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SIMULATION METHODS FOR INTERTEMPORAL ECONOMIC MODELS*

by

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Two classes of problems arise in simulating intertemporal economic models. The first is that of finding a solution to the implicit difference equation,

$$(1) \qquad F(x_{t+1}, x_t) = 0 \quad ,$$

where F is an n-dimensional vector-valued function of the n-dimensional vectors x_{t+1} and x_t, which represent the state of the economy in periods t + 1 and t, respectively. This class of problems has received a great deal of attention in the last few years because of the large amount of interest in simulating econometric models of the whole economy; these models can always be put in the form of equation 1 by an appropriate

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set of substitutions. The reader is referred to McGettigan [2] for a summary and evaluation of the methods for solving this kind of problem. In general, it is possible to solve equation 1 for x_{t+1} given the particular values of x_t by one or a combination of iterative methods, as long as F is moderately well-behaved.

There is, however, an important class of intertemporal economic models which cannot be put into the form of equation 1. These models are characterized by the property that one of the groups of participants in the economy makes plans extending into the future beyond period t + 1. If this is the case, the simulation problem involves two steps: The first is to solve the prediction and planning problem of the group which makes plans; this gives values for some or all of the elements of the vector x_{t+1}. The second step (which will vanish if the planning group determines all of the variables in the economy) is to find the remaining elements of x_{t+1} by solving an equation of the form of equation 1. The purpose of this paper is to discuss some methods for solving the prediction and planning problem. The mathematical discussion of the first section is supplemented by an economic example in the second section.

1. Simulation Methods

Prediction and planning models generally turn out to involve the solution of two-point boundary value problems. The typical problem of this kind can be formulated in the following way: Conditional on this period's plan being correct, next period's plan is given by the difference equation,

(2) $$x_{t+1} = G(x_t) \quad .$$

Two additional sets of requirements complete the statement of the characteristics of the plan. First, certain elements of the first step in the plan are fixed by initial conditions

(3) $$A_1 x_1 = \bar{x}_1 \quad ,$$

where A_1 is a matrix of m rows and n columns and $\bar{x}_1$ is a vector of m elements usually stating initial economic resources. Equation 3 expresses m linear constraints embodying the initial conditions of the plan. Second, there are terminal conditions at the time horizon, T:

(4) $$A_T x_T = \bar{x}_T \quad ,$$

where A_T is a matrix of n - m rows and n columns and $\bar{x}_T$ is a vector of n - m elements. Finally, the domain, P, of the function G expressing the set of permissable values of x_t is usually restricted in some way--for example, consumption and the capital stock must be nonnegative.

The problem, then, is to find a set of values for x_1 which meet the constraint of equation 3, and such that T - 1 successive applications of the difference equation, 2, brings the economic plan to a point, x_T, which meets the constraint of equation 4. We will discuss two methods for solving this problem.[1] The first is the obvious method of picking

[1] See Athans, [1], for a discussion and bibliography of methods for solving this kind of problem in an engineering context.

an initial point, x_1, from among those which meet the initial constraint of equation 3, calculating the corresponding x_T, and updating the initial guess according to some function of the terminal error, which is defined as

$$(5) \qquad E_T = A_T x_T - \bar{x}_T \quad .$$

To begin the discussion of this method, we note that the initial constraint can also be written as

$$(6) \qquad x_1 = B\hat{x} + C \quad ,$$

where B is a matrix of n rows and n - m columns, $\hat{x}$ is a vector of dimension n - m, and C is a vector of dimension n. This representation is not generally unique, but this will not introduce any difficulties. Our problem may then be restated as one of finding a vector $\hat{x}$ such that the vector of terminal errors, E_T, is zero. An example of a possible iterative method is the Newton-Raphson method, in which a value $\hat{x}^{(j)}$ is updated by the iteration equation,

$$(7) \qquad \hat{x}^{(j+1)} = \hat{x}^{(j)} - \left(\frac{\partial E_T^{(j)}}{\partial \hat{x}^{(j)}}\right)^{-1} E_T^{(j)}$$

The matrix of derivatives, $\frac{\partial E_T^{(j)}}{\partial \hat{x}^{(j)}}$, can be calculated by the following process. First,

$$(8) \qquad \frac{\partial E_T^{(j)}}{\partial \hat{x}^{(j)}} = A_T \frac{\partial x_T^{(j)}}{\partial \hat{x}^{(j)}} .$$

Second, we can calculate the derivative of the path with respect to the initial conditions by differentiating the difference equation governing the plan:

$$(9) \qquad \frac{\partial x_{t+1}^{(j)}}{\partial \hat{x}^{(j)}} = \frac{\partial G(x_t^{(j)})}{\partial \hat{x}^{(j)}} \cdot \frac{\partial x_t^{(j)}}{\partial \hat{x}^{(j)}} .$$

This is a linear matrix difference equation which can be calculated without any difficulty to give the value of $\frac{\partial E_T^{(j)}}{\partial \hat{x}^{(j)}}$ along the path $x_t^{(j)}$. Initial conditions are obtained by differentiating equation 6:

$$(10) \qquad \frac{\partial x_1^{(j)}}{\partial \hat{x}^{(j)}} = B.$$

This method has the property of quadratic convergence common to all applications of Newton's method. Unfortunately, its usefulness is limited to cases of short horizons (small values of T). The difficulty is that for most choices of the initial vector $\hat{x}$, repeated applications of the difference equation bring the economy to a point outside the permissable region, P, at a time before T. The difference equation which characterizes a plan is always <u>unstable</u>--the planned state of the economy at time t, x_t, becomes more and more sensitive to x as t becomes larger. This observation can be put another way by standing it on its head: Today's plan, as expressed by $\hat{x}$, becomes less and less sensitive to the desired future state of the economy, x_t, as the futurity of that state increases. The infinite future is completely irrelevant, so in the limit as t approaches infinity, x_t is infinitely sensitive to x:

$$\lim_{t\to\infty} \frac{\partial x_t}{\partial \hat{x}} = \begin{bmatrix} \infty & \cdots & \infty \\ \vdots & & \vdots \\ \infty & \cdots & \infty \end{bmatrix}$$

The instability of the basic difference equation has several implications for iterative processes for finding the initial value, $\hat{x}$. It is usually almost impossible to find a value of $\hat{x}^{(0)}$ to start the iterative process which corresponds to a trajectory $x_t^{(0)}$ which is always within the permissable set. Further, if such a value of $x^{(0)}$ is found, usually the first iteration will yield a value of $x^{(1)}$ whose trajectory fails to lie within the permissable set. Clearly any iterative process based on E_T and x must be supplemented by a method for handling initial values whose trajectories enter the forbidden region at a time before T. It is not easy to design methods for handling this problem which have a reasonable prospect for speedy convergence.

A second method for solving two-point boundary value problems is based on a somewhat more subtle approach. It has been applied with complete success to some simple intertemporal economic models, but it is not known whether it will work as well with more complicated models. We refer to the method of quasilinearization.[2] The basic idea of this method is that the two-point boundary value problem can be solved directly in systems of nonautonomous inhomogeneous linear difference equations.

[2] See McGill and Kenneth [3].

This suggests the following strategy: (1) Pick an arbitrary trajectory $x_t^{(0)}$ which meets the initial condition and the constraint $x_t^{(0)} \varepsilon P$, but not the difference equation: $x_{t+1}^{(0)} \neq G(x_t^{(0)})$. (2) Use this trajectory to calculate a nonautonomous linear approximation to the difference equation. (3) Solve the boundary value problem in the linear system. (4) Use the new trajectory as the basis for a new approximation and continue the next iteration from step 2.

This method has two properties which make it particularly attractive in applications to economic models. First, the linear approximation can often be interpreted in market terms: In a market equilibrium, each participant faces a linear economy which is tangent to the true curved economy at the point of his demand. The slopes are, of course, the prices. Second, the turnpike property common to almost all intertemporal economic models can be used to great advantage in choosing the initial trajectory $x_t^{(0)}$. Since the middle part of every trajectory will lie close to the turnpike, x^*, a natural choice for $x_t^{(0)}$ is a trajectory which goes straight to x^* in a few steps, stays there without moving until a few periods before T, and then moves to a point satisfying the terminal conditions in a few more steps. Using this initial trajectory makes the number of iterations to convergence more or less independent of the horizon, T--500-year plans can be calculated in 4 or 5 iterations. In contrast, it is possible to show that the number of iterations to convergence for any method for updating the initial condition by the terminal error becomes larger and larger without limit as T becomes larger.

The first step in presenting the details of the method of quasilinearization is to show how to solve the two-point boundary value problem in the approximating linear system. The linear system is

$$(11) \qquad \tilde{x}_{t+1} = G(x_t^{(j)}) + J_t^{(j)}(\tilde{x}_t - x_t^{(j)})$$

where

$$J_t^{(j)} = \frac{\partial G(x_t^{(j)})}{\partial x_t^{(j)}} \quad .$$

The solution to the linear system is

$$(12) \qquad \tilde{x}_T = \left[\prod_{t=1}^{T-1} J_t^{(j)}\right] \tilde{x}_1 + \left[\sum_{t=1}^{T-1} \left(\prod_{\tau=t+1}^{T-1} J_\tau^{(j)}\right)(G(x_t^{(j)}) - J_t^{(j)} \cdot x_t^{(j)})\right] \quad .$$

Next we define the terminal error along a trajectory in the linear system,

$$(13) \qquad \tilde{E}_T = A_T \tilde{x}_T - \bar{x}_T \quad ;$$

our goal is to make this a vector of zeroes. We also have

$$(14) \qquad \tilde{x}_1 = B\hat{x} + C \quad ,$$

since we want $\tilde{x}_1$ to meet the constraint on initial resources.

The value of $\hat{x}$ which solves the boundary value problem in the linear system is the solution to the linear system of equations,

$$\text{(15)} \qquad \tilde{E}_T = A_T \cdot \left[\prod_{t=1}^{T-1} J_t^{(j)} \right] \left[B\hat{x} + C \right] + A_T \cdot \left[\sum_{t=1}^{T-1} \left(\prod_{\tau=t+1}^{T-1} J_\tau^{(j)} \right) \left(G(x_\tau^{(j)}) - J_t^{(j)} x_t^{(j)} \right) \right] - \bar{x}_T = 0 \ .$$

If we let

$$\text{(16)} \qquad Q = A_T \cdot \left[\prod_{t=1}^{T-1} J_t^{(j)} \right] \cdot B$$

and

$$\text{(17)} \qquad R = A_T \cdot \left[\prod_{t=1}^{T-1} J_t^{(j)} \right] C + A_T \cdot \left[\sum_{t=1}^{T-1} \left(\prod_{\tau=t+1}^{T-1} J_\tau^{(j)} \right) \left(G(x_t^{(j)}) - J_t^{(j)} x_t^{(j)} \right) \right] - \bar{x}_T \ ,$$

then the equation is

$$\text{(18)} \qquad \tilde{E}_T = Q\hat{x} + R = 0 \ ,$$

or

(19) $$\hat{x} = - Q^{-1}R \quad .$$

The second step in each iteration in the method of quasilinearization is to take as the new trajectory the solution to the approximating linear system corresponding to the value of x calculated from equation 19; that is,

(20) $$x_t^{(j+1)} = \tilde{x}_t \quad ,$$

and

(21) $$x_1 = B\hat{x} + C \quad .$$

After the first iteration, each trajectory $x_t^{(j)}$ meets both the initial and terminal boundary conditions exactly. Each iteration takes the trajectory closer and closer to the unique trajectory which meets both conditions and is also a solution to the difference equation, G. Since the method is based on the Newton-Raphson method, it converges quadratically; in simple applications, 5 digits of accuracy in $\hat{x}$ can be reached in 4 or 5 iterations from a turnpike-type initial trajectory.

2. A Simple Example--The Ramsey Model

Consider an economy in which consumers maximize a utility function,

$$(22) \qquad U = \sum_{t=1}^{T} (1 + \rho)^{-t} \log c_t \quad .$$

One of the conditions for the maximum is the difference equation,

$$(23) \qquad c_{t+1} = \frac{1 + r_t}{1 + \rho} c_t$$

where r_t is the interest rate. With the usual one-sector technology, we have

$$(24) \qquad k_{t+1} = f(k_t) + (1 - \delta)k_t - c_{t+1}$$

and

$$(25) \qquad r_t = f'(k_t) - \delta \quad .$$

The competitive equilibrium in this economy is determined by the initial condition on total resources,

$$(26) \qquad k_1 + c_1 = \bar{k}$$

and a terminal condition which is the consumers' budget constraint,

$$(27) \qquad k_T = 0 \quad .$$

In terms of the notation of the first section, this model is:

$$(28) \qquad x_t = \begin{bmatrix} c_t \\ \\ k_t \end{bmatrix} ,$$

$$(29) \qquad G(x_t) = \begin{bmatrix} \dfrac{1 + f'(k_t) - \delta}{1 + \rho} \, c_t \\ \\ f(k_t) + (1 - \delta)k_t - \dfrac{1 + f'(k_t) - \delta}{1 + \rho} \, c_t \end{bmatrix} ,$$

$$(30) \qquad A_1 = [1, 1] ,$$

$$(31) \qquad \bar{x}_1 = \bar{k} ,$$

$$(32) \qquad A_T = [0, 1] ,$$

$$(33) \qquad E_T = k_T ,$$

$$(34) \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} ,$$

$$(35) \qquad C = \begin{bmatrix} 0 \\ \bar{k} \end{bmatrix} ,$$

and

$$(36) \qquad \hat{x} = c_1 .$$

With this way of setting up the problem, we are looking for a value of initial consumption so that terminal capital is exactly zero.

The matrix of derivatives, J_t, is

$$(37) \qquad J_t = \begin{bmatrix} \dfrac{1 + f'(k_t) - \delta}{1 + \rho} & \dfrac{f''(k_t)}{1 + \rho} c_t \\ -\dfrac{1 + f'(k_t) - \delta}{1 + \rho} & 1 + f'(k_t) - \delta - \dfrac{f''(k_t)}{1 + \rho} c_t \end{bmatrix} .$$

The approximating linear system reduces after some simplification to

$$(38) \qquad \tilde{c}_{t+1} = \frac{1 + r_t}{1 + \rho} \tilde{c}_t + \frac{f''(k_t)}{1 + \rho} c_t \cdot (\tilde{k}_t - k_t)$$

$$(39) \qquad \tilde{k}_{t+1} = f(k_t) + (1 + r_t)(\tilde{k}_t - k_t) + (1 - \delta)k_t - \tilde{c}_{t+1} \quad .$$

Thus on each iteration we are finding the competitive equilibrium in a surrogate economy with the following characteristics: Consumers maximize utility (approximately) as if the rate of interest at which they borrow or lend rises linearly with the amount they borrow. The technology, on the other hand, is taken to be linear, with the net marginal product of capital (r_t) variable over time but independent of the amount of capital, $\tilde{k}_t$.

It is interesting to inquire whether it is really necessary to include the interest rate adjustment term $\frac{f''}{1 + \rho} c_t \cdot (\tilde{k}_t - k_t)$ in the consumption planning equation, 38. If it were omitted, the quasi-linearization procedure would have the following interpretation: Consumers are presented with an arbitrary interest rate trajectory. They make consumption and savings plans for this interest rate; then the capital stock is set equal to consumers' demand for assets. This generates a new trial interest rate by the marginal product condition, and the process can be repeated. Unfortunately this simplification often fails to converge, so its use is not recommended.

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