

The Specification of Technology with Several Kinds of Output

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Econometric models encounter difficulties in handling more than a single kind of output. The absence of data on the flows of intermediate goods makes it impossible to estimate separate production functions for more than one product. Within the unifying framework of the modern theory of cost functions, this paper considers two alternative specifications that can accommodate disaggregation on the output side without requiring data on intermediate goods. The economic implications of these specifications are discussed, and a general specification is proposed in which it is possible to test the two specifications.

The theory of production functions suggests a straightforward way to specify the technology of an economy that produces more than one product. This specification has separate production functions for each product, taking as arguments the inputs of primary factors and of other products serving as intermediate goods. For the economies of the United States and most other countries, however, data are available only for the gross output of each product (including the amount used as inputs for other products), net output (deliveries to final demand), and direct inputs of primary factors. Except in scattered years, the data on the interindustry flows of goods necessary to estimate full production functions for each kind of output are unavailable. The absence of time series for the interindustry flows of goods has forced the proprietors of almost all complete econometric models of the U.S. economy to adopt the most aggregative specification possible, that of a single sector of production. The purpose of this paper is to discuss specifications of multisectoral technologies which can be estimated using the limited data available. These specifications are suitable for inclusion in a complete econometric model.

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Among well-known econometric models, only the Brookings Model confronts the problem of disaggregation directly and uses a truly multi-sectoral specification of the production (Fisher, Klein, and Shinkai 1965; Kresge 1969). We begin here with a brief discussion of the shortcomings of the Brookings scheme. In that scheme, the measure of the output of each product is the value added in the industry producing the product. The model incorporates a function relating the value added of an industry to its direct inputs of primary factors. This specification is at best a rough approximation, since only under very restrictive conditions does an exact functional relationship of this kind exist. Further, value added is not itself a useful measure of the output of a product. The demand functions in the Brookings Model, as in any econometric model, refer to deliveries to final demand, that is, to various categories of consumption and investment. In the Brookings Model, a fixed (or, recently, time-varying) fraction of each component of demand is allocated to each industry, and the value added in an industry is taken as the sum of all components allocated in this way. Although this is a purely mechanical expedient, it is probably about the best that can be done to unite the components of demand with a conventional industry classification of value added. No theoretically satisfactory solution is possible in the framework of the Brookings Model.

In this paper we return to the measure of output suggested by the theory of production, deliveries of goods to final demand. Since the data necessary to carry out direct estimation of separate production functions are invariably lacking, it is necessary to find a specification that relates output to observable measures of primary inputs. Although data on direct primary inputs are available for a quite highly disaggregated industry classification, this disaggregation is not useful because data on indirect primary inputs are not available. Recent progress has been achieved through the development of specifications based on measurements of *total* primary factor inputs to the whole economy. This paper presents a unified theoretical treatment of two approaches to the specification of multisectoral technologies based on the principle of relating disaggregated measures of the different kinds of output to aggregate quantities of primary factors. The first approach deals with the production possibility frontier under certain restrictive assumptions. The second deals with the set of demand functions for primary factors associated with the technology when factor markets are competitive.

The first approach was originally proposed by Mundlak (1963). He suggested estimation of the production possibility frontier giving an implicit relation between a vector of outputs, say y , and a vector of total inputs, say x . In general, a production possibility frontier can be defined in terms of a *transformation function*: $t(y, x) = 0$. In the absence of further restrictions, this formulation of technology permits arbitrary kinds of

interaction between total factor intensities and the trade-off between the various types of output. Mundlak proposed a substantive restriction on the form of the transformation function; he assumed that it can be written in the following way: $t(y, x) = -g(y) + f(x) = 0$. This restriction, known as *separability*, has a number of important implications whose study is one of the main purposes of the present paper. Two conclusions should be noted. First, separability almost always means that outputs are produced jointly. The only case in which the underlying production structure of the economy can be portrayed by separate production functions for each kind of output is the case where all the production functions are identical. Second, separability implies that output price ratios or marginal rates of transformation are independent of factor intensities or factor prices. This rather undesirable property makes it apparent that a specification of joint production with separability is no more general, in at least this crucial respect, than the one-sector specification.

A second contribution to the specification of production of several kinds of output is that of Diewert (1971). He observes that, if production functions and factor markets are competitive, then the amount of each factor employed by each industry can be written as a function of the output of that industry and the prices of all factors. With constant returns to scale, this amounts to saying that each output-factor ratio in each industry is determined solely by factor prices. The set of these functions for a given factor can be added across all industries to get a set of equations expressed entirely in observable variables:

$$\begin{aligned} x_1 &= \phi_1(y_1, \dots, y_M, w_1, \dots, w_N) \\ &\vdots \\ x_N &= \phi_N(y_1, \dots, y_M, w_1, \dots, w_N). \end{aligned}$$

Here w_1, \dots, w_N are the prices of the N factors x_1, \dots, x_N . If the factor demand functions ϕ_1, \dots, ϕ_N are derived from individual production functions, as they are in Diewert's work, then this system of equations embodies the substantive restriction, relative to the general transformation function, that production is *nonjoint*. A technology expressed by a transformation function is said to be joint if there is no way to portray it in terms of separate production functions, and nonjoint if it can be so portrayed.

The present paper investigates the implications of the restrictions of separability and nonjointness. Both are characterized in terms of the properties of the joint cost function. The method of analysis is based on principles of duality discovered by Shephard (1953), Uzawa (1964), and McFadden (1973). The paper goes on to propose a functional form for the joint cost function and related joint factor demands in which the two restrictions can be imposed parametrically. This function, the

generalized linear-generalized Leontief or hybrid Diewert joint cost function, makes it possible to carry out formal statistical tests of the two restrictions.

1. Approaches to the Specification of Joint Technologies

A well-behaved technology can be described equally well in terms of relations between quantities or relations between prices, as long as markets are competitive and profits are maximized. The basic relation among quantities for our purposes is the transformation function, $t(y, x) \geq 0$ if y can be produced with x . We assume that $t(y, x)$ is defined and continuous for all nonnegative y and x and that it is decreasing in y and increasing in x . We define the *joint cost function* as the function giving the minimum cost at which outputs y_1, \dots, y_M can be produced when factor prices are w_1, \dots, w_N . Since it serves as the basis of all of our results, we provide here an explicit statement of the duality between the joint cost function and its underlying technology:

Shephard-Uzawa-McFadden Duality Theorem for Joint Cost Functions

Suppose the transformation function $t(y, x)$ has a strictly convex input structure; that is, the input requirement set $V(y) = \{x \mid t(y, x) \geq 0\}$ is closed and strictly convex.¹ Then there is a unique joint cost function $C(y, w)$, differentiable in w , defined by

$$C(y, w) = \min_{x \in V(y)} w \cdot x.$$

Further, $C(y, w)$ is positive linear homogeneous, nondecreasing, and concave² in the factor prices, w . Finally, it obeys Shephard's lemma,

$$t\left(y, \frac{\partial C(y, w)}{\partial w}\right) = 0;$$

that is, the vector of cost-minimizing factor inputs is equal to the vector of derivatives of the cost function with respect to the factor prices.

The proof of this theorem is given by McFadden (1973). We note without proof that when the transformation function $t(y, x)$ is differentiable in outputs, y , the following condition also holds:

$$\frac{\partial C(y, w)/\partial y_i}{\partial C(y, w)/\partial y_j} = \frac{\partial t(y, x)/\partial y_i}{\partial t(y, x)/\partial y_j};$$

¹ This rules out the case of factors that are perfect substitutes or perfect complements. All of our results are valid for the latter case, however.

² Since there is occasional confusion on the subject, it is worth noting that the concavity of the cost function does not follow from the convexity of the technology. All cost functions are concave, irrespective of the characteristics of the underlying technology.

that is, the ratio of the marginal costs of two goods is equal to the marginal rate of transformation between them. Thus the production possibility frontier is tangent to the isocost surface at the point where production takes place.

The transformation function describes the technology uniquely, but more than one transformation function can describe the same technology. For this reason, Shephard, and subsequently McFadden, have found it convenient to deal with a normalized form of the transformation function known as the *distance function*. It is related to the transformation function by the following identity in x and y :

$$t\left(y, \frac{1}{D(y, x)} x\right) = 0.$$

That is, the distance, $D(y, x)$, is the amount by which an arbitrary vector of inputs, x , must be scaled down so that it will exactly produce a vector of outputs, y .³

2. Characterization of the Joint Cost Function When the Technology Is Separable or Nonjoint and Has Constant Returns to Scale

In this section we show that straightforward criteria exist for determining whether the technology underlying a joint cost function is separable or nonjoint. Throughout, we assume that the transformation function is differentiable in outputs, y ; this rules out perfect complementarity of outputs. We do not require differentiability of $t(y, x)$ in x ; the differentiability of $C(y, w)$ in w depends, rather, on the strict convexity of the input requirement set.⁴ We also assume that the technology has constant returns to scale: $t(y, x) = 0$ implies $t(\lambda y, \lambda x) = 0$, all $\lambda \geq 0$. More general results, and proofs of several results used here are presented in the Appendix.

We begin with a

Theorem on Separability

A necessary and sufficient condition for separability ($t(y, x) = -g(y) + f(x)$) is that the joint cost function be multiplicatively separable: $C(y, w) = H(y)\psi(w)$.

³ McFadden uses the term "transformation function" for what we call the distance function. Our departure from his terminology is necessary in order to discuss separability of our $t(y, x)$.

⁴ A slightly weaker assumption that would extend all of the following results to the technology with fixed input coefficients is that $C(y, w)$ is differential for strictly positive factor prices only.

Proof:

i) *Necessity*.—We show first that $f(x)$ is homothetic; that is, there exists a strictly increasing positive function $h(\cdot)$ such that $h[f(x)]$ is linearly homogeneous. Consider any $x^{(1)}, x^{(2)}$ on the same isoquant: $f(x^{(1)}) = f(x^{(2)})$. Then $f(\lambda x^{(1)}) = f(\lambda x^{(2)})$ for any $\lambda > 0$, since both equal $g(\lambda y)$ by constant returns to scale. Then by the lemma of the Appendix, $f(x)$ is homothetic.

Next we examine the distance function $D(y, x)$, defined by

$$f\left(\frac{1}{D(y, x)} x\right) = g(y)$$

or

$$h\left(f\left(\frac{1}{D(y, x)} x\right)\right) = h(g(y)).$$

By the homogeneity of $h[f(\cdot)]$,

$$\frac{1}{D(y, x)} h(f(x)) = h(g(y)),$$

and

$$D(y, x) = h(f(x)) \frac{1}{h(g(y))},$$

which we observe is multiplicatively separable. Finally, by McFadden's lemma 10 (1973, p. 73 in the manuscript), $C(y, w)$ is also multiplicatively separable: $C(y, w) = H(y)\psi(w)$, where $H(y) = h(g(y))$.

ii) *Sufficiency*.—Separability of the joint cost function implies separability of the distance function: $D(y, x) = D^{(1)}(y)D^{(2)}(x)$. Now $t(y, x) = 0$ if and only if $D(y, x) = 1$, so $t(y, x) = D^{(1)}(y) - 1/D^{(2)}(x)$ is one transformation function representing the technology of $D^{(1)}(y)D^{(2)}(x)$; it is separable, as required.

Corollary

If the technology is separable, the ratios of any two marginal costs are independent of factor prices.

In competitive equilibrium, prices equal marginal costs, so under separability, output price ratios are independent of factor prices or factor intensities. We see, therefore, that separability represents a generalization of the one-sector technology, in that output price ratios can vary as the output mix varies. However, the interesting and possibly important feature of two-sector and more elaborate technologies—dependence of output price ratios on factor prices—is entirely absent. This suggests that separability may not be a suitable specification for a complete econometric model.

We turn now to the study of nonjointness as a restriction on the general joint cost function. We begin with the

Definition of a Nonjoint Technology

A technology with transformation function $t(y, x)$ is *nonjoint* if there exist functions $f^{(1)}(x^{(1)}), \dots, f^{(M)}(x)$ (interpreted as individual production functions) with the properties: (i) There are no economies of jointness: if x can produce $y(t(y, x) \geq 0)$, there is a factor allocation $x^{(1)} + \dots + x^{(M)} = x$ such that $f^{(i)}(x^{(i)}) \geq y_i$, $i = 1, \dots, M$. (ii) There are no diseconomies of jointness: if $y_i = f^{(i)}(x^{(i)})$, all i , then $x = x^{(1)} + \dots + x^{(M)}$ can produce y .

To show that a technology is nonjoint, we must exhibit the individual functions $f^{(1)}, \dots, f^{(M)}$ and show that they meet both of these requirements. Note that nonjointness requires only that the $f^{(i)}$ exist as functions; there is no requirement that there be physically separate processes producing the various outputs, y_i . Thus the observation that more than one output is produced in the same plant is not sufficient to rule out nonjointness.

Although we have given the natural definition of nonjointness, it does not turn out to be a useful characterization of it. There is no obvious way to translate this definition into an econometric restriction that can be imposed on a more general specification of the technology. The problem of providing an alternative characterization has been studied previously by Samuelson (1966), who states his results in terms of the derivatives of the transformation function. His results do not seem to lead to any econometrically useful restrictions, but the following alternative characterization of nonjointness in terms of the joint cost function does seem to be useful:

Theorem on Nonjointness

A necessary and sufficient condition for nonjointness is that the total cost of producing all outputs be the sum of the costs of producing each separately:

$$C(y, w) = y_1 \phi^{(1)}(w) + \dots + y_M \phi^{(M)}(w),$$

where $\phi^{(i)}(w)$ is the cost of producing a unit of output i .

Proof:

The more general theorem on nonjointness in the Appendix shows that the technology is nonjoint if $C(y, w) = C^{(1)}(y_1, w) + \dots + C^{(M)}(y_M, w)$. It remains to show that if the whole technology has constant returns to scale, then the individual cost function for a typical output, say the first,

has the form of constant returns to scale: $C^{(1)}(y_1, w) = y_1 \phi^{(1)}(w)$. Now $C(y, w)$ is linearly homogeneous: $C(\lambda y, w) = \lambda C(y, w)$. If $y_i = 0$, $i = 2, \dots, M$, and $C(y, w)$ has the additive form of nonjointness, then

$$\begin{aligned} C^{(1)}(\lambda y_1, w) + C^{(2)}(0, w) + \dots + C^{(M)}(0, w) \\ = \lambda [C^{(1)}(y_1, w) + C^{(2)}(0, w) + \dots + C^{(M)}(0, w)]. \end{aligned}$$

By taking $\lambda = 0$, we see that $C^{(i)}(0, w) = 0$, $i = 1, \dots, M$, since each is nonnegative. Finally, by taking $\lambda = 1/y_1$, we have

$$C^{(1)}(y_1, w) = y_1 C^{(1)}(1, w) = y_1 \phi^{(1)}(w).$$

Corollary

If the technology is nonjoint, the marginal cost of each output is independent of the level of any output.

Now in the case of separability of technology, the ratios of the marginal costs depend *only* on the output mix, while with nonjointness, marginal costs are *independent* of the output mix. This suggests that the overlap between the two restrictions is very small. We now give the

Impossibility Theorem for Separable Nonjoint Technologies

No multiple-output technology with constant returns to scale can be both separable and nonjoint. That is, the individual production functions in such a technology are identical except for a scalar multiple, implying that there is effectively only a single kind of output.

Proof:

The general theorem on separable nonjoint technologies in the Appendix establishes that the joint cost function has the form

$$C(y, w) = [g^{(1)}(y_1) + \dots + g^{(M)}(y_M)]\phi(w).$$

On the other hand, the theorem on nonjointness under constant returns implies that $C(y, w)$ can be written as the sum of separate costs:

$$C(y, w) = y_1 \phi^{(1)}(w) + \dots + y_M \phi^{(M)}(w).$$

By setting all but the i th element of y equal to zero, we have

$$\frac{\phi^{(i)}(w)}{\phi(w)} = \frac{1}{y_i} \left[g^{(i)}(y_i) + \sum_{j \neq i} g^{(j)}(0) \right] = \alpha_i,$$

a constant independent of w and y . Thus $C(y, w) = (\alpha_1 y_1 + \dots +$

$\alpha_M y_M) \phi(w)$. The individual production functions of the separable non-joint technology are

$$f^{(i)}(x^{(i)}) = \frac{1}{\alpha_i} f^*(x^{(i)}),$$

where $f^*(\cdot)$ is the production function whose unit cost function is $\phi(w)$.

Nontrivial separable technologies are inherently joint, and their use in empirical work forecloses investigation of the hypothesis of nonjointness. Since there are few strong reasons a priori to believe in jointness at the level of aggregation of complete econometric models, this is a second serious drawback of separable specifications of technology.

3. A Functional Form for Joint Cost Functions That Contains Separability and Nonjointness as Parametric Restrictions

The family of functional forms introduced by Diewert (1971) is a logical source for a convenient specification for joint cost functions.⁵ Diewert has proposed the generalized Leontief cost function for a single output:

$$C(y, w) = y \sum_{i=1}^N \sum_{j=1}^N a_{ij} \sqrt{w_i w_j},$$

and has shown that the technology underlying this cost function can approximate, locally, the curvature or substitution properties of almost any technology. He has also proposed a generalized linear production function,

$$y = \sum_{i=1}^N \sum_{j=1}^N a_{ij} \sqrt{x_i x_j},$$

and again has shown that it can approximate almost any technology. Since the joint cost function is a hybrid dual concept, it is natural to use a hybrid Diewert function as a specification. Our choice is the

Generalized Linear–Generalized Leontief (Hybrid Diewert) Joint Cost Function

$$C(y, w) = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^M \sum_{l=1}^M a_{ijkl} \sqrt{y_k y_l} \sqrt{w_i w_j}$$

This function has a generalized linear relation among outputs and a generalized Leontief relation among inputs. It has a rather large number of parameters, but this is made necessary by the ambitious requirement

⁵ Other families of joint cost functions would probably serve equally well and give very similar empirical results. The quadratic log function of Christensen, Jorgenson, and Lau (1971; in press) would be a sensible alternative, for example.

that both separability and nonjointness be obtainable by parametric restrictions.

For separability, the coefficients must obey the following restriction: $a_{ijkl} = \beta_{ij}\alpha_{kl}$; this makes the joint cost function separable as follows:

$$C(y, w) = \left(\sum_{k=1}^M \sum_{l=1}^M \alpha_{kl} \sqrt{y_k y_l} \right) \left(\sum_{i=1}^N \sum_{j=1}^M \beta_{ij} \sqrt{w_i w_j} \right).$$

In terms of the separable transformation function $t(y, x) = -g(y) + f(x)$, the function $g(y)$ is the generalized linear function appearing in the joint cost function, and $f(x)$ is the production function corresponding to the generalized Leontief cost function appearing there. The form of $f(x)$ is rather complicated and cannot be given in a closed expression: the reader is referred to Diewert (1971) for a discussion on this point.

For nonjointness, all of the coefficients in the joint cost function corresponding to interaction among the y 's must vanish: $a_{ijkl} = 0$ unless $k = l$. Then the coefficients a_{ijkk} are the coefficients of a separate generalized Leontief cost function for industry k . Under this restriction, each industry still has an arbitrary production function.

Estimation of the hybrid Diewert joint cost function in any of its three forms is conveniently based on the multivariate system obtained by setting each observed factor input equal to the factor demand derived from the cost function, and each price equal to the marginal cost derived from the cost function:

$$\begin{aligned} x_1 &= \frac{\partial C(y, w)}{\partial w_1} \\ &\vdots \\ x_N &= \frac{\partial C(y, w)}{\partial w_N} \\ p_1 &= \frac{\partial C(y, w)}{\partial y_1} \\ &\vdots \\ p_M &= \frac{\partial C(y, w)}{\partial y_M}. \end{aligned}$$

With two factors and two outputs, the cost function is

$$\begin{aligned} C(y_1, y_2, w_1, w_2) &= a_{1111}y_1w_1 + a_{1122}y_2w_1 \\ &+ a_{2211}y_1w_2 + a_{2222}y_2w_2 \\ &+ 2a_{1211}y_1\sqrt{w_1w_2} + 2a_{1222}y_2\sqrt{w_1w_2} \\ &+ 2a_{1112}w_1\sqrt{y_1y_2} + 2a_{2212}w_2\sqrt{y_1y_2} \\ &+ 4a_{1212}\sqrt{y_1y_2}\sqrt{w_1w_2}. \end{aligned}$$

The corresponding estimating equations are

$$\begin{aligned}
 x_1 &= a_{1111}y_1 + 2a_{1112}\sqrt{y_1y_2} + a_{1122}y_2 \\
 &\quad + \sqrt{w_2/w_1} (a_{1211}y_1 + 2a_{1212}\sqrt{y_1y_2} + a_{1222}y_2); \\
 x_2 &= \sqrt{w_1/w_2} (a_{1211}y_1 + 2a_{1212}\sqrt{y_1y_2} + a_{1222}y_2) \\
 &\quad + a_{2211}y_1 + 2a_{2212}\sqrt{y_1y_2} + a_{2222}y_2; \\
 p_1 &= a_{1111}w_1 + a_{2211}w_2 \\
 &\quad + 2a_{1211}\sqrt{w_1w_2} \\
 &\quad + a_{1112}w_1\sqrt{y_2/y_1} + a_{2212}w_2\sqrt{y_2/y_1} \\
 &\quad + 2a_{1212}\sqrt{y_2/y_1}\sqrt{w_1w_2}; \\
 p_2 &= a_{1122}w_1 + a_{2222}w_2 \\
 &\quad + 2a_{1222}\sqrt{w_1w_2} \\
 &\quad + a_{1112}w_1\sqrt{y_1/y_2} + a_{2212}w_2\sqrt{y_1/y_2} \\
 &\quad + 2a_{1212}\sqrt{y_1/y_2}\sqrt{w_1w_2}.
 \end{aligned}$$

This system has nine parameters after taking account of three restrictions between the equations.

The null hypothesis of separability is:

$$\begin{aligned}
 a_{1111}a_{2222} &= a_{2211}a_{1122}; \\
 a_{1122}a_{2222} &= a_{2212}a_{1122}; \\
 a_{1211}a_{2222} &= a_{1222}a_{2211}; \\
 a_{1212}a_{2222} &= a_{1222}a_{2212}.
 \end{aligned}$$

In practice, this is tested by estimating the separable specification,

$$\begin{aligned}
 x_1 &= \alpha_{11}y_1 + 2\alpha_{12}\sqrt{y_1y_2} + \alpha_{22}y_2 \\
 &\quad + \sqrt{w_2/w_1} (\beta_{12}\alpha_{11}y_1 + 2\beta_{12}\alpha_{12}\sqrt{y_1y_2} + \beta_{12}\alpha_{22}y_2); \\
 x_2 &= \sqrt{w_1/w_2} (\beta_{12}\alpha_{11}y_1 + 2\beta_{12}\alpha_{12}\sqrt{y_1y_2} + \beta_{12}\alpha_{22}y_2) \\
 &\quad + \beta_{22}\alpha_{11}y_1 + 2\beta_{22}\alpha_{12}\sqrt{y_1y_2} + \beta_{22}\alpha_{22}y_2; \\
 P_1 &= \alpha_{11}w_1 + 2\alpha_{11}\beta_{12}\sqrt{w_1w_2} + \alpha_{11}\beta_{22}w_2 \\
 &\quad + \sqrt{y_2/y_1} (\alpha_{12}w_1 + 2\alpha_{12}\beta_{12}\sqrt{w_1w_2} + \alpha_{12}\beta_{22}w_2) \\
 P_2 &= \sqrt{y_1/y_2} (\alpha_{12}w_1 + 2\alpha_{12}\beta_{12}\sqrt{w_1w_2} + \alpha_{12}\beta_{22}w_2) \\
 &\quad + \alpha_{22}w_1 + 2\alpha_{22}\beta_{12}\sqrt{w_1w_2} + \alpha_{22}\beta_{22}w_2.
 \end{aligned}$$

Here we have used the normalization $\beta_{11} = 1$, leaving five parameters.

The sum of squared residuals from this mildly nonlinear regression can be compared to the sum of squares from the unrestricted linear regression, and the null hypothesis tested with an approximate F -test.

The null hypothesis of nonjointness has the simple linear form:

$$a_{1112} = 0;$$

$$a_{1212} = 0;$$

$$a_{2212} = 0.$$

The test is simply a test of the joint significance of the output interaction terms $\sqrt{y_1 y_2}$, $\sqrt{w_2/w_1} \sqrt{y_1 y_2}$, and $\sqrt{w_1/w_2} \sqrt{y_1 y_2}$ in the regression.

4. Conclusion

We have studied two alternative families of specifications that relate the vector of outputs produced by a technology to the vector of primary inputs employed in production. The first exhibits mathematical separability of the transformation function describing the technology. This family has two undesirable properties: it requires that the technology be joint, so that it requires that the cost of producing one kind of output depend on the amount of other kinds of output produced, and it requires that the ratios between output prices be independent of factor prices. The second family exhibits additive separability of the cost function of the technology. It requires that the technology be nonjoint, so it rules out interaction among the productive processes except through the primary factors. We have shown that these two families of restrictions are mutually exclusive—a technology cannot have a separable transformation function and be nonjoint. Finally, we have proposed a function form suitable for econometric investigation of technologies with more than one kind of output. In it, separability of the transformation function and nonjointness of the technology are available as parametric restriction and thus may be tested with the usual methods of statistical inference.

Appendix

Results for Technologies That Do Not Have Constant Returns to Scale

We begin by stating a well-known property of homothetic functions:

Lemma:

A function $f(x)$ is homothetic if $f(x) = f(x')$ implies $f(\lambda x) = f(\lambda x')$ for all $\lambda \geq 0$ and all vectors x and x' in the domain of f .

Proof:

A function $f(x)$ is homothetic if there is a function h such that $h(f(x))$ is linearly homogeneous. A function with this property is obtained as follows: Let \bar{x} be an

arbitrary point in the domain of f . Then define $h(q)$ as follows: $h(q) = z$ such that $f(z\bar{x}) = q$. Now $h[f(\lambda x)] = z$ such that $f(z\bar{x}) = f(\lambda x)$. By hypothesis $f[(z/\lambda)\bar{x}] = f(x)$, or $h[f(x)] = z/\lambda$. Thus $h[f(\lambda x)] = \lambda h[f(x)]$, and $h[f(z)]$ is linearly homogeneous.

Next we give the

General Theorem on Separability

Suppose that a technology is separable: $t(y, x) = -g(y) + f(x)$. Then the joint cost function has the form $C(y, w) = C^*(g(y), w)$.

Proof:

Consider two vectors of outputs y and y' such that $g(y) = g(y')$. We seek to show that $C(y, w) = C(y', w)$. The input requirement set for y is

$$V(y) = \{x \mid f(x) \geq g(y)\} = \{x \mid f(x) \geq g(y')\} = V(y'),$$

and

$$C(y, w) = \min_{x \in V(y)} w \cdot x = \min_{x \in V(y')} w \cdot x = C(y', w),$$

as required.

Next we give a characterization of nonjoint technologies in terms of the input requirement sets:

General Theorem 1 on Nonjointness

A technology is nonjoint if and only if there exist separate input requirement sets $V^{(1)}(y_1), \dots, V^{(M)}(y_M)$ such that $V(y) = V^{(1)}(y_1) + \dots + V^{(M)}(y_M)$.

Proof:

i) Suppose the technology is nonjoint. Let $V^{(i)}(y_i)$ be the input requirement set of $y_i = f^{(i)}(x^{(i)})$. Now suppose there is a y and an x such that $x \in V(y)$ but there is no allocation $x^{(1)} + \dots + x^{(M)} = x$ such that $x^{(i)} \in V^{(i)}(y_i)$. This contradicts part (i) of the definition of nonjointness. Suppose, on the other hand, that there is a y and an allocation $x^{(1)}, \dots, x^{(M)}$ such that $x^{(i)} \in V^{(i)}(y_i)$ but $x^{(1)} + \dots + x^{(M)}$ is not in $V(y)$. This contradicts part (ii) of the definition. We conclude that $V(y) = V^{(1)}(y_1) + \dots + V^{(M)}(y_M)$.

ii) Suppose $V(y) = V^{(1)}(y_1) + \dots + V^{(M)}(y_M)$. Let $f^{(i)}(x^{(i)})$ be the production function corresponding to the input requirement set $V^{(i)}(y_i)$. Then parts (i) and (ii) of the definition are easily seen to hold.

This restatement of the definition of nonjointness enables us to draw on a result of McFadden to give a useful (and obvious) characterization of nonjoint technologies in terms of the joint cost function:

General Theorem 2 on Nonjointness

A technology is nonjoint if and only if the joint cost function can be written as the sum of independent cost functions for each kind of output: $C(y, w) = C^{(1)}(y_1, w) + \dots + C^{(M)}(y_M, w)$.

Proof:

The proof follows directly from repeated application of McFadden's composition rule 5, "Summation of Input Requirement Sets" (1973, table 2).

Finally we give the

General Theorem on Separable Nonjoint Technologies

A technology is both separable and nonjoint if and only if its joint cost function has the form $C(y, w) = [g^{(1)}(y_1) + \dots + g^{(M)}(y_M)]\phi(w)$, that is, the separate production functions are identical except for pure scale effects.

Proof:

The technology is separable, so $t(y, x) = -g(y) + f(x)$. We show first that $f(x)$ is homothetic. Let I_i be the i th column of the $M \times M$ identity matrix. Consider a vector of inputs, x , and define y_1 and y_2 by $g(I_1 y_1) = g(I_2 y_2) = f(x)$. We seek to show that $g(I_1 y_1 + I_2 y_2) = f(2x)$. By nonjointness, $g(I_1 y_1 + I_2 y_2) \leq f(2x)$. Suppose $g(I_1 y_1 + I_2 y_2) < f(2x)$. Then there is a factor allocation $x^{(1)} + x^{(2)} = 2x$ such that $f(x^{(1)}) > g(I_1 y_1)$ and $f(x^{(2)}) > g(I_2 y_2)$, and thus $f(x) < f(x^{(1)})$ and $f(x) < f(x^{(2)})$. By the convexity of the input requirement set,

$$f[\tfrac{1}{2}(x^{(1)} + x^{(2)})] \geq \tfrac{1}{2}[f(x^{(1)}) + f(x^{(2)})] > \tfrac{1}{2}[f(x) + f(x)]$$

or $f(x) > f(x)$, a contradiction. We conclude that $g(I_1 y_1 + I_2 y_2) = f(2x)$.

Now suppose we have inputs x and \tilde{x} such that $f(x) = f(\tilde{x})$. By the previous result we have $f(2x) = f(2\tilde{x})$. Next we show that $f(\tfrac{1}{2}x) = f(\tfrac{1}{2}\tilde{x})$. Suppose, on the contrary, that $f(\tfrac{1}{2}x) > f(\tfrac{1}{2}\tilde{x})$. Consider y_1 and y_2 such that $g(I_1 y_1) = g(I_2 y_2) = f(\tfrac{1}{2}x)$. Then $g(I_1 y_1 + I_2 y_2) = f(x)$. On the other hand, $g(I_1 y_1) > f(\tfrac{1}{2}\tilde{x})$ and $g(I_2 y_2) > f(\tfrac{1}{2}\tilde{x})$. Consider an allocation of factors, $x^{(1)} + x^{(2)} = \tilde{x}$. By convexity, $f(\tfrac{1}{2}\tilde{x}) = f[\tfrac{1}{2}(x^{(1)} + x^{(2)})] > \tfrac{1}{2}[f(x^{(1)}) + f(x^{(2)})]$, so it is not possible that $f(x^{(1)}) > f(\tfrac{1}{2}\tilde{x})$ and $f(x^{(2)}) > f(\tfrac{1}{2}\tilde{x})$. We conclude that y_1 and y_2 cannot be produced with \tilde{x} , and that $g(I_1 y_1 + I_2 y_2) > f(\tilde{x})$. But this contradicts $g(I_1 y_1 + I_2 y_2) = f(x) = f(\tilde{x})$, so we must have $f(\tfrac{1}{2}x) \leq f(\tfrac{1}{2}\tilde{x})$. Equality is established by reversing x and \tilde{x} in the argument. By repetition of this and the previous result, we have $f(2^n x) = f(2^n \tilde{x})$, $-\infty < n < \infty$. Next we observe that if $f(x^{(i)}) = f(\tilde{x}^{(i)})$, $-\infty < i < \infty$, then

$$f\left(\sum_{i=-\infty}^{\infty} x^{(i)}\right) = f\left(\sum_{i=-\infty}^{\infty} \tilde{x}^{(i)}\right),$$

provided the sums exist, by the continuity of f . Now consider the binary representation of an arbitrary positive number, λ :

$$\lambda = \sum_{i=-\infty}^{\infty} \delta_i 2^i, \quad \delta_i = 0 \text{ or } 1.$$

Let $x^{(i)} = \delta_i 2^i x$ and $\tilde{x}^{(i)} = \delta_i 2^i \tilde{x}$. By our earlier results, if $f(x) = f(\tilde{x})$, $f(x^{(i)}) = f(\tilde{x}^{(i)})$. Thus

$$f\left(\sum_{i=-\infty}^{\infty} \delta_i 2^i x\right) = f\left(\sum_{i=-\infty}^{\infty} \delta_i 2^i \tilde{x}\right),$$

or $f(\lambda x) = f(\lambda \tilde{x})$ for any $\lambda \geq 0$. By the lemma, this establishes that f is homothetic. Without loss of generality we may assume that it is linearly homogeneous, that is, that the one-sector technology $q = f(x)$ has constant returns to scale. Let $\phi(w)$ be the unit cost function of $f(x)$. Then the individual cost function for output i is $C^{(i)}(y_i, w) = g(I_i y_i) \phi(w)$, and the joint cost function is $C(y, w) = g^{(1)}(y_1) = \dots = g^{(M)}(y_M) \phi(w)$ as asserted, where $g^{(i)}(y_i) = g(I_i y_i)$.

References

- Christensen, L. R.; Jorgenson, D. W.; and Lau, L. J. "Conjugate Duality and the Transcendental Logarithmic Production Function." *Econometrica* 39, no. 4 (July 1971): 255-56.
- . "Transcendental Logarithmic Production Frontiers." *Rev. Econ. and Statis.* (in press).
- Diewert, W. E. "An Application of the Shephard Duality Theorem: A Generalized Leontief Production Function." *J.P.E.* 79 (May/June 1971): 481-507.
- Fisher, F. M.; Klein, L.; and Shinkai, Y. "Price and Output Aggregation in the Brookings Econometric Model." In *The Brookings Quarterly Econometric Model*, edited by J. Duesenberry et al. Amsterdam: North-Holland, 1965.
- Kresge, D. T. "Price and Output Conversion: A Modified Approach." In *The Brookings Model—Some Further Results*, edited by J. Duesenberry et al. Chicago: Rand McNally, 1969.
- McFadden, D. "Cost, Revenue and Profit Functions." In *An Econometric Approach to Production Theory*. Amsterdam: North-Holland, 1973, in press.
- Mundlak, Y. "Specification and Estimation of Multiproduct Production Functions." *J. Farm Econ.* 45 (May 1963): 433-43.
- Samuelson, P. A. "The Fundamental Singularity Theorem for Non-Joint Production." *Internat. Econ. Rev.* 7 (January 1966): 34-41.
- Shephard, R. *Theory of Cost and Production Functions*. Princeton, N.J.: Princeton Univ. Press, 1953.
- Uzawa, H. "Duality Principles in the Theory of Cost and Production." *Internat. Econ. Rev.* 5 (May 1964): 216-20.