0, \tag{B8}$$

$$\vdots$$

$$F_0^iF_{i-1}^i+S_1^iS_i^i=0. \tag{B9}$$

It should be pointed out that $C(\omega)$ and $D(\omega)$ also satisfy conservation of energy, giving the equations

$$(C_0)^2+(C_1)^2+\cdots+(C_n)^2+(D_0)^2+(D_1)^2+\cdots +(D_n)^2=(I_0^0)^2, \tag{B10}$$

$$C_0C_1+C_1C_2+\cdots+C_{n-1}C_n+D_0D_1+D_1D_2+\cdots +D_{n-1}D_n=0, \tag{B11}$$

$$C_0C_2+C_1C_3+\cdots+C_{n-2}C_n+D_0D_2+D_1D_3+\cdots +D_{n-2}D_n=0, \tag{B12}$$

$$\vdots$$

$$C_0C_n+D_0D_n=0. \tag{B13}$$

APPENDIX C

This Appendix gives a systematic and rapid method of calculating the input to a crystal once the output is known. This is simply a formalized procedure of solving for the F^{i-1} and S^{i-1} of Eq. (13d) once the F^i and S^i are known. In matrix form, the expressions are

$$\begin{bmatrix} F_0^{i-1}\\ F_1^{i-1}\\ F_2^{i-1}\\ \vdots\\ F_{i-1}^{i-1} \end{bmatrix} = \frac{1}{\{(F_{i-1}^i)^2+(S_i^i)^2\}^{\frac{1}{2}}} \begin{bmatrix} F_0^i & S_1^i\\ F_1^i & S_2^i\\ \vdots & \vdots\\ \vdots & \vdots\\ F_{i-1}^i & S_i^i \end{bmatrix} \begin{bmatrix} S_i^i\\ -F_{i-1}^i \end{bmatrix}. \tag{C1}$$

$$\begin{bmatrix} S_0^{i-1}\\ S_1^{i-1}\\ \vdots\\ S_{i-1}^{i-1} \end{bmatrix} = \frac{1}{\{(F_{i-1}^i)^2+(S_i^i)^2\}^{\frac{1}{2}}} \begin{bmatrix} F_0^i & S_1^i\\ F_1^i & S_2^i\\ \vdots & \vdots\\ F_{i-1}^i & S_i^i \end{bmatrix} \begin{bmatrix} F_{i-1}^i\\ S_i^i \end{bmatrix}. \tag{C2}$$

One convenient check is that the calculated values F_{i-1}^{i-1} and S_0^{i-1} should always be zero.

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