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The trend in econometric analysis of distributed lags is clearly in the direction of more general, less restrictive specification of lag functions. This has come about partly because of the gradual accumulation of longer and longer economic time series, providing more and more degrees of freedom for estimation, and partly because the results obtained from very restrictive lag specifications have not been altogether satisfactory. The purpose of this paper is to present a new specification of a lag distribution which is flexible in its shape for short lags and yet admits an infinitely long tail. None of the distributed lag specifications which have previously appeared in the econometric literature has combined both of these desirable properties.

1. Review of Previous Distributed Lag Specifications[1]

In general, a distributed lag model can be written as:

$$(1) \qquad y_t = \sum_{\tau=0}^{p-1} \beta_\tau x_{t-\tau}$$

where y_t and x_t are time series and β_τ are the coefficients of the lag structure. Often the number of periods, p, covered by the lag function is so large that the individual coefficients β_τ cannot be estimated with sufficient accuracy. In other cases the number of periods exceeds the number of observations on x_t and quite frequently p is infinite. In such cases we usually estimate the coefficients subject to some restrictive hypothesis, which requires that β_τ be in some sense a smooth function of τ.

The best-known such hypothesis is that of L. M. Koyck [14], requiring that β_τ have the special form

$$(2) \qquad \beta_\tau = a\lambda^\tau, \quad 0 < \lambda < 1.$$

This specification is quite attractive because it has a simple linear estimator of the form

$$(3) \qquad y_t = ax_t + \lambda y_{t-1} \quad .$$

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[1] For a more comprehensive review of the literature see Zvi Griliches [9].

Such geometrically declining distributed lags arise in several popular classes of econometric models; the adaptive expectations (Cagan [3]), partial adjustment (Nerlove [17], [18]), and "permanent-income" (Friedman [7]) models, are well-known examples. However, Griliches has shown that use of lagged dependent variable estimators, as in equation (3), will produce biased estimates in the presence of serial correlation, a most likely occurence, and to compound the difficulties all the tests for the presence of serial correlation will be biased toward disguising it [8]. As a result a considerable amount of effort has been devoted to the econometric aspects of obtaining unbiased estimates of the parameters of the Koyck distributed lag, though without achieving any general agreement about how to estimate them.[2]

But quite apart from the problem of bias when using the linear estimator, is the objection that for many contexts a geometric distributed lag is too restrictive. Particularly when the length of the period between observations in a time series is less than a year the Koyck requirement that the maximum response is felt within the coincident time period seems unrealistic. In such cases, a more realistic shape for the lag structure would be one which rises to a peak before declining toward zero.

For example, the lag between appropriations and expenditures in the investment process is quite clearly of this humped form (Almon [1]), as would seem likely also in the case of the lag

[2]See, for example, Klein [13], Liviatan [15], Hannan [10], Jorgenson [11], and Dhrymes [6]. Thomas Sargent presents the results of a battery of Monte Carlo experiments on a number of alternative techniques without clear-cut conclusions as to the superiority of any of them [19].

between orders and shipments (Kareken and Solow [12]). Therefore, many investment expenditure models require a delayed response which would be reflected in hump-shaped lag distributions (Bischoff[2]). In other econometric work, expectational models more complicated than the adaptive variety also require a distributed lag estimator which can rise before falling, and in addition they may even require that the lag weights change sign (Modigliani and Sutch [16]).

In order to obtain more flexible specifications a number of techniques have been suggested. A finite inverted V lag was used by Frank de Leeuw [4] as was the sum of two finite geometric lags with different rates of decline and coefficients of opposite sign [5]. Both of these techniques require estimation of one or more nonlinear parameters and therefore considerable computation.

Solow [20] suggested the specification that the points on a lag structure lie along a Pascal probability distribution; however, computational difficulties have prevented its widespread use.[3] Solow in his own empirical work in collaboration with Kareken [12] suggested a computationally simpler lag function which can be estimated by generalizing equation (3) to incorporate several lags of the dependent variable. The properties of this estimator were later described by Jorgenson [11] who named the class of resulting lag functions Rational since they can be described as the ratio of two finite polynomials in the

[3]There is no published example of a Solow-Pascal estimate of which we are aware.

lag operator.[4]

Restraining the lag weights of formula (1) to be rational allows us to rewrite it as

$$(4) \qquad y_t = \frac{a_0 + a_1 L + \dots + a_m L^m}{1 + b_1 L + \dots + b_n L^n} \; x_t$$

where m and n are the orders of the two finite polynomials. m can be zero or any non-negative integer. If n is zero the distributed lag is finite (with m + 1 periods); when n is a positive integer the distributed lag is infinite.

Equation (4) can be written in an alternative form, as follows:

$$(5) \qquad y_t = \left[c_0 + c_1 L + c_2 L^2 + \dots + c_m L^m + \frac{c_{m+1} L^{m+1}}{1 + b_1 L + \dots + b_n L^n} \right] x_t \; .$$

This can be established by long division; $c_{m+1}L^{m+1}$ is the remainder from the division operation. Equation (5) demonstrates that the rational distributed lag consists of a "head"

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[4]The lag operator, L, is defined by

$$\begin{aligned} Lx_t &= x_{t-1} \\ L^2 x_t &= x_{t-2} \\ &\vdots \\ L^n x_t &= x_{t-n} \; . \end{aligned}$$

giving the lag coefficients for short lags, in which each lag coefficient appears as a separately estimated parameter, and an infinitely long "tail" whose shape is determined solely by the parameters in the denominator of equation (4). Our basic criticism of the rational distributed lag is that too many parameters are required to obtain the necessary flexibility in specifying the shape of the head of the lag distribution.

An alternative lag specification which has been widely used in recent econometric work, is a finite distributed lag whose coefficients are restricted to lie on a polynomial of low order. The technique was first suggested and used by Shirley Almon [1]. Modifications which change the restrictions upon the polynomial have been introduced by Bischoff [2] and also employed by Modigliani and Sutch [16]. In its unrestrained form the lagged weights are given by

$$\beta_\tau = a_0 + a_1\tau + a_2\tau^2 + \dots + a_N\tau^N \tag{6}$$

where N is the order of the polynomial. Estimators for the polynomial lag do not employ lagged dependent variables and thus are consistent in the presence of serial correlation. However, the estimators require p - 1 observations on x_t before the beginning of the sample period, so in applications which specify a very long distributed lag the frequent lack of long continuous data series prevents or seriously limits their use.

2. Specification of the Infinite Distributed Lag

This paper presents a new distributed lag specification which is essentially a combination of a polynomial function for the head and a rational function for the tail. The advantage of this specification is that the head of the distribution has the considerable flexibility (for relatively few parameters) offered by the polynomial function, while the tail can be infinitely long, as described by the rational function. After setting forth this specification in greater detail, we shall discuss a simple nonlinear estimator for our lag function which does not employ lagged dependent variables. Our lag function, together with this estimation technique, allows us to specify and estimate equations with different lag structures for different independent variables. The estimator is also consistent in the presence of any form of serial correlation. The lag structure has an infinite number of terms yet uses a minimum of pre-sample observations and thus allows the use of long lag distributions even when the available time series are relatively short.

The functional form which we shall impose on the basic distributed lag of equation (1) is the following:

$$(7) \qquad y_t = \sum_{\tau=0}^{q-1}(\alpha_0 + \alpha_1\tau + \alpha_2\tau^2 + \cdots + \alpha_N\tau^N)\, x_{t-\tau} + \alpha_{N+1}\sum_{\tau=0}^{\infty}\lambda^{\tau}x_{t-\tau} \quad .$$

That is, we require that the lag coefficient, β_τ, have the form

$$(8) \qquad \beta_\tau = \begin{cases} (\alpha_0 + \alpha_1\tau + \dots + \alpha_N\tau^N) + \alpha_{N+1}\lambda^\tau & \text{for } 0 \leq \tau \leq q - 1 \\ \alpha_{N+1}\lambda^\tau & \text{for } \tau \geq q. \end{cases}$$

N is the degree of the polynomial and usually will be chosen between 2 and 4; it must be less than q.

It is natural to impose an additional restriction on the lag function of equation (7) which requires that the joint at $\tau = q$ be smooth. This can be accomplished by restraining the polynomial part of the function to pass through zero at $\tau = q$; that is,

$$(9) \qquad \alpha_0 + \alpha_1 q + \dots + \alpha_N q^N = 0.$$

Solving equation (9) for α_0 and substituting the result in formula (8), we obtain

$$(10) \qquad \beta_\tau = \alpha_1(\tau - q) + \alpha_2(\tau^2 - q^2) + \dots + \alpha_N(\tau^N - q^N) + \alpha_{N+1}\lambda^\tau \quad , \quad \text{for } 0 \leq \tau \leq q - 1.$$

We note parenthetically that this lag function is very closely related to an alternative, in which the head is specified by a polynomial alone, rather than the sum of a polynomial and a geometric term. In the range we are considering, a polynomial approximation to λ^τ is fairly close; we write it as

$$(11) \qquad \alpha_0^* + \alpha_1^*\tau + \alpha_2^*\tau^2 + \dots + \alpha_N^*\tau^N = \alpha_{N+1}\lambda^\tau$$

A restriction which makes the joint smooth for this form is

$$(11a) \qquad \alpha_0^* + \alpha_1^* q + \dots + \alpha_N^* q^N = \alpha_{N+1}\lambda^q$$

so that

$$(12) \qquad \beta_\tau = \alpha_{N+1}\lambda^q + (\alpha_1 + \alpha_1^*)(\tau - q) + (\alpha_2 + \alpha_2^*)(\tau^2 - q^2) + \dots$$

$$+ (\alpha_N + \alpha_N^*)(\tau^N - q^N), \qquad \text{for } 0 \leq \tau \leq q - 1 \quad .$$

We prefer the overlapping form expressed in equation (10), however, because it is somewhat easier to handle computationally.

3. Estimation

Our distributed lag model now has the form

$$(13) \qquad y_t = \sum_{\tau=0}^{q-1} \left[\alpha_1(\tau - q) + \alpha_2(\tau^2 - q^2) + \dots + \alpha_N(\tau^N - q^N)\right] x_{t-\tau}$$

$$+ \alpha_{N+1} \sum_{\tau=0}^{\infty} {}^{\tau}x_{t-\tau} \quad .$$

We define N new variables, z_{tj}, as follows:

$$(14) \qquad z_{tj} = \sum_{\tau=0}^{q-1} (\tau^j - q^j)x_{t-\tau}, \quad j = 1, \dots, N.$$

In terms of these weighted moving averages, equation (13) has the simple form

$$(15) \qquad y_t = \sum_{j=1}^{N} \alpha_j z_{tj} + \alpha_{N+1} \sum_{\tau=0}^{\infty} \lambda^\tau x_{t-\tau} \quad .$$

This technique of reducing the polynomial portion of the distributed lag to N new variables, each a moving weighted average of past observations on x_t, is closely related to Mrs. Almon's technique of distributed lag estimation [1]. However, her approach uses Lagrangian interpolation polynomials to obtain weights for the moving averages, in which case the coefficients of the weighted averages will be points on the estimated polynomial, rather than its parameters. We prefer the more direct technique, which produces identical results, because it is somewhat more straightforward. Moreover, it has the advantage that the addition of another z_{tj} variable increases N by one without the necessity of recomputing the other z_{tj}'s, as would be the case with the Lagrangian weights.

We must now turn our attention to the last term of equation (15). If the summation were over a finite rather than an infinite number of terms and if we knew λ, we could proceed as with the other terms and compute a new variable which would be a weighted moving average of x_t. Even estimating λ would not be a major problem. Since λ is bounded between zero and one we could simply search this interval for the value which produced the minimum sum of squared residuals. Alternatively, we could use a simple nonlinear regression algorithm to estimate the relation directly. However, this approach has the disadvantage of requiring a large number of pre-sample observations on x_t. The polynomial portion has already used $q - 1$ of these and a finite geometric portion would have to use many more. Instead, following Klein [13], we suggest a technique which does not require additional pre-sample observations beyond the q - 1 already used and which is capable of estimating the infinite lag implied by equation (15).

Suppose that the first observation in the pre-sample period used by the polynomial portion of the distributed lag is at date θ; and that observations before this date are unobtainable. At each subsequent date we can divide the infinite sum into two components, one observable and one unobservable; that is,

$$(16) \qquad \sum_{\tau=0}^{\infty} \lambda^{\tau} x_{t-\tau} = \sum_{\tau=0}^{t-\theta} \lambda^{\tau} x_{t-\tau} + \sum_{\tau=t-\theta+1}^{\infty} \lambda^{\tau} x_{t-\tau} \quad .$$

The first term on the right-hand side can be computed from the available data. By shifting the indexes and factoring, we have

$$(17) \qquad \sum_{\tau=0}^{\infty} \lambda^{\tau} x_{t-\tau} = \sum_{\tau=0}^{t-\theta} \lambda^{\tau} x_{t-\tau} + \lambda^{t-\theta} \sum_{\tau=1}^{\infty} \lambda^{\tau} x_{\theta-\tau} \quad .$$

Notice that the term $\sum_{\tau=1}^{\infty} \lambda^{\tau} x_{\theta-\tau}$ does not involve t and thus is a constant, independent of the date. We now define two new variables as follows:

$$(18) \qquad w_{t1} = \sum_{\tau=0}^{t-\theta} \lambda^{\tau} x_{t-\tau}$$

$$w_{t2} = \lambda^{t-\theta} \quad .$$

Next we replace the last term of equation (15) to obtain

$$(19) \qquad y_t = \alpha_1 z_{t1} + \alpha_2 z_{t2} + \cdots + \alpha_N z_{tN} + \alpha_{N+1} w_{t1} + \alpha_{N+2} w_{t2} \quad .$$

The new coefficient, α_{N+2}, bears the following relation to the other parameters of the model:

$$(20) \quad \alpha_{N+2} = \alpha_{N+1} \sum_{\tau=1}^{\infty} \lambda^{\tau} x_{\theta-\tau} \quad .$$

Equation (19) can be estimated in a straightforward fashion. The only problem is the nonlinearity which arises because the parameter λ is required for the computation of w_{t1} and w_{t2}. This nonlinearity is easily handled by a standard nonlinear estimation method; for a detailed discussion of this estimation problem, see Hall [9a].

4. The Selection of q and N

The discussion above assumed that q, the number of periods which are included in the head of the distribution, and N, the order of the polynomial, were known *a priori*. Of course, this is rarely the case. Therefore, we now turn to the discussion of techniques for the selection of q and N.

First we note that quite obviously N must be less than q. Since we need N of the Z variables to estimate an Nth order polynomial, if the order were larger than q we would require more Z variables than we had coefficients to estimate in the head of the distribution. If this were the case we could simply enter the q observations on $X_{t-\tau}$ directly into the regression equation and estimate the head completely unrestrained. This would also be the appropriate procedure if q = N. However, in most applications q will be substantially larger than N.

For most econometric work a second-, third-, or fourth-degree polynomial is probably quite appropriate. As a regular practice it would seem advisable to start with a fourth-degree polynomial and reduce its order if the higher-order terms contribute insignificantly to the explanation of the dependent variable. The appropriate test of significance in this case is the t-test on the Nth Z variable's coefficient.

It is important that when employing this step-wise prodecure we begin with a relatively high-order polynomial and reduce its richness if this is warranted, rather than beginning with a low-order polynomial and increasing its order. In the latter case it would be erroneous to conclude that a fourth-order term would be insignificant had it been added to a third-order polynomial estimator from the evidence that the third-order term was insignificant.

One of the drawbacks to the use of the finite Almon polynomial estimator is that the length of the lag must be specified in advance. This means typically that an experimentor is required to run a number of regressions identical save for the length of lag. He then chooses that lag structure which fits the data most closely in terms of standard error of estimate or which has the most plausible lag shape. When more than one independent variable has a distributed lag the complexity of the search increases considerably.

The technique of lag estimation we propose in this paper has the analogous problem of specifying q, the point on the lag structure where the polynomial portion of the distribution ends and the declining tail begins. Fortunately, the search for this integer is less difficult here than in the finite Almon case, for basically two reasons. First, the estimates are, in well behaved cases, insensitive to the exact placement of q as long as q is chosen sufficiently large. Within a certain range it is unimportant whether that portion of the lag is estimated by the declining tail or the polynomial part--both can approximate the specified lag distribution in a reasonable way.

In addition, with the Almon distribution, often negative (but statistically insignificant) tails appear when q is chosen too long and N is small, and yet shortening q without increasing N worsens the fit and increasing N loses degrees of freedom. This problem arises, we think, because a low-ordered finite polynomial (particularly if it is restrained to go to zero at the end) is not capable of producing an inflection near the tail while still fitting the head adequately. With the infinite tail, the polynomial estimator does not have this problem and therefore does not seem to lead to ambiguities of choice between goodness of fit criteria and the plausibility of the lag shape.

The procedure we suggest for the selection of q is to pick a fairly large value for q and then, by examining the shape of the estimated distribution, shorten the q point accordingly. The standard error of estimate is the appropriate criterion for choosing q.

5. Unscrambling the Lag Distribution

After obtaining estimates of the α coefficients of equation (19) we can compute the lag coefficients β_τ and some associated statistics. For this purpose it is convenient to express the relation between the lag coefficients and the regression coefficients in matrix form:

$$(21)\quad \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{q-1} \\ \beta_q \\ \beta_{q+1} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} -q & -q^2 & \cdot\ \cdot\ \cdot & -q^N & 1 \\ 1-q & 1-q^2 & \cdot\ \cdot\ \cdot & 1-q^N & \lambda \\ 2-q & 2-q^2 & \cdot\ \cdot\ \cdot & 2^N-q^N & \lambda^2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 1 & (q-1)^2-q^2 & \cdot\ \cdot\ \cdot & (q-1)^N-q^N & \lambda^{q-1} \\ 0 & 0 & \cdot\ \cdot\ \cdot & 0 & \lambda^q \\ 0 & 0 & \cdot\ \cdot\ \cdot & 0 & \lambda^{q+1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_N \\ \alpha_{N+1} \end{bmatrix}$$

or

$$(22)\quad \beta = A\alpha \ .$$

Letting $\hat{\alpha}$ denote the vector of estimates of α, we obtain estimates $\hat{\beta}$ of β by the formula,

$$(23) \qquad \hat{\beta} = \hat{A}\hat{\alpha} \quad ,$$

where $\hat{A}$ is identical to A except in the last column, which has powers of the estimate, $\hat{\lambda}$, instead of powers of λ itself.

The next step is to obtain an approximate variance-covariance matrix for the estimated coefficients, $\hat{\beta}$. First, we define

$$(24) \qquad \Phi = \begin{bmatrix} \hat{\alpha}_1 \\ \cdot \\ \cdot \\ \cdot \\ \hat{\alpha}_{N+1} \\ \hat{\lambda} \end{bmatrix}$$

and

$$(25) \qquad \Psi = \hat{\alpha}_{N+1} \begin{bmatrix} 0 \\ 1 \\ 2\hat{\lambda} \\ 3\hat{\lambda}^2 \\ \cdot \\ \cdot \\ \cdot \\ q\hat{\lambda}^{q-1} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad .$$

Now

$$(26) \qquad \frac{\partial\beta}{\partial\Phi} = \left[\hat{A} \ \vdots \ \Psi \right] .$$

Finally, if $V(\Phi)$ is the variance-covariance matrix of the estimated coefficients, the asymptotic variance-covariance matrix of $\hat{\beta}$ is

$$(27) \qquad V(\hat{\beta}) = \left(\frac{\partial\beta}{\partial\Phi} \right) V(\Phi) \left(\frac{\partial\beta}{\partial\Phi} \right)' .$$

The square roots of the diagonal elements of this matrix are the asymptotic standard errors of the lag coefficients β_τ and can be used in carrying out statistical inference about single lag coefficients. One statistic of general interest is the sum of the lag coefficients, defined by

$$(28) \qquad s = \sum_{\tau=0}^{\infty} \beta_\tau .$$

Under our lag specification,

$$(29) \qquad s = \sum_{\tau=0}^{q-1} \sum_{j=1}^{N} \hat{\alpha}_j (\tau^j - q^j) + \hat{\alpha}_{N+1} \sum_{\tau=0}^{\infty} \hat{\lambda}^\tau .$$

By evaluating the infinite geometric series and rearranging the polynomial part, we have

$$(30) \qquad s = \sum_{j=1}^{N} \hat{\alpha}_j \left[\sum_{\tau=0}^{q-1} (\tau^j - q^j) \right] + \frac{\alpha_{N+1}}{1 - \hat{\lambda}} .$$

The row vector of derivatives with respect to the estimates is

$$(31) \qquad \frac{\partial s}{\partial \Phi} = \left[\sum_{\tau=0}^{q-1} (\tau - q), \ldots, \sum_{\tau=0}^{q-1} (\tau^N - q^N), \frac{1}{1-\hat{\lambda}}, \frac{\alpha_{N+1}}{(1-\hat{\lambda})^2} \right]$$

so the asymptotic variance of s is

$$(32) \qquad V(s) = \left(\frac{\partial s}{\partial \Phi} \right) V(\Phi) \left(\frac{\partial s}{\partial \Phi} \right)' .$$

Another statistic which is sometimes of interest is the mean lag, m, which is defined by

$$(33) \qquad m = \frac{1}{s} \left[\sum_{\tau=0}^{q-1} \tau \sum_{j=1}^{N} \hat{\alpha}_j (\tau^j - q^j) + \frac{\hat{\alpha}_{N+1}}{(1-\hat{\lambda})^2} \right] .$$

The vector of derivatives is

$$(34) \qquad \frac{\partial m}{\partial \Phi} = \frac{1}{s} \left[\sum_{\tau=0}^{q-1} \tau(\tau - q), \ldots, \sum_{\tau=0}^{q-1} \tau(\tau^N - q^N), \frac{1}{(1-\hat{\lambda})^2}, \frac{2\alpha_{N+1}}{(1-\hat{\lambda})^3} \right] - \frac{m}{s^2} \frac{\partial s}{\partial \Phi} ,$$

so the asymptotic variance is

$$(35) \qquad V(m) = \left(\frac{\partial m}{\partial \Phi} \right) V(\Phi) \left(\frac{\partial m}{\partial \Phi} \right)' .$$

The mean lag is a meaningful statistic only when the lag coefficients are all of the same sign.

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