Models of Technology Diffusion and Growth

Chris Tonetti

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Today's Class

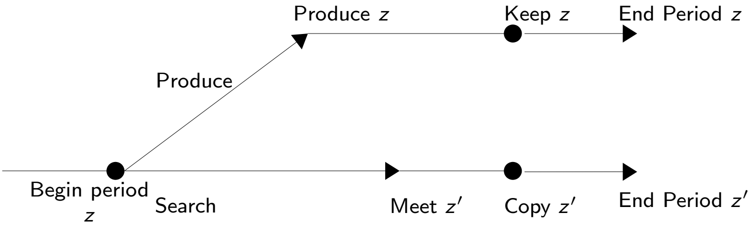
Equilibrium Imitation and Growth (JPE 2014)

Jesse Perla and Chris Tonetti

Key Mechanism

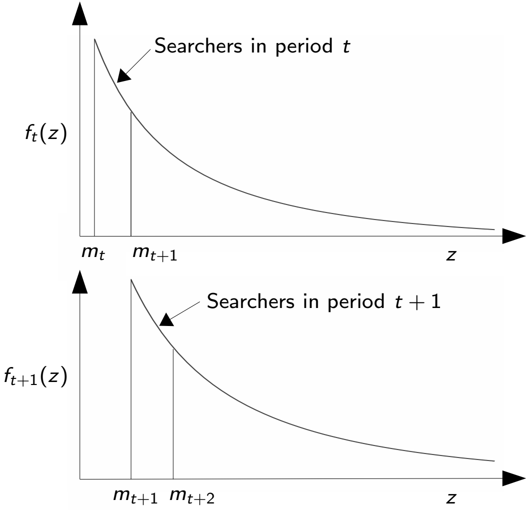
- Heterogeneous firms: produce or search for a new productivity
- Searchers randomly meet and copy a producing firm in the existing productivity distribution
- Selective search endogenously evolves distribution, shifting weight to more productive
- Aggregate state = productivity distribution, F_t, where $\min \text{support}\{F_t\} = m_t$

Intra-Period Timing



Evolution of the Productivity Distribution

f_t = productivity pdf, $m_t := \min \text{support}\{f_t\}$



Consumers

- $t = 0, 1, \ldots, \infty$
- Infinitely lived agents
- Representative consumer owns aggregate output Y_t
- Utility: $\sum_{t=0}^{\infty} \beta^t \frac{Y_t^{1-\gamma}}{1-\gamma}$, $\gamma \geq 0$
- Interest rate: $\frac{1}{1+r_t} := \beta \left(\frac{Y_{t+1}}{Y_t} \right)^{-\gamma}$

Firm Problem

Measure 1, linear production, aggregate state F, idiosyncratic z

$$V_t(z) = \max_{\{produce, adopt\}} \left\{ z + \tfrac{1}{1+r_t} V_{t+1}(z), \tfrac{1}{1+r_t} \int V_{t+1}(z') dF_t(z' | z' \geq \hat{z}_t) \right\} \tag{1}$$

- Solution is reservation productivity each period: m_{t+1}
- Firms uses forecast of $\hat{z}_t$ to calculate value
- In RE equilibrium, $\hat{z}_t = m_{t+1}$
- Discount with consumer's interest rate

Evolution of F is a Truncation

F_{t+1} is F_t truncated at m_{t+1}:

$$f_{t+1}(z) = f_t(z) + f_t(z \,|\, z \geq m_{t+1}) F_t(m_{t+1}) = \frac{f_t(z)}{1 - F_t(m_{t+1})} \quad (2)$$

Given an initial condition F_0, $m_0 \equiv \min \mathrm{support}\{F_0\}$, and a sequence $\{m_{t+1}\}$:

$$f_t(z) = \frac{f_0(z)}{1 - F_0(m_t)} \quad (3)$$

Firm Problem with Law of Motion

$$V_t(z) = \max_{\{produce, adopt\}} \left\{ z + \tfrac{1}{1+r_t} V_{t+1}(z), \tfrac{1}{1+r_t} \int_{m_{t+1}}^{\infty} V_{t+1}(z') \frac{f_0(z')}{1 - F_0(m_{t+1})} \mathrm{d}z' \right\} \tag{4}$$

- Optimal adoption policy is a sequence $\{m_t\}_{t=1}^{\infty}$
- Aggregate production: $Y_t = \int_{m_{t+1}}^{\infty} z\, f_t(z) \mathrm{d}z$
- Mass of Searchers: $S_t = \int_{m_t}^{m_{t+1}} f_t(z) \mathrm{d}z$

Equilibrium

A competitive equilibrium is a $\{m_t, V_t(\cdot), r_t\}_{t\geq 0}$, such that

i) given $\{r_t\}$, $\{m_{t+1}\}$ are the reservation productivities, with $\{V_t(\cdot)\}$ the associated value functions

ii) given $\{m_t\}$, $\{r_t\}$ are consistent with consumer IMRS

BGP with Heterogeneous Agents

- BGP for scalars is easy. e.g. $Y_{t+1} = gY_t$
- BGP for the growing distribution $F_t(z)$ is more complicated
- **Scale Invariant**
  A set of distributions, $\{F_t\}$, and scales, $\{m_t\}$, are scale invariant if

$$F_t(\tilde{z} m_t) \quad \text{are identical for all } t \geq 0, \tilde{z} \in [1, \infty)$$

BGP Equilibrium

A BGP Equilibrium is a Competitive Equilibrium, with a constant growth factor $g > 1$, such that

i) $Y_{t+1} = gY_t$

ii) $\{f_t\}$ with $\{m_t\}$ are scale invariant

Computing a BGP

Proposition

Given a Pareto initial condition and parameter restrictions (i.e. $F_0(z) = 1 - \left(\frac{m_0}{z}\right)^{\alpha}$)

An equilibrium exists with the following properties

i) *The growth rate is:* $g = \left(\beta \frac{\alpha}{\alpha - 1}\right)^{\frac{1}{\gamma - 1 + \alpha}}$

ii) *Minimum of Support:* $m_t = m_0 g^t$

iii) *Production:* $Y_t = \frac{\alpha}{\alpha - 1} g^{1-\alpha} m_t$

iv) *Searchers:* $S_t = 1 - g^{-\alpha}$

v) *The value function is piecewise-linear, with kinks at* $\{m_{t+1}\}$. *That is,* $\forall\, s \in \mathbb{N}$

$$V_t(z) = \frac{1+r}{r}\left(1 - \left(\frac{1}{1+r}\right)^{s}\right) z + \left(\frac{1}{1+r}\right)^{s} \bar{W} g^{t+s},\ z \in [m_0 g^{t+s}, m_0 g^{t+s+1}]$$

BGP Proof Sketch - Existence by Construction

Guesses:

- Pareto(m_0, α) will fulfill BGP requirements for f_0
  - $f_0(z; m_0, \alpha) = \alpha m_0^{\alpha} z^{-\alpha-1}$ with $\text{support}\{f_0\} = [m_0, \infty)$
  - $\implies f_t(z) = \alpha m_t^{\alpha} z^{-\alpha-1}$
- Reservation productivity: $m_{t+1} = g m_t$
- Value of adoption grows geometrically. For some constant $\bar{W}$:

$$V_t(z) = m_t \bar{W}, \text{ for } z \in [m_t, g m_t]$$

Verify and Solve:

i) Plug the Pareto guess into the indifference equation

ii) Simplify to a system of 2 equations containing g and $\bar{W}$

iii) Solve for g and $\bar{W}$, confirming they are not functions of t

Firm Problem with Guesses

$$V_t(z) = \max\left\{z + \tfrac{1}{1+r}V_{t+1}(z), \tfrac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^{\infty} V_{t+1}(z')z'^{-\alpha-1}\mathrm{d}z'\right\}$$

Indifference at m_{t+1}:

$$\begin{aligned} V_t(gm_t) =& gm_t + \tfrac{1}{1+r}V_{t+1}(gm_t) \\ =& \tfrac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^{\infty} V_{t+1}(z')z'^{-\alpha-1}\mathrm{d}z' \end{aligned}$$

Linear value of search guess gives 2 equalities:

$$\begin{aligned} m_t\bar{W} =& gm_t + \tfrac{1}{1+r}gm_t\bar{W} && \text{(EQ1)} \\ =& \tfrac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^{\infty} V_{t+1}(z')z'^{-\alpha-1}\mathrm{d}z' && \text{(EQ2)} \end{aligned}$$

Solving EQ1

Equate the first of the two equalities

$$m_t \bar{W} = g m_t + \frac{1}{1+r} g m_t \bar{W}$$

Solving for W

$$\bar{W} = \frac{g}{1 - g/(1+r)}$$

... independent of t, as required.

EQ2: Split Integral

Trick: Split the integral in EQ2 at *next* period's indifference point $(g^2 m_t)$ and use decision rule:

$$\begin{aligned} gm_t + \tfrac{1}{1+r} gm_t \bar{W} = {} & \tfrac{1}{1+r}\alpha(gm_t)^\alpha \int_{gm_t}^{g^2 m_t} V_{t+1}(z')z'^{-\alpha-1}\mathrm{d}z' \\ & + \tfrac{1}{1+r}\alpha(gm_t)^\alpha \int_{g^2 m_t}^{\infty} V_{t+1}(z')z'^{-\alpha-1}\mathrm{d}z' \end{aligned}$$

. . . computing the integrals separately.

EQ2: First Integral

By the decision rule, firms search at $t+1$ if $z \leq g^2 m_t$ with value $g m_t \bar{W}$

$$\begin{aligned}\int_{g m_t}^{g^2 m_t} V_{t+1}(z') z'^{-\alpha-1} \mathrm{d}z' &= g m_t \bar{W} \int_{g m_t}^{g^2 m_t} z'^{-\alpha-1} \mathrm{d}z' \\ &= \frac{g m_t \bar{W}}{\alpha} (g m_t)^{-\alpha} (1 - g^{-\alpha})\end{aligned}$$

EQ2: Second Integral

By the decision rule, firms will produce at $t+1$ if $z > g^2 m_t$

$$\begin{aligned}\int_{g^2 m_t}^{\infty} V_{t+1}(z') z'^{-\alpha-1} \mathrm{d}z' &= \int_{g^2 m_t}^{\infty} \left[z' + \tfrac{1}{1+r} V_{t+2}(z') \right] z'^{-\alpha-1} \mathrm{d}z' \\ &= \frac{1}{\alpha-1} (g^2 m_t)^{1-\alpha} + \tfrac{1}{1+r} \int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} \mathrm{d}z'\end{aligned}$$

…one last integral for $V_{t+2}(\cdot)$

EQ2: Third Integral

Trick: Using the indifference equation at $t+1$, where the reservation productivity is $g^2 m_t$.

$$\begin{aligned} V_{t+1}(g^2 m_t) =& g^2 m_t + \tfrac{1}{1+r} g^2 m_t \bar{W} \\ =& \tfrac{1}{1+r}\alpha (g^2 m_t)^{\alpha} \int_{g^2 m_t}^{\infty} V_{t+2}(z') z'^{-\alpha-1} \mathrm{d}z' \end{aligned}$$

EQ2: Collect Integrals

Combining all of the integrals into EQ2 and simplify

$$(1+r)g^{\alpha} = -\bar{W} + \frac{\alpha}{\alpha-1}g + g(1+\tfrac{1}{1+r}\bar{W})$$

... independent of t, as required.

Solve System for g, W

The system of equations is

$$\bar{W} = \frac{g}{1 - g/(1+r)}$$

$$(1+r)g^{\alpha} = -\bar{W} + \frac{\alpha}{\alpha - 1}g + g(1 + \tfrac{1}{1+r}\bar{W})$$

The solution, given parameter restrictions, is

$$g = \left[\tfrac{1}{1+r}\left(\frac{\alpha}{\alpha - 1}\right)\right]^{\frac{1}{\alpha - 1}}$$

Finally: Use Consumer's IMRS

- Given a fixed r, this is a solution for a constant g
- Given a fixed g, consumer problem gives $\frac{1}{1+r} = \beta g^{-\gamma}$

Substitute and rearrange

$$g = \left[\beta\left(\frac{\alpha}{\alpha-1}\right)\right]^{\frac{1}{\gamma-1+\alpha}}$$

Comparative Statics of Growth Rate

Proposition

The following properties hold for a solution to the BGP:

i) $\frac{\partial g}{\partial \beta} > 0$ *and* $\frac{\partial g}{\partial \gamma} < 0$

ii) g *is independent of* $\min \text{support}\{F_0\}$

iii) $\frac{\partial g}{\partial \alpha} < 0$

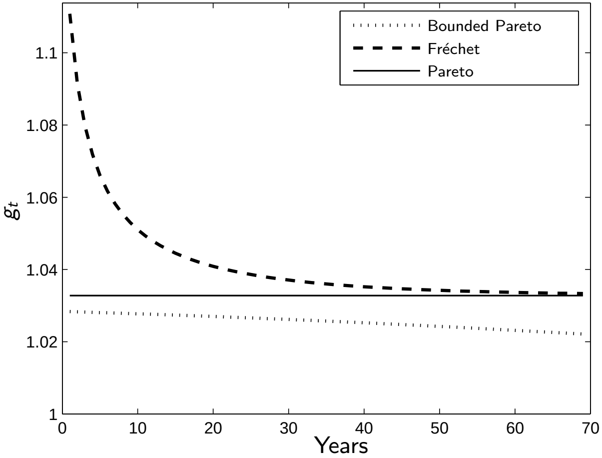
- $\downarrow \alpha$ *is* $\uparrow$ *inequality in Pareto*
- *Interpret* $\downarrow \alpha$ *as broader opportunities in the economy*
- *Fatter tail generates higher growth*

Numerical Solutions for the Dynamics

From a solution to the consumer's search problem, $\{m_t\}$:

- $f_t(z) = \frac{f_0(z)}{1-F_0(m_t)}$ and $Y_t = \int_{m_{t+1}}^{\infty} z\, dF_t$
- $g_t \equiv \frac{Y_{t+1}}{Y_t}$ may diverge, converge, or not be defined
- $\frac{1}{1+r_t} := \beta g_t^{-\gamma}$
- F_t may converge to a "degenerate" distribution

Dynamic Example with Alternate Distributions



Planner's Problem

- The planner makes the search vs. produce decision
- Describe recursively, with $f(\cdot)$ the state with $\min \operatorname{support}\{f\} = m(f)$
- Chooses the growth rate $g(f) \geq 1$ such that $m' = g(f)m(f)$, where $m(f) \equiv \min \operatorname{support}\{f\}$
- Maximizes the consumer's utility

$$U(f) = \max_{g \geq 1} \left\{ \frac{\left(\int_{gm(f)}^{\infty} z\, f(z) \mathrm{d}z \right)^{1-\gamma}}{1-\gamma} + \beta U(f') \right\}$$

$$\text{s.t. } f'(z) = \frac{f(z)}{1 - F(g\, m(f))}$$

Planner Proof Sketch

Guesses:

- The Pareto(m_0, α) will fulfill distribution requirements
- $U(m) = -Am^{1-\gamma}$, where $A > 0$

Verify and Solve:

i) Plug in guesses:

$$U(m) = \max_{g \geq 1} \left\{ \frac{\left(\frac{\alpha}{\alpha-1} g^{1-\alpha} m\right)^{1-\gamma}}{1-\gamma} + \beta U(gm) \right\}$$

$$-Am^{1-\gamma} = \max_{g \geq 1} \left\{ \left(\frac{\alpha}{\alpha - 1}\right)^{1-\gamma} \frac{g^{(1-\alpha)(1-\gamma)}}{1-\gamma} m^{1-\gamma} - \beta A g^{1-\gamma} m^{1-\gamma} \right\}$$

ii) Get the first order condition for g. Confirm m drops out

iii) Use the first order condition and Bellman to solve for A, g

iv) Find conditions on parameters such that the objective is globally concave

Planner vs. Competitive Equilibrium

Comparing first-best to competitive equilibrium:

$$g_{fb} = \left(\beta \frac{\alpha}{\alpha - 1}\right)^{\frac{1}{\gamma - 1}}, \quad g_{ce} = \left(\beta \frac{\alpha}{\alpha - 1}\right)^{\frac{1}{\gamma - 1 + \alpha}}$$

- $g_{fb} > g_{ce}$
- Signs of $\frac{\partial g}{\partial \beta}$, $\frac{\partial g}{\partial \gamma}$ and $\frac{\partial g}{\partial \alpha}$ same as the CE
- The wedge increases with higher inequality: $\frac{d(g_{fb}/g_{ce})}{d\alpha} < 0$

Extra Material

- Constrained planner problem
- Deriving the value function
- Normalization to stationary environment
- Unconditional draws
- Analytic results on dynamics
- Computational material for dynamics

Mechanism in Continuous Time

- Model setup used by:
  - Perla, Tonetti, Waugh (2014)
  - Benhabib, Perla, Tonetti (2015)
- Translate the discrete-time version as directly as possible
- Describe as an optimal stopping problem
- Describe as a free boundary problem
- Some notation and parameters:
  - r, discount rate
  - x, search cost
  - $V'(t) = \frac{\mathrm{d}V(t)}{\mathrm{d}t}$

Firm's Problem Summary

- As before, firm chooses to adopt vs. produce
  i) If it produces, it earns flow value Z
  ii) If it adopts, it pays xZ and draws with certainty
- Define the following to cast as an optimal stopping problem

  $V(Z,t)$: Value of production (i.e continuation)
  $V_s(t)$: Value of search (i.e. stopping) *before* costs
  $M(t)$: Optimal solution s.t. $Z \leq M(t)$ searches
  $S(t)$: *Flows* of adopters at time t
- As before, adopters only meet non-adopters

$$f(t, Z | Z > M(t))$$

Sequential Problem

- Given a $V_s(t)$ *gross* value of search at time t
- **Choice:** Define $T(Z,t)$ as the (absolute) time to search

$$M(t) \equiv \max\{Z | T(Z,t) = t\}$$

$$\begin{aligned} V(t,Z) &= \max_{T \geq t} \left\{ \int_t^T e^{-r(\tau - t)} Z \mathrm{d}\tau + e^{-r(T-t)} \left[V_s(T) - xZ \right] \right\} \\ &= \boxed{\max_{T \geq t} \left\{ \frac{1 - e^{-r(T-t)}}{r} Z - x e^{-r(T-t)} Z + e^{-r(T-t)} V_s(T) \right\}} \end{aligned}$$

Agent searches immediately at the point where $T(t,Z) = t$

Searching

- Agents will draw from the distribution of non-adopters:

$$f(t, Z|Z > M(t)) = \frac{f(t,Z)}{1-F(t,M(t))}$$

- Support of $f(t, Z)$ evolves with $M(t)$

$$\lim_{\Delta \to 0} \inf \text{support}\{f(t+\Delta, \cdot)\} = M(t)$$

$$\inf \text{support}\{f(t, \cdot)\} = M(t), \quad \text{at points of continuity}$$

- Hence, where $M(t)$ is continuous, $F(t, M(t)) = 0$
  - Draw directly from a distribution arbitrarily close to $f(t, Z)$
  - Only a flow of agents search at these points
  - Assume $M(t)$ is continuous $\forall t > 0$

Value of Search

- Gross value is the expected continuation value of the new draw

$$V_s(t) = \int_{M(t)}^{\infty} V(t,\tilde{Z}) f(t,\tilde{Z} | \tilde{Z} > M(t)) \mathrm{d}\tilde{Z}$$

At points of continuity of $M(t)$,

$$\boxed{V_s(t) = \int_{M(t)}^{\infty} V(t,\tilde{Z}) f(t,\tilde{Z}) \mathrm{d}\tilde{Z}}$$

Searchers as a Flow

- Searchers *cross* $M(t)$ barrier in each instant
- Method: “flux” or “probability current” for stochastic processes
- Reference frame:
  - Z doesn't change in the continuation region (i.e. $\mathrm{d}Z = 0 \cdot \mathrm{d}t$)
  - $M(t)$, the absorbing barrier, moves
- Change of variables to ensure $\tilde{Z} = 0$ at the barrier $\forall$ t

$$\tilde{Z} \equiv Z - M(t)$$
$$\tilde{f}(t, \tilde{Z}) = f(t, \tilde{Z} + M(t)) \implies \qquad \tilde{f}(t, 0) = f(t, M(t))$$

Using Ito's Lemma (no diffusion term) or direct Taylor series:

$$\mathrm{d}\tilde{Z} = -M'(t)\mathrm{d}t$$

Searchers

- The "probability current" at $\tilde{Z}$ is:

$$J(t,\tilde{Z}) = -M'(t)\tilde{f}(t,\tilde{Z})$$

- The flow of searchers is the probability current at $\tilde{Z} = 0$, where -1 is the "backwards" direction

$$\begin{aligned} S(t) &= -1 \times J(t,0) \\ &= M'(t)\tilde{f}(t,0) \\ &= \boxed{M'(t)f(t,M(t))} \end{aligned}$$

- See Gardiner (2009) equation 5.1.13, for more advanced cases

Law of Motion for $f(t, Z)$

- Flow $S(t)$ searchers draw in proportion to $f(t, Z)$
- No other changes in Z, with a "conservation" condition

$$\frac{\partial f(t,Z)}{\partial t} = S(t) f(t,Z), \forall Z > M(t) \qquad \int_{-\infty}^{\infty} f(t,Z) = 1, \forall t$$

Using $S(t)$ formula gives a "Kolmogorov Forward Equation"

$$\boxed{\frac{\partial f(t,Z)}{\partial t} = f(t, M(t)) M'(t) f(t,Z)}, \forall Z > M(t)$$

Solution: For any $M(t)$ and $f_0(Z)$ initial condition

$$\boxed{f(t,Z) = \frac{f_0(Z)}{1 - F_0(M(t))}, \quad \forall\, Z > M(t)}$$

A truncation, solution works *any* $M(t)$ (i.e. off-BGP)

Solution Approach

i) Simplify the sequential problem to generate an ODE in $V_s(t)$

ii) Use $V_s(t)$ to eliminate $V(t, Z)$ and get an integral equation

iii) Using the system of equations in $V_s(t)$, $M(t)$

- Guess and verify a BGP solution
- Solve for a non-BGP with arbitrary ICs

Using the First-Order Condition

$$V(t,Z) = \max_{T \geq t} \left\{ \frac{1-e^{-r(T-t)}}{r} Z - xe^{-r(T-t)} Z + e^{-r(T-t)} V_s(T) \right\}$$

Taking the FOC for T

$$0 = e^{-r(T-t)} \left(Z + xrZ - rV_s(T) + V_s'(T) \right)$$

Evaluate at indifference point, $M(t)$, where $T = t$:

$$\boxed{rV_s(t) = (1+xr)M(t) + V_s'(t)}$$

Or, with an asset pricing interpretation

$$r\left(V_s(t) - xM(t)\right) = M(t) + V_s'(t)$$

Unknowns: $V_s(t), M(t)$

Simplifying the Sequential Problem

- By definition, at the optimum $T(t,Z) = M^{-1}(Z)$
- Substitute and drop the max

$$V(t,Z) = \frac{1-e^{-r(M^{-1}(Z)-t)}}{r} Z - x e^{-r(M^{-1}(Z)-t)} Z + e^{-r(M^{-1}(Z)-t)} V_s(M^{-1}(Z))$$

Simplifying the Value of Search

Substitute $V(t, Z)$ and LOM into $V_s(t)$

$$V_s(t) = \int_{M(t)}^{\infty} V(t, \tilde{Z}) f(t, \tilde{Z}) \mathrm{d}\tilde{Z}$$
$$= \int_{M(t)}^{\infty} \left(\tfrac{1}{r} Z - \tfrac{1+xr}{r} e^{-r(M^{-1}(Z)-t)} Z + e^{-r(M^{-1}(Z)-t)} V_s(M^{-1}(Z)) \right) \frac{f_0(Z)}{1 - F_0(M(t))} \mathrm{d}Z$$

Unknowns: $V_s(t), M(t)$(and $M^{-1}(Z)$ indirectly)

Summary of Equations

Given parameters x, r and initial condition: $f_0(\cdot)$

$$rV_s(t) = (1 + xr)M(t) + V_s'(t)$$

$$V_s(t) = \int_{M(t)}^{\infty} \left(\tfrac{1}{r} Z - \tfrac{1+xr}{r} e^{-r(M^{-1}(Z)-t)} Z + e^{-r(M^{-1}(Z)-t)} V_s(M^{-1}(Z)) \right) \frac{f_0(Z)}{1 - F_0(M(t))} d$$

An integral-differential system in $M(t), V_s(t)$

Balanced Growth Path Guess

Guess and Verify:

- $F_0(Z) = 1 - \left(\frac{M(0)}{Z}\right)^{\alpha}$, a Pareto distribution
- $V_S(t) = V_S(0)e^{gt}$
- $M(t) = M(0)e^{gt}$
  - Note that $M(0)$ is chosen as the minimum of support
- Hence $M^{-1}(Z) = \frac{1}{g}\log\left(\frac{Z}{M(0)}\right)$

Plug into our system of 2 equations and use undetermined coefficients to solving for g and $V_S(0)$,

BGP Solution

Proposition

If $\alpha > 1$ *and* $1/(\alpha+1) < xr(\alpha-1) < 1$ *then:*

$$\boxed{g = \frac{1 - xr(\alpha-1)}{x\alpha(\alpha-1)}}$$

$$\boxed{V_s(0) = M(0)\frac{x\alpha(\alpha-1)(1+xr)}{xr(\alpha^2-1)-1}}$$

Substituting these into the sequential $V(t, Z)$*,*

$$\boxed{V(t,Z) = \frac{Z}{r} + \frac{g(1+xr)}{r-g}\frac{Z}{r}\left(\frac{Z}{M(0)}\right)^{-r/g} e^{rt}}$$

Interpreting the Value Function

$$V(t, Z) = \underbrace{\frac{Z}{r}}_{\text{Production Forever}} + \underbrace{CZ^{1-r/g}e^{rt}}_{\text{Option Value of Search}}$$

The option value of search:

- Increasing in time
- Decreasing in Z for a fixed t due to longer wait until search
  - Asymptotically linear in Z for a fixed t
- Makes the value function convex in Z for a fixed t

Recursive Continuation Value

$$V(t,Z) = Z\Delta + \frac{1}{(1+r\Delta)} V(t+\Delta, Z)$$

Multiply by $(1+r\Delta)$, subtract V(t,Z), and divide by Δ

$$rV(t,Z) = Z + \frac{V(t+\Delta,Z)-V(t,Z)}{\Delta}$$

Take the limit

$$rV(t,Z) = Z + \frac{\partial V(t,Z)}{\partial t}$$

Optimal Stopping Sufficiency Conditions

$V(t, Z)$, $M(t)$, and $V_s(t)$ must satisfy:

$$rV(t, Z) = Z + \frac{\partial V(t, Z)}{\partial t}$$

$$V(t, M(t)) = V_s(t) - xM(t)$$

$$\frac{\partial V(t, M(t))}{\partial Z} = \frac{\partial (V_s(t) - xM(t))}{\partial Z} = -x$$

$$V_s(t) = \int_{M(t)}^{\infty} V(t, \tilde{Z}) f(t, \tilde{Z}) \mathrm{d}\tilde{Z}$$

$$\frac{\partial f(t, Z)}{\partial t} = f(t, M(t)) M'(t) f(t, Z)$$

Normalizing to Stationary Environment: PDF

Recall Scale Invariance.
For convenience let Φ be unnormalized CDF and ϕ be unnormalized pdf

$$z := \frac{Z}{M(t)},$$
$$F(z,t) := \Phi(Z,t)$$
$$f(z,t) = M(t)\phi(Z,t)$$

Normalizing to Stationary Environment

Divide everything by M(t), since M'(t)/M(t)=g.

$$v(z,t) := \frac{V(Z,t)}{M(t)},$$

then

$$\begin{aligned} V(Z,t) &= M(t)v(Z/M(t),t) \\ \frac{dV(Z,t)}{dt} &= \frac{d(M(t)v(Z/M(t),t))}{dt} \\ \frac{dV(Z,t)}{dZ} &= M(t)\frac{dv(Z/M(t),t))}{dz} \end{aligned}$$

Normalizing Bellman: dt

$$
\begin{aligned}
\frac{dV(Z,t)}{dt} &= \frac{d(M(t)v(Z/M(t),t))}{dt} \\
&= \frac{dM(t)}{dt}v(z,t) + M(t)\frac{dv(z,t)}{dt} \\
&= M'(t)v(z,t) + M(t)\left[\frac{\partial v(z,t)}{\partial t} + \frac{\partial v(z,t)}{\partial z}\frac{\partial z}{\partial M(t)}\frac{\partial M(t)}{\partial t}\right] \\
&= M'(t)v(z,t) + M(t)\left[\frac{\partial v(z,t)}{\partial t} - \frac{\partial v(z,t)}{\partial z}\frac{z}{M(t)}M'(t)\right]
\end{aligned}
$$

$$
\frac{1}{M(t)}\frac{dV(Z,t)}{dt} = gv(z,t) + \frac{\partial v(z,t)}{\partial t} - gz\frac{\partial v(z,t)}{\partial z}
$$

Normalizing Bellman: Synthesis

$$rV(Z,t) = Z + \frac{\partial V(Z,t)}{\partial t}$$

Divide by M(t) and subsitute using definitions

$$\begin{aligned}
rv(z,t) &= z + \frac{1}{M(t)}\frac{\partial V(Z,t)}{\partial t} \\
rv(z,t) &= z + gv(z,t) + \frac{dv(z,t)}{dt} - gz\frac{\partial v(z,t)}{\partial z} \\
(r-g)v(z,t) &= z + \frac{\partial v(z,t)}{\partial t} - gz\frac{\partial v(z,t)}{\partial z}
\end{aligned}$$

Normalizing Smooth Pasting: dz

$$
\begin{aligned}
\frac{dV(Z,t)}{dZ} &= M(t)\frac{dv(Z/M(t),t))}{dZ} \\
&= M(t)\frac{\partial v(z,t)}{\partial z}\frac{1}{M(t)} \\
&= \frac{M(t)}{M(t)}\frac{\partial v(z,t)}{\partial z} \\
\frac{dV(Z,t)}{dZ} &= \frac{\partial v(z,t)}{\partial z}
\end{aligned}
$$

then smooth pasting is

$$
\frac{\partial v(1,t)}{\partial z} = -x
$$

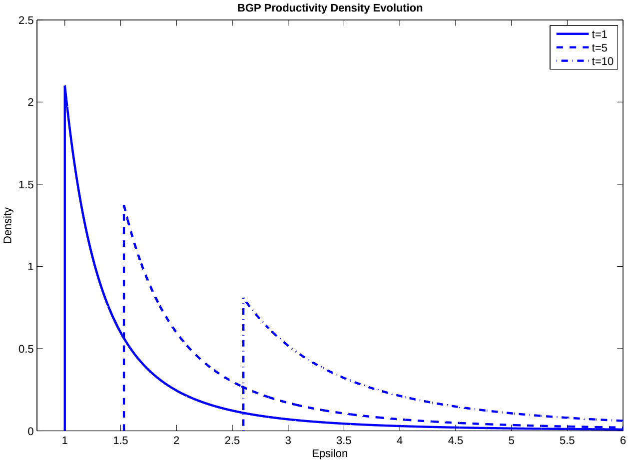
BGP: From PDE to ODE

On the BGP, normalized functions should not depend on time.

$$(r-g)v(z) = z - gz\frac{\partial v(z)}{\partial z}$$
$$\frac{\partial v(1)}{\partial z} = -x$$
$$v(1) = \int_1^\infty v(\tilde{z})f(\tilde{z})\mathrm{d}\tilde{z} - x$$

Appendix

BGP Evolution of the Productivity Distribution



Mass of Searchers

Mass of searchers are those below the $h(t)$ threshold

$$\mathbb{S}(t) := F(t, h(t))$$

At points of continuity,

$$\mathbb{S}(t) = F(t, \inf \text{support}(t, \cdot)) = 0$$

Law of Motion at Discontinuities

At points of discontinuity in $h(t)$, a mass $\mathbb{S}(t)$ "exit" and draw from $\lim_{\Delta \to 0} F(\cdot, t + \Delta)$

$$F(z, t+) = \underbrace{F(z, t)}_{\text{Was below } z} - \underbrace{\mathbb{S}(t)}_{\text{Searched}} + \underbrace{\mathbb{S}(t)F(z, t+)}_{\text{Searched and drew} \leq z}, \quad \text{for } z \geq h(t+) \tag{5}$$

$$F(z, t+) - F(z, t) = -(1 - F(z, t+))F(h(t), t) \tag{6}$$