NEAR-INDETERMINACY OF OUTPUT UNDER CONSTANT MARGINAL COST

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Goal: Derive bounds on profit lost by producing sub-optimal output. Sharp bounds showing that very little profit is lost by fairly large departures from profit-maximizing output. The bounds are over all members of broad classes of demand functions.

I. The class of all linear demand functions

Demand function: $p = a - bq$

Profit maximizing output and price:

$$q^* = (a - k)/2b$$

$$p^* = (a + k)/2b$$

Profit-maximizing sales:

$$S^* = (a^2 - k^2)/4b$$

Profit:

$$R(q^*) = (a - k)^2/4b$$

Lost profit in relation to sales:

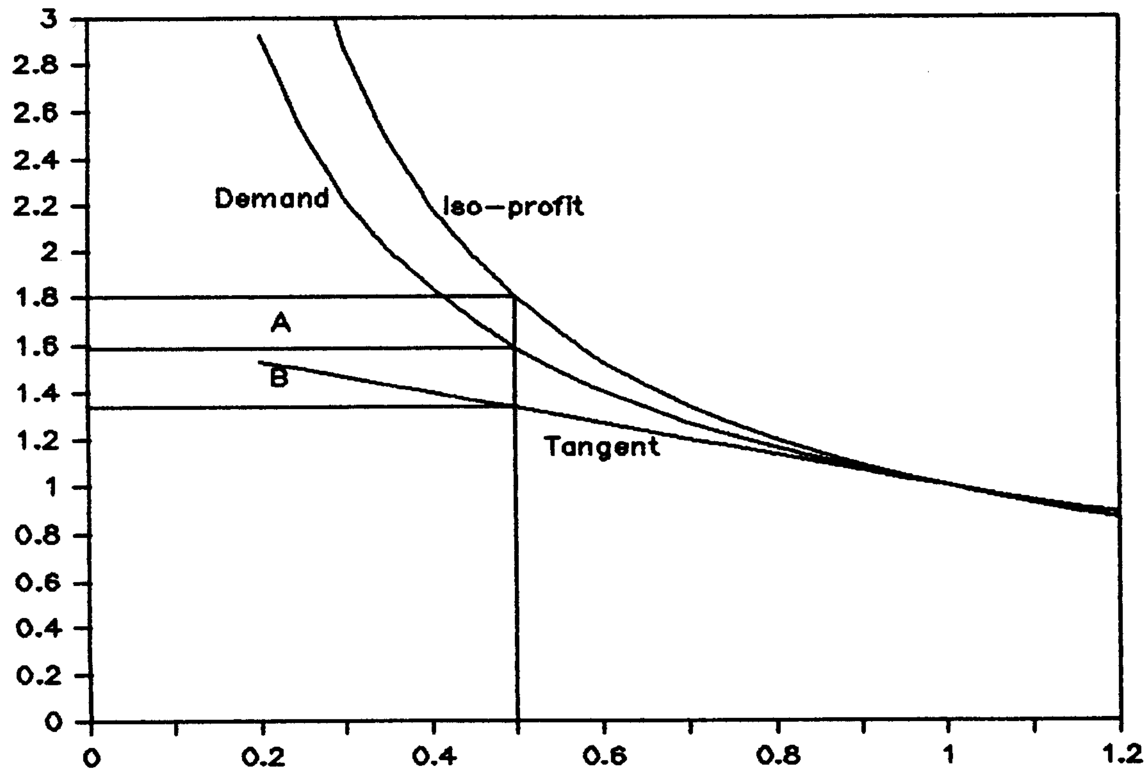
$$L(k,a,b,d) = (R(q^*) - R((1-d)q^*))/S^*$$

$$= ((a - k)/(a + k))\ d^2$$

Over the parameters a and k, the loss is maximal at $k = 0$, in which case the value of a is irrelevant. Thus a bound is available over all demand functions and all value of marginal cost:

Theorem 1: $L(k,a,b,d) \leq d^2$

The figure shows that any convex demand schedule involves lost profit no greater than the linear demand schedule tangent to it. Thus the bound also applies in the class of convex demand schedules.


Demand
Iso-profit
Tangent
A
B
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2

II. The class of all constant-elasticity demand schedules

Demand: $p = (q/a)^{-1/n}$

Profit-maximizing price and quantitiy:

$$p* = (n/(n-1))k$$

$$q* = a((n/(n-1))k)^{-n}$$

Sales:

$$S* = a((n/(n-1))k)^{1-n}$$

Profit:

$$R(q) = ((q/a)^{-1/n} - k)q$$

Lost profit in relation to sales:

$$L(k,a,n,d) = 1 - (1-d)^{1-1/n} - d(1 - 1/n)$$

Let

$$F(x) = (1-d)^{x} + dx$$

where x is 1 - 1/n. The minimum of F(x) occurs at

$$x(d) = (\log d - \log(-\log(1-d)))/\log(1-d)$$

x is virtually invariant to d. It is .498 at d=.05, .493 at d=.15, and .485 at d=.3. In other words, the worst case occurs at an elasticity of just under two, for all values of the parameters.

This gives

Theorem 2. $L(k,a,n,d) < 1 - F(x(d))$ among all constant-elasticity demand schedules.

The table shows that this bound is remarkably tight. Even a shortfall of output of 30 percent cannot lower profit by more than 1.3 percent.

Table 1. Bounds on lost profit from non-optimal output

Output shortfall d	Linear demand	Constant elasticity x	Elas.	bound
	Lost profit relative to sales			
5%	0.3%	0.498	1.99	.0%
10%	1.0%	0.496	1.98	0.1%
15%	2.3%	0.493	1.97	0.3%
20%	4.0%	0.491	1.96	0.6%
25%	6.3%	0.488	1.95	0.9%
30%	9.0%	0.485	1.94	1.3%
35%	12.3%	0.482	1.93	1.9%
40%	16.0%	0.479	1.92	2.5%