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Conversion of Frequency-Modulated Light to Space-Modulated Light

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The conversion of a frequency-modulated light signal to a spatially-modulated light signal via an ideal linear dispersing element is considered. The purpose of the paper is to examine discrepancies between the results of an analysis based on the exact but complicated spectral approach to frequency modulation, as opposed to the results of an analysis based on the intuitive and often approximately correct varying-frequency approach. The spatial Fourier transform of the dispersed signal is calculated and then used as a moment generating function to obtain expressions for the centroid motion and width of the optical beam. Conditions for the validity of the varying-frequency approach are given.

INTRODUCTION

IF a frequency-modulated light signal is incident on an optical dispersing element such as the prism shown in Fig. 1, it is intuitively expected that the ray angle of the dispersed light, and, therefore, the position of the beam-spot, sweeps back and forth at the modulating frequency. This expectation is based on what is known in electrical engineering terminology as the varying-frequency or quasisteady-state viewpoint of frequency modulation,[1] wherein a frequency-modulated signal is considered as a sinusoidal signal whose instantaneous frequency varies with time.

It is well known, however, that a frequency-modulated signal may be described from a Fourier or spectral viewpoint, wherein the signal is seen to consist of a carrier and a number of sidebands. From this point of view, it appears that Fig. 1 should be replaced by a figure such as Fig. 2, in which each sideband or spectral component of the signal is dispersed to a different angle. Each light ray in the figure corresponds to an optical frequency differing by the modulating frequency from the optical frequency of adjacent rays, i.e., each ray is a separate sideband. The width of each spectral component, and therefore the amount of overlap of the various spectral components, is determined by the radiation pattern of the dispersing element used. The relative amplitudes and phases of the sidebands are determined by the depth and phase of the modulation and are not indicated in the figure.

In Fig. 2, it is not obvious that the beam will swing back and forth at the modulating frequency. Furthermore, if we consider the situation in Fig. 3, where the resolution of the dispersing element is sufficiently high that the sidebands are essentially separated, then the amplitude of each spectral component is constant in time; here there is certainly no beam swinging back and forth at the modulating frequency. Though the situation of Fig. 3, where the sidebands are spatially resolved, would be hard to realize at lower carrier frequencies (e.g., radio frequencies), at optical frequencies it is not hypothetical. An experiment has recently been performed by Kaminow[2] in which an optical maser signal was phase-modulated at 10 Gc and was then incident on a Fabry–Perot interferometer whose resolution was such that the spectral components of the signal were clearly separated.

The purpose of this paper is to consider the discrepancies between the results of an analysis of the con-

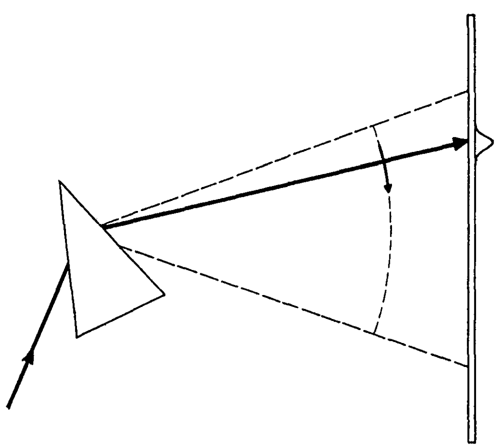

FIG. 1. Varying-frequency viewpoint.

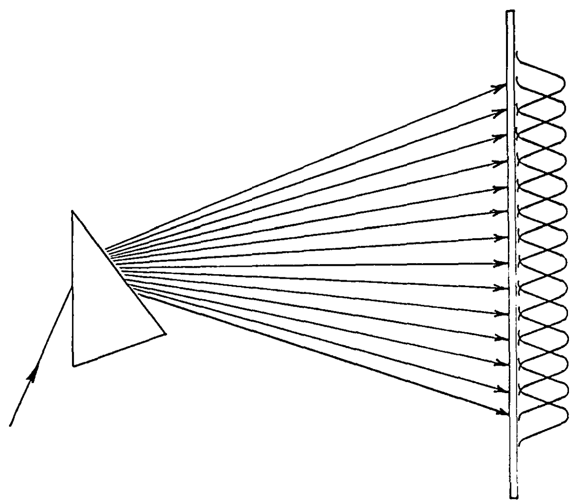

FIG. 2. Spectral viewpoint—considerable overlap of spectral components.

[1] F. E. Terman, *Electronic and Radio Engineering* (McGraw-Hill Book Company, Inc., New York, 1955), pp. 596–600.

[2] I. P. Kaminow, Appl. Phys. Letters 2, 41 (1963).

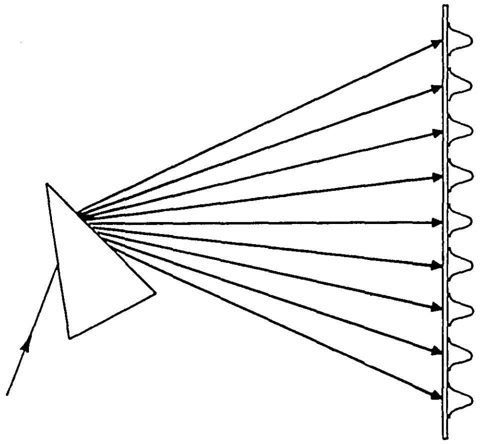

FIG. 3. Spectral viewpoint—spectral components essentially resolved.

version process based on the exact but complicated spectral approach of Figs. 2 and 3, as opposed to the results of an analysis based on the often approximately correct varying-frequency approach of Fig. 1.[3] Though this problem was treated earlier, in part, in connection with a discriminator phototube proposed by Siegman for the demodulation of frequency-modulated light,[4] the treatment in the present paper is considerably more general. In the following analysis, a general radiation pattern for the optical dispersing element is assumed, and an expression for the radiated intensity distribution as a function of space and time is found. Its Fourier transform with respect to the spatial variable is calculated and is then used as a moment generating function to find: the total intensity, the position of the centroid, and the instantaneous variance or width of the radiated intensity distribution. It is shown that if the modulating frequency or the optical resolution is sufficiently low that the spectral components overlap considerably, the Fourier transform based on the spectral viewpoint approximates the Fourier transform based on the varying-frequency viewpoint; and that, therefore, for a sufficiently low modulating frequency, these viewpoints agree in their consequences. On the other hand, if the modulating frequency or optical resolution is large enough that the spectral components are essentially separated, as in the case in Fig. 3, then all time variation of the optical intensity envelope will disappear. Of course, in this case, the varying-frequency approach is completely incorrect. For the general case of Fig. 2, where the spectral over-lap is considerable but not complete, the results of the analysis are interesting and perhaps surprising.

[3] A related question that has been considered by electrical engineers for many years is that of the validity of the varying-frequency approach when studying the effect of a linear network on a frequency modulated signal. A good review is given by R. F. Brown, Proc. IEE **104**, 52 (1957).

[4] S. E. Harris and A. E. Siegman, IRE Trans. Electron Devices **ED-9**, 322 (1962).

ANALYSIS

It is useful to write the optical signal which is applied to the dispersing element in a complex form known as the analytic signal.[5] We take the incident light signal to be sinusoidally frequency modulated and write it as

$$\begin{aligned} E(t) &= \sqrt{2}\exp[j(\omega_c t+\delta\sin\omega_m t)] \\ &= \sqrt{2}\exp(j\omega_c t)\sum_{n=-\infty}^{+\infty} J_n(\delta)\exp(jn\omega_m t), \end{aligned} \tag{1}$$

where ω_c is the optical carrier frequency, ω_m is the modulating frequency, and δ is the peak phase deviation.[6] The normalization is chosen such that the average intensity, i.e., $\frac{1}{2}E(t)E^*(t)$, is unity. For purposes of later comparison, we may note that the instantaneous phase of $E(t)$ is

$$\phi(t)=\omega_c t+\delta\sin\omega_m t, \tag{2}$$

and, therefore, the instantaneous frequency is

$$\begin{aligned} \omega(t) &= d\phi/dt=\omega_c+\delta\omega_m\cos\omega_m t \\ &= \omega_c+\omega_d\cos\omega_m t, \end{aligned} \tag{3}$$

where ω_d is the maximum frequency deviation.

We assume a dispersing element whose dispersion is linear with frequency and whose radiation pattern is one-dimensional and may be a complex function of position. We denote the radiation pattern of the element by $U(x-x_0)$, where x is the position coordinate in its focal plane. That is, $U(x-x_0)$ is the envelope of the radiated electric field strength when a monochromatic signal is incident on the dispersing element. We take $x_0=0$ for the optical carrier frequency $\omega=\omega_c$, and suppose that the dispersion is such that x_0 is displaced by a distance $\Delta x=a$ when the optical frequency shifts by an amount $\Delta\omega=\omega_m$. Thus the spectral components of the frequency-modulated signal are spaced a distance a apart in the focal plane.

The optical electric field strength as a function of space and time may then be written

$$E(x,t)=\sum_{n=-\infty}^{+\infty} J_n(\delta)U(x-na)\exp[j(\omega_c+n\omega_m)t]. \tag{4}$$

The optical intensity, or radiated power density in the focal plane of the dispersing element as a function of space and time is then

$$I(x,t)=\tfrac{1}{2}[E(x,t)E^*(x,t)]. \tag{5}$$

[5] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Inc., New York, 1959), pp. 492–496.

[6] Equation (1) is really only an approximation of the true analytical signal, but to the extent that ω_c,ω_m, and δ are such that there is essentially no spectral energy below zero frequency, the approximation is satisfactory. See, for instance, J. Dugundji, IRE Trans. Inform. Theory **IT-4**, 53 (1958).

Substituting Eq. (4), we find that $I(x,t)$ becomes

$$I(x,t)=\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} J_n(\delta)J_m(\delta)U(x-na)U^*(x-ma)\times\exp[j(n-m)\omega_m t]. \quad (6)$$

It may be noted that at each point of space $I(x,t)$ may contain components at all harmonics of the modulating frequency. The central idea of this paper is to analyze the properties of the intensity distribution described by Eq. (6).

The Fourier transform or moment generating function of $I(x,t)$ with respect to the space variable x is then defined as

$$Q(t,p)\equiv\int_{-\infty}^{+\infty}\exp(jxp)I(x,t)dx. \quad (7)$$

A change of variable, $q=n-m$, is next made, and $Q(t,p)$ becomes

$$Q(t,p)=\sum_{q=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} J_{q+m}(\delta)J_m(\delta)\exp(jq\omega_m t)\times\int_{-\infty}^{+\infty}\exp(jxp)U[x-(m+q)a]U^*[x-ma]dx. \quad (8)$$

To sum over m, we make the change of variable

$$y=x-ma-\tfrac{1}{2}qa \quad (9)$$

in order to remove m from under the integral. We then define[7]

$$R(p,q)\equiv\int_{-\infty}^{+\infty}\exp(jyp)U\left(y-\frac{qa}{2}\right)U^*\left(y+\frac{qa}{2}\right)dy. \quad (10)$$

With these changes, $Q(t,p)$ becomes

$$Q(t,p)=\sum_{q=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} J_{q+m}(\delta)J_m(\delta)\exp(jq\omega_m t)R(p,q)\times\exp\left[j\left(map+\frac{qap}{2}\right)\right]. \quad (11)$$

Use is then made of a form of Graf's generalization of Newmann's addition theorem for Bessel functions which is[8]

$$\exp\left[jq\left(\frac{\pi}{2}-\frac{ap}{2}\right)\right]J_q\left(2\delta\sin\frac{ap}{2}\right)=\sum_{m=-\infty}^{+\infty}J_{m+q}(\delta)J_m(\delta)\exp(jmap), \quad (12)$$

and which allows us to sum over m. The moment generating function then takes the considerably simpler form

$$Q(t,p)=\sum_{q=-\infty}^{+\infty}R(p,q)J_q\left(2\delta\sin\frac{ap}{2}\right)\times\exp\left[jq\left(\omega_m t+\frac{\pi}{2}\right)\right]. \quad (13)$$

Before proceeding, we consider the function $R(p,q)$ in more detail. This function is the Fourier transform of the complex correlation function of the radiation patterns of two spectral components which are spaced a distance qa apart. The function and its derivatives with respect to p are, in general, complex and at times are written in the polar form

$$R^{(n)}(p,q)=|R^{(n)}(p,q)|\exp j[\theta^{(n)}(p,q)], \quad (14)$$

where

$$R^n(p,q)\equiv(d^n/dp^n)R(p,q).$$

We note for future use that

$$R^{(n)}(0,-q)=(-1)^n[R^{(n)}(0,q)]^*, \quad (15)$$

which is obtained by examination of Eq. (10).

The total intensity at any instant of time may now be rapidly calculated. Thus, from Eq. (7), we have

$$I(t)=\int_{-\infty}^{+\infty}I(x,t)dx=Q(t,p)\Big|_{p=0}. \quad (16)$$

By using Eq. (13), and noting that $J_q(0)=1$ for $q=0$, and $J_q(0)=0$ for $q\neq0$, we then have

$$I(t)=R(0,0)=\int_{-\infty}^{+\infty}U(y)U^*(y)dy. \quad (17)$$

Thus as is expected from energy considerations, we find that the total intensity is independent of time; that is, the mixing between the various spectral components does not introduce any amplitude modulation of the total intensity. This result checks with our intuition from the varying-frequency viewpoint, wherein the beam swings back and forth with no variation in amplitude.

We next consider the motion of the centroid of the distribution $I(x,t)$. The first moment of the distribution, as a function of time, is

$$m(t)=\int_{-\infty}^{+\infty}xI(x,t)dx=-j\left[\frac{d}{dp}Q(t,p)\right]\Big|_{p=0}. \quad (18)$$

The evaluation of Eq. (18) involves the use of Eq. (15) with $n=0$ and $q=1$, and some common identities for the derivatives of Bessel functions. The result is

$$m(t)=a\delta|R(0,1)|\cos[\omega_m t+\theta(0,1)]-jR'(0,0). \quad (19)$$

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[7] It may be noted that $R(p,q)$ is of the same form as the radar ambiguity function; however, the author has not been able to find further significance in this relationship. P. M. Woodward, *Probability and Information Theory with Applications to Radar* (Pergamon Press, Inc., New York, 1953).

[8] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (University Press, Cambridge, England, 1952).

First, consider the second term on the right, which from Eq. (10) is given by

$$-jR'(0,0)=\int_{-\infty}^{+\infty} yU(y)U^*(y)dy. \qquad (20)$$

This term is the first moment of the radiation pattern of the dispersing element and will vanish if, as we shall suppose, its origin is taken at its centroid. If the total intensity of Eq. (17) is unity, then the centroid of $I(x,t)$ as a function of time is

$$x_{cg}(t)=a\delta|R(0,1)|\cos[\omega_m t+\theta(0,1)]. \qquad (21)$$

We may note that the centroid motion, on the basis of the intuitive varying-frequency point of view, is given by $x=a\delta\cos\omega_m t$. From Eq. (10) it is seen that $R(0,1)$ is simply the complex correlation function of two identical spectral components spaced a distance a apart. The spectral analysis has then given the result that the true centroid motion is equal to the motion obtained from the varying-frequency viewpoint, reduced in magnitude by the factor $|R(0,1)|$ and shifted in phase by $\theta(0,1)$. If the sidebands overlap greatly, then $|R(0,1)|$ approaches unity and $\theta(0,1)$ approaches zero, and the motion is given correctly by the varying-frequency viewpoint. As the overlap of the spectral components is reduced to zero, $|R(0,1)|$ approaches zero and the swinging motion disappears. We note that the motion predicted by the spectral analysis need not be in phase with the motion predicted by the varying-frequency approach. In fact, if $\theta(0,1)$ should equal π, then at every instant the beam would move in the direction opposite to that predicted by the varying-frequency approach. It is worth noting that the validity of the varying-frequency approach as applied to the determination of the centroid of the distribution is not affected by the magnitude of the peak phase deviation δ.

We next seek information as to the effective width of the swinging beam, and therefore set out to find its variance.[9] To start, the second moment $W(t)$ of $I(x,t)$ is given by

$$W(t)=\int_{-\infty}^{+\infty} x^2 I(x,t)dx=-\left[\frac{d^2}{dp^2}Q(t,p)\right]\bigg|_{p=0}. \qquad (22)$$

This expression is evaluated with the aid of two forms of Eq. (15); in one $n=1$, $q=1$; and in the other $n=0$, $q=2$. The result is

$$\begin{aligned}W(t)=&-R''(0,0)+\tfrac{1}{2}a^2\delta^2R(0,0)\\&+\tfrac{1}{2}a^2\delta^2|R(0,2)|\cos[2\omega_m t+\theta(0,2)]\\&+a\delta|R^1(0,1)|\sin[\omega_m t+\theta^1(0,1)]. \qquad (23)\end{aligned}$$

From Eq. (10), the last term of (23) is

$$R'(0,1)=j\int_{-\infty}^{+\infty} yU\left(y-\frac{a}{2}\right)U^*\left(y+\frac{a}{2}\right)dy. \qquad (24)$$

This integral is zero if $U(y-a/2)\,U^*(y+a/2)$ is an even function of y, which is the case if $U(y)$ is a Hermitian function of y, i.e., if $U^*(y)=U(-y)$. Since this is often true, we neglect this term in the following expressions.[10]

For normalized total intensity, the variance of $I(x,t)$ is then given by

$$\sigma^2(t)=W(t)-m^2(t). \qquad (25)$$

Substituting from Eqs. (19) and (23), and neglecting the last term of each of these equations, we then have the result

$$\begin{aligned}\sigma^2(t)=&-R''(0,0)+\tfrac{1}{2}a^2\delta^2[1-|R(0,1)|^2]\\&+\tfrac{1}{2}a^2\delta^2\{|R(0,2)|\cos[2\omega_m t+\theta(0,2)]\\&-|R(0,1)|^2\cos[2\omega_m t+\theta(0,1)]\}. \qquad (26)\end{aligned}$$

When the spectral overlap is very large, $|R(0,1)|$ and $|R(0,2)|$ approach unity, and $\theta(0,1)$ and $\theta(0,2)$ approach zero. The variance then reduces to

$$\sigma^2_{\text{varying-frequency}}(t)=-R''(0,0)=\int_{-\infty}^{+\infty} y^2U(y)U^*(y)dy. \qquad (27)$$

This is just the variance that would be obtained if the analysis were approached from the varying-frequency viewpoint, since from this point of view the width of the beam would remain constant as it swings back and forth at the modulating frequency. The other limiting case which we consider is that in which the resolution is allowed to become large enough that the spectral components are completely separated. In this situation $|R(0,1)|$ and $R(0,2)$ are zero, and Eq. (26) reduces to

$$\sigma^2_{\text{resolved}}(t)=\frac{a^2\delta^2}{2}+\int_{-\infty}^{+\infty} y^2U(y)U^*(y)dy. \qquad (28)$$

As a check on this work we could compute the variance for this case directly from Fig. 3, and with the help of the Bessel identity

$$\sum_0^\infty n^2J_n^2(\delta)=\delta^2/4,$$

we would obtain (28).

Returning to Eq. (26), we see that in the general case the variance is somewhat greater than that given by the varying-frequency viewpoint, and we also note the presence of a second-harmonic term. That is, the swinging beam contracts and expands slightly at twice the modulating frequency.

It is of interest to inquire as to the conditions which

[9] Unfortunately, the variance of certain line shapes of interest such as $\sin x/x$ diverges. For such line shapes, variance is not an appropriate measure of width.

[10] $U(y)$ will be Hermitian in the important case of Fraunhofer diffraction by an amplitude object. This results since the process of Fraunhofer diffraction is a Fourier transformation, and the Fourier transform of a real function is necessarily Hermitian. See M. Born and E. Wolf, Ref. 5, p. 400.

must be satisfied such that the moment generating function based on the spectral viewpoint, i.e. $Q(t,p)$ of Eq. (13), will reduce to a moment generating function based on the varying-frequency viewpoint; for, to the extent that two generating functions are equivalent, the properties of their respective distributions are equivalent. We show next (though not rigorously), that the conditions which must be satisfied for this equivalence are:

$$R(p,q) \cong R(p,0) \tag{29a}$$

for all integers q such that $J_q[2\delta \sin(ap/2)]$ has significant amplitude for any p; and

$$\sin ap \cong ap;\text{ i.e., } |p| \ll 1/a \tag{29b}$$

for all p such that $R(p,0)$ has significant amplitude. Equations (29a) and (29b) describe the situation of essentially complete spectral overlap. The first of these equations states that the Fourier transform of the complex correlation function of two spectral components which are spaced a distance qa apart should be approximately the same as the Fourier transform of the intensity distribution of a single component. The range of q over which this requirement must be satisfied is determined by the magnitude of the peak phase deviation δ; for δ determines the number of spectral components which have significant amplitude. Though δ did not influence the validity of the varying-frequency approach as applied to the determination of the centroid of the distribution, it does, in general, influence its validity as applied to higher moments. Next, consider the condition Eq. (29b). On account of the reciprocal-width property of Fourier transforms, $R(p,0)$ generally falls almost to zero for p greater than a few times the reciprocal of the linewidth of $U(x)U^*(x)$. Thus the condition that p be $\ll 1/a$, where $R(p,0)$ has significant amplitude is a statement to the effect that the linewidth of $U(x)$ should be many times the distance a between the spectral components; in other words, it is a statement that the spectral components should overlap appreciably. We now substitute the conditions Eqs. (29a) and (29b) into Eq. (13), and we find that the moment generating function then becomes

$$Q_{\text{varying-frequency}}(t,p) \cong \sum_{q=-\infty}^{+\infty} R(p,0) J_q(\delta ap) \times \exp\left[jq\left(\omega_m t + \frac{\pi}{2}\right)\right]. \tag{30}$$

This expression may be summed by use of the Bessel identity

$$\sum_{n=-\infty}^{+\infty} J_n(x) \exp(jn\theta) = \exp[j(x \sin\theta)], \tag{31}$$

with the result

$$Q_{\text{varying-frequency}}(t,p) \cong R(p,0) \exp[ja\delta p \cos\omega_m t]. \tag{32}$$

The moment generating function of Eq. (32) is just what is obtained from the varying-frequency viewpoint. Using that viewpoint, we find the radiated optical intensity as a function of space and time to be

$$I_{\text{varying-frequency}}(x,t) = |U(x - a\delta \cos\omega_m t)|^2, \tag{33}$$

from which, by use of the definition of the moment generating function [Eq. (7)], Eq. (32) readily follows. We have thus shown that the conditions (29) are those for which the varying-frequency approach is valid.

In the other limiting case, that of extremely high optical resolution, we have $R(p,q) \cong 0$ for $q \neq 0$, and thus be substituting into Eq. (13) we find

$$Q_{\text{resolved}}(t,p) \cong R(p,0) J_0(2\delta \sin\tfrac{1}{2}ap). \tag{34}$$

In this case, all of the moments of the intensity distribution will be independent of time, and we have the condition of Fig. 3.

SUMMARY OF RESULTS

When the optical resolution or modulation frequency is sufficiently low that the overlap of the sideband spectral components is very large, then all of the results of the spectral analysis reduce approximately to the results of an analysis using the varying-frequency point of view. For insufficient overlap of the spectral components, the magnitude of the centroid motion is smaller than that predicted by the varying-frequency approach, and may differ in phase from that predicted by the varying-frequency approach. The variance of the beam at any instant, in general, exceeds that predicted by the varying-frequency approach and, in general, contains a second-harmonic component.

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