The Dynamic Effects of Fiscal Policy in an Economy with Foresight[1]

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This paper presents the results of a study of several related dynamic fiscal problems in a simple economic model whose distinguishing feature is that the participants in the economy are assumed to make plans by looking into the future. The competitive equilibrium in the presence of a changing fiscal policy is calculated, and compared to the intertemporal equilibrium in the absence of any change. Throughout, we use the term equilibrium in its Walrasian sense, to indicate that all markets clear. In every case, we calculate the full general equilibrium in the economy, over all time periods and all markets. Part of the contribution of the paper, it is hoped, is in suggesting some methods for the characterization of intertemporal competitive equilibrium. Except for very general properties—existence of equilibrium and turnpike behaviour—this area of study has received very little attention in the literature.

Since the actual calculation of the equilibrium is rather tediously algebraic, we have adopted the following organization: In part 1, we describe the model and indicate some of its properties. In part 2, we present, without proof, descriptions of the dynamic effects of a variety of fiscal policies. Finally, in part 3, we develop the analytic method applied in section 2.

Four fiscal policies are treated: a recurrent head tax, a consumption tax, an interest tax, and an investment credit. In almost every case, there are important anticipatory effects on the economy in advance of the imposition of the policy. Generally, the anticipatory effects help to smooth the jolt caused by the policy. For example, savings gradually increase in anticipation of the head tax, enabling individuals to maintain a smooth level of consumption in spite of the tax, although eventually consumption must be reduced by the whole amount of the tax. Even when the policy change causes a jolt in the real flows of the economy, as in the case of the consumption tax, the anticipatory effect is in the opposite direction to the jolt.

It is also instructive to note how the competitive price system acts to cushion the economy against the impact of a policy change. This is most strikingly illustrated in the case of the investment credit (defined as a negative excise tax on investment goods). In a partial equilibrium analysis ignoring the effect of the credit on interest rates, it appears that no rational entrepreneur would hold capital goods at the instant the credit became available. He could escape the capital loss induced by the credit, seemingly, by selling his capital goods just before the credit became available and buying them back an instant later. In fact, however, the simultaneous desire of all capital owners to sell and buy back causes a dramatic fall in the interest rate, exactly large enough to remove all the incentive to sell in the first place.

1. AN ECONOMY WITH FORESIGHT

From the point of view of constructing a model in which today's equilibrium depends on future economic developments, consumers play the principal role. We suppose that

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there are a large number of identical individuals, each seeking to maximize an intertemporal utility function of the Ramsey form:

$$u(c_{-T}, \ldots, c_0, \ldots, c_T) = \sum_{t=-T}^{T} \left(\frac{1}{1+\rho}\right)^t u(c_t). \qquad \ldots(1)$$

Here c_t is the individual's consumption of the single consumption good assumed to be produced, ρ is his rate of impatience, assumed to be positive, and $u(.)$ is an increasing, concave, twice-differentiable function. The economy exists only over the period from $-T$ to T; there is no population change over time. This is likely to be a good approximation to a more realistic model with partially overlapping generations if the lifespan is very long compared to the time scale of economic events.

Further, we assume that there is a credit market in which each individual can lend any amount between periods t and $t+1$ at an interest rate r_t. Then to maximize utility, each individual will adjust his consumption plan to equate the marginal rate of substitution between consumption in any adjacent time periods to the price ratio as given by the interest rate:

$$\frac{u'(c_{t+1})/(1+\rho)}{u'(c_t)} = \frac{1}{1+r_t}. \qquad \ldots(2)$$

Among all the consumption plans meeting this requirement, the most expensive one that can be financed by the consumer's wealth is chosen. The source of his wealth is discussed below.

On the producers' side, we simplify matters by assuming that there is only a single input to production called capital.[1] Gross output y_t is given by a twice-differentiable production function $f(k_t)$ with strictly decreasing returns to scale. We assume that consumption goods and capital goods are perfect substitutes in production, so there is no ambiguity in measuring gross output and capital goods in units of the consumption good. If a fraction δ of the capital stock has to be replaced each period, the following equation governs capital accumulation:

$$k_{t+1} = f(k_t)+(1-\delta)k_t - c_{t+1}. \qquad \ldots(3)$$

Note that consumption in period $t+1$ comes out of the gross output of period t. We assume that all profits are distributed evenly among all the consumers; this is the source of consumers' wealth mentioned above. Now the interest rate in this economy can be shown by a simple arbitrage argument to equal the net marginal product of capital:

$$r_t = f'(k_t)-\delta. \qquad \ldots(4)$$

We suppose that the economy begins with an endowment of resources $\bar{q}$ to be divided between immediate consumption and the initial capital stock k_{-T}. Finally, since no consumer will lend in period T, we require that k_T be zero.

If the aggregates in equation 3 are measured *per capita*, then the capital accumulation equation (equation 3) and the consumer marginal rate of substitution equation with the net marginal product of capital in place of the interest rate (equations 2 and 4), together with the initial and terminal conditions just mentioned, form a complete description of the competitive equilibrium:

$$k_{t+1} = f(k_t)+(1+\delta)k_t - c_{t+1}, \qquad \ldots(5)$$

$$u'(c_{t+1}) = \frac{1+\rho}{1+f'(k_t)-\delta}\, u'(c_t), \qquad \ldots(6)$$

$$k_{-T}+c_{-T} = \bar{q}, \qquad \ldots(7)$$

$$k_T = 0. \qquad \ldots(8)$$

[1] The results would remain unchanged if inelastically supplied labour also entered as an input.

We note that any solution to this system of equations must be a general equilibrium: producer equilibrium is guaranteed by the equality of the interest rate and the net marginal product of capital. To show that consumers are in equilibrium, we observe that the marginal rate of substitution condition has already been imposed; it remains to show that the budget constraint is met exactly for each consumer. The latter can be established by the following argument: let A_t be a typical consumer's assets at time t. Then $A_{-T} = \bar{q} - c_{-T}$. One useful way to state the budget constraint is in the form that terminal assets be zero: $A_T = 0$. If π_t is the individual's share of profit, and $r_t A_t$ is his interest income, his asset accumulation equation is

$$A_{t+1} = \pi_t + r_t A_t - c_{t+1}. \qquad \ldots(9)$$

His share of profits is the difference between net output and interest payments, *per capita*:

$$\pi_t = f(k_t) - \delta k_t - r_t k_t. \qquad \ldots(10)$$

Substituting this into the asset accumulation equation and comparing the result with the capital accumulation equation, we find that since $A_{-T} = k_{-T}$, they are equal in all periods, and, in particular, they are both zero in the last period, as required.

This establishes the equivalence between a solution to equations 5 through 8 and a competitive equilibrium. Existence and uniqueness of the equilibrium follow from general theorems of Debreu [3] for existence and Arrow and Hurwicz [1] for uniqueness. It is easily seen that the present model meets the hypotheses of these theorems.

Only a very limited characterization of the competitive equilibrium in this model will be given here, as the main results will be described in section 2 after the model has been extended to take account of fiscal action. In the first place, there is a unique stationary state in the model, say k^* and c^*, defined by

$$f'(k^*) - \delta = \rho \qquad \ldots(11)$$

and

$$c^* = f(k^*) - \delta k^*. \qquad \ldots(12)$$

That is, consumption remains stationary when the interest rate (the inducement to postpone consumption) is equal to the rate of impatience. The actual level of consumption is the net output produced by the capital stock held by producers at this interest rate. Second, the competitive equilibrium displays the turnpike property—even if k_{-T} is very low, k_t arches toward k^* before dropping to zero, and, in fact, tends to remain close to k^* for an increasing fraction of the time as T becomes larger. A formalization of this property and a proof for the differential equation version appear in Cass [2].[1]

2. THE RESPONSE TO FISCAL POLICY

In this section we describe the response of the economy to four different kinds of fiscal policy. In each case, there are no taxes for the period $-T$ through 0, and then one tax (or credit) is imposed suddenly beginning in period 1 and continuing until period T at the same rate. The fact that the policy will take effect in period 1 is known even in period $-T$. The four policies are: (i) a recurrent head tax, (ii) a consumption tax, (iii) an interest tax, and (iv) an investment credit.

In the discussion of the results it is assumed, implicitly, that the lifetime of the economy, $2T$, is long compared to the period in which there is a perceptible response to the policy. Further, the actual length of a single period is assumed to be very short, so that the equilibrium will appear to be a smooth function of time, except in the cases where the policy induces a jump from period 0 to period 1.[2]

1 Cass treats the formally equivalent problem of optimal growth with a Ramsey social welfare function.

2 In an earlier draft, the intellectually more satisfactory case of continuous time and an infinite lifetime was treated. This introduced a number of deep but essentially spurious difficulties, which, although not insoluble, could only obscure the main points of the paper.

Existence and uniqueness of the competitive equilibrium in the model with taxes follow by obvious extensions of the results cited in the previous section. The important properties of the excess demand functions—continuity and satisfaction of the weak axiom of revealed preference—are unaffected by the introduction of a proportional gap between prices paid by consumers and prices received by producers.

We employ the following notation in describing the effects of the various policies under consideration:

λ_1 = positive characteristic root of the approximating linear system, as defined in section 3;

λ_2 = negative characteristic root;

A_c = departure of consumption from stationary level just before tax change; measures anticipatory effect. $A_c = c_0 - c^*$;

A_k = anticipatory effect on capital. $A_k = k_0 - k^*$;

J_c = jump in consumption induced by tax change. $J_c = c_1 - c_0$;

L_c = long-run departure of consumption from its stationary level. $L_c = \lim_{t\to\infty} c_t - c^*$;

L_k = long-run departure of capital from its stationary level. $L_k = \lim_{t\to\infty} k_t - k^*$.

In section 3 we show that to a good approximation, the path followed by the economy perturbed by a fully anticipated discrete change in fiscal policy is

$$\left.\begin{aligned} c_t &= c^* + A_c e^{\lambda_1 t} \quad &\text{...(13)}\\ k_t &= k^* + A_k e^{\lambda_1 t} \quad &\text{...(14)}\end{aligned}\right\} t \leqq 0$$

$$\left.\begin{aligned} c_t &= c^* + L_c + (A_c + J_c - L_c)e^{\lambda_2 t} \quad &\text{...(15)}\\ k_t &= k^* + L_k + (A_k - L_k)e^{\lambda_2 t} \quad &\text{...(16)}\end{aligned}\right\} t > 0.$$

Starting in the distant past at a point close to the stationary levels of consumption and capital, the economy follows an exponential path toward the point $c_0 = c^* + A_c$, $k_0 = k^* + A_k$ in anticipation of the policy change. The change itself may induce a jump in consumption.[1] After the jump, the economy follows an exponential path to a new stationary point, possibly differing from the old one because of permanent influences of the new taxes. Note that the response of the economy to perturbations is stable over time—$e^{\lambda_1 t}$ vanishes as t becomes large and negative and $e^{\lambda_2 t}$ vanishes as t becomes large and positive.

We begin with the case of a recurrent head tax requiring each taxpayer to pay a fixed quantity, γ_h, of goods to the government at the beginning of each period starting with period 1. In this and succeeding cases, we assume that the government disposes of the proceeds of the tax in a way that leaves consumers and producers unaffected. The anticipatory effects of the head tax are

$$A_c = \frac{\lambda_2}{\lambda_1 - \lambda_2}\gamma_h < 0, \qquad \text{...(17)}$$

$$A_k = \frac{1}{\lambda_1 - \lambda_2}\gamma_h > 0. \qquad \text{...(18)}$$

Consumption declines and capital rises in anticipation of the tax; this builds up a buffer stock of capital out of which the tax is paid initially. The quantity $\frac{-\lambda_2}{\lambda_1 - \lambda_2}$ lies between 0 and $\frac{1}{2}$, so consumption declines by not more than half the amount of the tax.

[1] Policies other than the ones discussed here could induce a jump in the capital stock. This could be incorporated by defining a parameter J_k. An example of such a policy is a capital levy.

No jump is induced by the head tax at the time of its imposition. In the competitive equilibrium, consumers pay the tax at first entirely out of accumulated wealth. As the capital stock is drawn down, however, more and more of the tax is paid by reducing consumption. In the long run, capital declines to its earlier stationary level, and consumption declines to a stationary level less than its earlier level by the full amount of the tax:

$$L_c = -\gamma_h, \qquad \text{...(19)}$$

$$L_k = 0. \qquad \text{...(20)}$$

To summarize the effects of the head tax, we give the expressions for the path of consumption:

$$c_t = c^* - \frac{-\lambda_2}{\lambda_1 - \lambda_2}\gamma_h e^{\lambda_1 t}, \quad t \leqq 0,$$

$$= c^* - \left(1 - \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 t}\right)\gamma_h, \quad t > 0. \qquad \text{...(21)}$$

Our second case is that of a consumption tax which raises the effective price of one unit of consumption to $1+\gamma_c$ in period 1 and every succeeding period. The only substitution effect of this tax between adjacent periods is between c_0 and c_1 where it induces a downward jump in consumption. The general equilibrium accommodation to this jump affects consumption and capital in every period, but, in contrast to the head tax, the consumer is not cushioned against the jump itself.

To isolate the general equilibrium substitution effects of the consumption tax, we suppose first that it is combined with a negative head tax of equal yield. The anticipatory effects of this combination are

$$A_c = \frac{-\lambda_2}{\lambda_1 - \lambda_2}\frac{c^*}{E_c}\gamma_c > 0, \qquad \text{...(22)}$$

$$A_k = \frac{-1}{\lambda_1 - \lambda_2}\frac{c^*}{E_c}\gamma_c < 0, \qquad \text{...(23)}$$

where E_c is the absolute value of the elasticity of marginal utility with respect to consumption, evaluated at c^*:

$$E_c = -c^* \frac{u''(c^*)}{u'(c^*)}. \qquad \text{...(24)}$$

The jump in consumption is

$$J_c = -\frac{c^*}{E_c}\gamma_c < 0. \qquad \text{...(25)}$$

The anticipatory effect on consumption is no larger than one-half of the magnitude of the jump. With the compensating negative head tax, the consumption tax has no long run effects. Thus as long as its rate is not changed, the consumption tax is perfectly neutral, in the sense that it has no substitution effect. But a fully anticipated change in the tax rate causes a serious jolt in the economy, one which the price system does not absorb, so the jolt appears in the real flows of the economy. Consumption rises and investment falls before the tax is imposed. At the moment of its imposition, consumption falls discontinuously and investment rises by the same amount. The capital stock and consumption gradually rise thereafter to their previous stationary values.

In the absence of the offsetting negative head tax, the response of the economy is similar, except that the anticipatory effect on consumption is smaller because of the opposite anticipatory effect of the income loss caused by the tax:

$$A_c = \frac{\lambda_2}{\lambda_1-\lambda_2}\left(1-\frac{1}{E_c}\right)c^*\gamma_c, \qquad \text{...(26)}$$

$$A_k = \frac{1}{\lambda_1-\lambda_2}\left(1-\frac{1}{E_c}\right)c^*\gamma_c. \qquad \text{...(27)}$$

If the elasticity of marginal utility is unity (for example, if $u(c) = \log c$), then the substitution and income effects exactly balance, and the economy gives no outward indication of its expectation of the policy change.

The consumption jump caused by the consumption tax is the same in the absence of the income offset as it was in the earlier case. The long run effects without offset are

$$L_c = -c^*\gamma_c, \qquad \text{...(28)}$$

$$L_k = 0. \qquad \text{...(29)}$$

Our third case is that of an interest tax which reduces the effective rate of interest earned by consumers from $f'(k)-\delta$ to $(1-\gamma_r)(f'(k)-\delta)$ for interest receipts in period 1 and every succeeding period. This might be imposed directly on consumers as a tax on interest income or on producers as a tax on capital earnings. The latter example is particularly interesting because it permits interpretation of our results as a dynamic theory of the shifting of the corporate income tax.

Once again we suppose that there is an offsetting negative head tax so that we can isolate the substitution effects of the tax. First, the anticipation effects are

$$A_c = \frac{\lambda_2^2}{\lambda_1-\lambda_2}\frac{k^*}{E_k}\gamma_r > 0, \qquad \text{...(30)}$$

$$A_k = \frac{\lambda_2}{\lambda_1-\lambda_2}\frac{k^*}{E_k}\gamma_r < 0. \qquad \text{...(31)}$$

E_k is the absolute value of the elasticity of the net marginal product of capital evaluated at the stationary level of capital:

$$E_k = -k^*\frac{f''(k^*)}{f'(k^*)-\delta}. \qquad \text{...(32)}$$

Consumption rises and capital falls in anticipation of the diminished return from capital after the tax. Significantly, no jump in either capital or consumption is induced by the sudden imposition of the tax. The capital stock continues its decline after the tax, and consumption begins to decline. Both approach levels in the long run that are lower than the earlier stationary levels:

$$L_c = -\rho\frac{k^*}{E_k}\gamma_r < 0, \qquad \text{...(33)}$$

$$L_k = -\frac{k^*}{E_k}\gamma_r < 0. \qquad \text{...(34)}$$

In spite of the income offset, long-run consumption is reduced by the substitution toward less capital caused by the interest tax. Compared to the consumption tax, the interest tax is better behaved in its impact effect but is costlier in the long run because of its permanent substitution effect.

If the interest tax is levied as a tax on capital earnings, it is natural to inquire into the extent to which the tax is shifted, that is, to what extent capital earnings per unit of capital increases to compensate for the tax. It is customary to define the rate of shifting, s, as the ratio of the amount of additional income before tax per unit of capital caused by the tax to the amount of tax per unit of capital if income were at the level that it would have had if there were no tax. In the present model,

$$s_t = \frac{f'(k_t) - \delta - \rho}{\rho \gamma_r}. \qquad \text{...(35)}$$

Using the linear approximation,

$$f'(k_t) - \delta - \rho \doteq -(k_t - k^*) \frac{E_k}{k^*}, \qquad \text{...(36)}$$

we have

$$s_t = 1 - \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 t}, \quad t > 0. \qquad \text{...(37)}$$

Thus the rate of shifting starts at $\frac{-\lambda_2}{\lambda_1 - \lambda_2}$, which lies between 0 and $\frac{1}{2}$, just after the tax is imposed, and gradually approaches unity. This is in accord with the general belief that the rate of shifting in the short-run is less than in the long-run. It should be noted that this theory of shifting does not depend on the existence of short-run disequilibrium to explain the disparity between short and long-run rates of shifting.

As a final case we consider an investment credit, which we take to be a negative excise tax at rate γ_k on investment goods, whether used for net investment or replacement. The investment credit has two distinct substitution effects. In the period that it first becomes available, it has the same substitution effect as a consumption tax at the equivalent rate, and, in fact, causes the same jump in consumption and investment. The credit also has a permanent substitution effect toward more capital, arising from its subsidy on replacement investment.

With an offsetting positive head tax the investment credit causes a response described by the following parameters:

$$A_c = \frac{\lambda_2}{\lambda_1 - \lambda_2}\left(\frac{c^*}{E_c} - \left(\left(\frac{\lambda_1}{\rho} - 1\right)\frac{\rho+\delta}{E_k}\right)k^*\right)\gamma_k, \qquad \text{...(38)}$$

$$A_k = \frac{1}{\lambda_1 - \lambda_2}\left(\frac{c^*}{E_c} - \left(\left(\frac{\lambda_1}{\rho} - 1\right)\frac{\rho+\delta}{E_k}\right)k^*\right)\gamma_k, \qquad \text{...(39)}$$

$$J_c = -\frac{c^*}{E_c}\gamma_k < 0, \qquad \text{...(40)}$$

$$L_c = \frac{k^*}{E_k}(\rho+\delta)\gamma_k > 0, \qquad \text{...(41)}$$

$$L_k = \frac{\rho+\delta}{\rho}\frac{k^*}{E_k}\gamma_k > 0. \qquad \text{...(42)}$$

There are two opposing anticipation effects on consumption, a positive effect arising from the consumption tax aspect of the credit, and a negative effect arising from the negative interest tax aspect. In the long run, the investment credit increases both consumption and capital, at the cost of consumption foregone around the time of the initiation of the credit.

Although the investment credit induces a jump in the real flows in the economy, the price system does a great deal to cushion the economy against the shock. Effectively, the rate of interest becomes almost infinitely negative for loans between periods 0 and 1. If the interest rate remained constant, individual entrepreneurs would sell their capital in period 0 and buy new capital with the benefit of the credit in period 1. In the competitive equilibrium, however, prices adjust to ensure that exactly the existing capital stock is held between periods 0 and 1.

3. ANALYTICAL METHOD

Our demonstration of the conclusions presented in the previous section proceeds in the following way. First, we develop an extended version of the model of equations 5 to 8. It differs from the earlier model in two respects: it incorporates parameters for the various tax policies that concern us, and it subdivides each time period into N subperiods. We then differentiate this model with respect to the tax parameters to get a system of equations describing the response of the competitive equilibrium to changes. We evaluate this system along the stationary path, and calculate the limit of the solution as the time span becomes large ($T \rightarrow \infty$) and the time interval becomes small ($N \rightarrow \infty$). We interpret the resulting expression as a linear approximation for finite changes in tax parameters, and this forms the basis for the conclusions of section 2. The validity of the approximation has been verified by explicit numerical calculation of the exact competitive equilibrium in a typical case for each policy.[1]

The complete model is the following:

$$\left.\begin{aligned} u'(c_{j+1}) &= \frac{1+\dfrac{\rho}{N}}{1+\dfrac{f'(k_j)-\delta}{N}}\, u'(c_j), && \quad \ldots(43)\\ k_{j+1} &= \frac{1}{N} f(k_j) + \left(1-\frac{\delta}{N}\right)k_j - \frac{1}{N} c_{j+1}, && \quad \ldots(44) \end{aligned}\right\} j<0$$

$$\left.\begin{aligned} u'(c_1) &= \frac{\left(1+\dfrac{\rho}{N}\right)(1+\theta_c)}{1+(1-\theta_r)\dfrac{f'(k_0)-\delta}{N} - \left(1-\dfrac{\delta}{N}\right)\theta_k}\, u'(c_0), && \quad \ldots(45)\\ k_1 &= \left(1-\frac{\delta}{N}\right)k_0 + \frac{1}{N}\frac{1}{1-\theta_k}(f(k_0)-c_1-\theta_h-\theta_c c_1-\theta_r(f'(k_0)-\delta)k_0), && \quad \ldots(46) \end{aligned}\right\} j=0$$

$$\left.\begin{aligned} u'(c_{j+1}) &= \frac{\left(1+\dfrac{\rho}{N}\right)(1-\theta_k)}{1+(1-\theta_r)\dfrac{f'(k_j)-\delta}{N} - \left(1-\dfrac{\delta}{N}\right)\theta_k}\, u'(c_j), && \quad \ldots(47)\\ k_{j+1} &= \left(1-\frac{\delta}{N}\right)k_j + \frac{1}{N}\frac{1}{1-\theta_k}(f(k_j)-c_{j+1}-\theta_h-\theta_c c_{j+1}-\theta_r(f'(k_j)-\delta)k_j), && \quad \ldots(48) \end{aligned}\right\} j>0$$

$$k_{-TN} + \frac{1}{N} c_{-TN} = k^* + \frac{1}{N} c^*, \qquad \ldots(49)$$

$$k_{TN} = k^*. \qquad \ldots(50)$$

The functions and parameters retain their definitions in terms of the original time units;

[1] The author is grateful to Michael Hurd for programming these calculations.

for example, $\frac{1}{N}f(k)$ is the output produced in one subperiod with capital k. Note that the terminal condition, equation 50, differs from the more natural terminal condition of the previous section. This is purely for the sake of analytical convenience, and has no effect on the results, since we consider only the limit as T and N become large. The tax parameters are defined as follows:

θ_h = amount of head tax, *per capita* per period,

θ_c = rate of consumption tax. The price, including tax, of one unit of consumption is $1+\theta_c$,

θ_r = interest tax rate,

θ_k = rate of investment credit. The effective price of one unit of investment is $1-\theta_k$.

To obtain a general expression for derivatives with respect to any of these parameters, we introduce a dummy parameter, z, with the following relation to the θ's: $\theta_h = \gamma_h z$; $\theta_c = \gamma_c z$; $\theta_r = \gamma_r z$, and $\theta_k = \gamma_k z$. We calculate derivatives with respect to z, and choose the γ's to represent one or another fiscal policy. This is especially convenient because the resulting expression in terms of the γ's can simply be reinterpreted as the linear approximation with the γ's serving as tax rates.

At this point we differentiate equations 42 to 49 with respect to z, and evaluate the resulting system at the point $z = 0$, for which we know $c_j = c^*$ and $k_j = k^*$ for all j. If we let

$$A(N) = \begin{bmatrix} 0 & -\dfrac{\rho}{1+\dfrac{\rho}{N}}\dfrac{c^*}{E_c}\dfrac{E_k}{k^*} \\ -1+\dfrac{1}{N}\dfrac{\rho}{1+\dfrac{\rho}{N}}\dfrac{c^*}{E_c}\dfrac{E_k}{k^*} & \rho \end{bmatrix}, \qquad \ldots(51)$$

$$S(N) = \begin{bmatrix} \dfrac{c^*}{E_c}\left(\gamma_c + \dfrac{1-\dfrac{\delta}{N}}{1+\dfrac{\rho}{N}}\gamma_k\right) \\ 0 \end{bmatrix}, \qquad \ldots(52)$$

$$b(N) = \begin{bmatrix} \dfrac{c^*}{E_c}\dfrac{\rho\gamma_r-(\rho+\delta)\gamma_k}{1+\dfrac{\rho}{N}} \\ -\gamma_h - c^*\gamma_c - \rho k^*\gamma_r + \delta k^*\gamma_k \end{bmatrix}, \qquad \ldots(53)$$

$$x_j = \begin{bmatrix} \dfrac{dc_j}{dz} \\ \dfrac{dk_j}{dz} \end{bmatrix}, \qquad \ldots(54)$$

then the resulting system is

$$x_{j+1} = \left(I+\frac{1}{N}A(N)\right)x, \quad j<0,$$

$$= \left(I+\frac{1}{N}A(N)\right)x_0+\frac{1}{N}b(N)+S(N), \quad j=0,$$

$$= \left(I+\frac{1}{N}A(N)\right)x_j+\frac{1}{N}b(N), \quad j>0; \qquad \text{...(55)}$$

$$\frac{1}{N}x_{1,-TN}+x_{2,-TN}=0, \qquad \text{...(56)}$$

$$x_{2,TN}=0. \qquad \text{...(57)}$$

Here I is a 2×2 identity matrix.

The vector $S(N)$ gives the structural shock associated with the consumption tax and the investment credit, and the vector $b(N)$ gives the permanent structural effects of the various taxes; the first element contains substitution effects of the interest tax and investment credit and the second element income effects of all the taxes. The problem at this point is to calculate the reduced form or solution of this set of structural equations. Since the system is linear with constant coefficients, there is no obstacle in principle to solving it directly. A great simplification is achieved, however, by calculating the limit of the solution as the time span becomes large (to eliminate end effects) and as the time interval becomes short (to eliminate nuisance terms that arise from the use of discrete time). Nothing of economic interest is lost by taking this limit.[1]

In the Appendix, it is shown that the limiting solution to systems of this kind has the following form:

$$x(t) = M_0e^{\lambda_1 t}\begin{bmatrix}\lambda_2\\1\end{bmatrix}, \quad t\leqq 0,$$

$$= M_1e^{\lambda_2 t}\begin{bmatrix}\lambda_1\\1\end{bmatrix}+\begin{bmatrix}m_1\\m_2\end{bmatrix}, \quad t>0, \qquad \text{...(58)}$$

where λ_1 and λ_2 are characteristic roots of $A = \lim_{N\to\infty} A(N)$, M_0, M_1, m_1, and m_2 are constants depending on $b = \lim_{N\to\infty} b(N)$, and $S = \lim_{N\to\infty} S(N)$ in the following way:

$$\begin{bmatrix}m_1\\m_2\end{bmatrix} = -A^{-1}b, \qquad \text{...(59)}$$

$$M_0 = \frac{S_1-\lambda_1S_2-m_1+\lambda_1m_2}{\lambda_1-\lambda_2}, \qquad \text{...(60)}$$

$$M_1 = \frac{S_1-\lambda_1S_2-m_1+\lambda_2m_2}{\lambda_1-\lambda_2}. \qquad \text{...(61)}$$

[1] It would be even simpler to calculate the solution to the limit of the system of equations, rather than the limit to the solution. Unfortunately the first limit does not exist, because of the shock term $S(N)$. This is the source of the difficulty mentioned in footnote 2 on p. 231.

In the present case,

$$A = \begin{bmatrix} 0 & -\rho \dfrac{c^*}{E_c} \dfrac{E_k}{k^*} \\ -1 & \rho \end{bmatrix}, \qquad \text{...(62)}$$

$$S = \begin{bmatrix} \dfrac{c^*}{E_c} (\gamma_c + \gamma_k) \\ 0 \end{bmatrix}, \qquad \text{...(63)}$$

$$b = \begin{bmatrix} \dfrac{c^*}{E_c} (\rho\gamma_r - (\rho+\delta)\gamma_k) \\ -\gamma_h - c^*\gamma_c - \rho k^* \gamma_r + \delta k^* \gamma_k \end{bmatrix}. \qquad \text{...(64)}$$

The characteristic roots of A are

$$\lambda_1 = \frac{\rho + \left(\rho^2 + 4\rho \dfrac{c^*}{E_c} \dfrac{E_k}{k^*}\right)^{\frac{1}{2}}}{2} > 0, \qquad \text{...(65)}$$

$$\lambda_2 = \frac{\rho - \left(\rho^2 + 4\rho \dfrac{c^*}{E_c} \dfrac{E_k}{k^*}\right)^{\frac{1}{2}}}{2} < 0. \qquad \text{...(66)}$$

Thus, the long-run effects are

$$L_c = m_1 = \frac{k^*}{E_k} (\rho\gamma_r - (\rho+\delta)\gamma_k) - \gamma_h - c^*\gamma_c - \rho k^* \gamma_r + \delta k^* \gamma_k, \qquad \text{...(67)}$$

$$L_k = m_2 = \frac{1}{\rho} \frac{k^*}{E_k} (\rho\gamma_r - (\rho+\delta)\gamma_k). \qquad \text{...(68)}$$

The general anticipatory effects are

$$A_k = M_0 = \frac{1}{\lambda_1 - \lambda_2} \left(\gamma_h + \left(\frac{1}{E_c} + 1\right) c^* \gamma_c + \left(\rho - \frac{\lambda_2}{E_k}\right) k^* \right) \gamma_r$$
$$+ \left(\frac{c^*}{E_c} - \left(\left(\frac{\lambda_1}{\rho} - 1 \right) \frac{\rho+\delta}{E_k} + \delta \right) k^* \right) \gamma_k \qquad \text{...(69)}$$

and

$$A_c = \lambda_2 M_0. \qquad \text{...(70)}$$

Finally,

$$J_c = S_1 = -\frac{c^*}{E_c} (\gamma_c + \gamma_k). \qquad \text{...(71)}$$

Next, we note how these results are used to obtain a linear approximation to the true competitive equilibrium with a particular fiscal policy. In general, the true equilibrium can be written

$$x_t = \bar{x}_t(\theta_h, \theta_c, \theta_r, \theta_k). \qquad \text{...(72)}$$

If each of these depends on the dummy parameter, z, then a linear approximation to x_t is

$$x_t \doteq \bar{x}_t(0, 0, 0, 0) + \left(\frac{\partial \bar{x}_t}{\partial \theta_h} \frac{d\theta_h}{dz} + \frac{\partial \bar{x}_t}{\partial \theta_c} \frac{d\theta_c}{dz} + \frac{\partial \bar{x}_t}{\partial \theta_r} \frac{d\theta_r}{dz} + \frac{\partial \bar{x}_t}{\partial \theta_k} \frac{d\theta_k}{dz} \right) z, \qquad \text{...(73)}$$

or

$$x_t \doteq \begin{bmatrix} c^* \\ k^* \end{bmatrix} + \left(\frac{\partial \bar{x}_t}{\partial \theta_h}\gamma_h + \ldots + \frac{\partial \bar{x}_t}{\partial \theta_k}\gamma_k\right) z. \qquad \ldots(74)$$

The expression inside the parentheses is exactly what we have derived above. By setting $z = 1$, we see that this expression itself serves as a linear approximation to the departure from the stationary state if each γ is interpreted as a tax rate itself.

The various policies studied are defined as follows

(i) Head tax: $\gamma_h > 0$; $\gamma_c = \gamma_r = \gamma_k = 0$.
(ii*a*) Consumption tax, with income offset: $\gamma_c > 0$; $\gamma_h = -c^*\gamma_c$; $\gamma_r = \gamma_k = 0$.
(ii*b*) Consumption tax, no offset: $\gamma_c > 0$, $\gamma_h = \gamma_r = \gamma_k = 0$.
(iii) Interest tax, income offset: $\gamma_r > 0$, $\gamma_h = -\rho k^* \gamma_r$, $\gamma_c = \gamma_k = 0$.
(iv) Investment credit, income offset: $\gamma_k > 0$, $\gamma_h = \delta k^* \gamma_k$, $\gamma_c = \gamma_r = 0$.

APPENDIX

Lemma. *Suppose*

$$x_{j+1} = \left(I + \frac{1}{N}A(N)\right)x_j, \quad 0 \leqq j \leqq TN-1, \qquad \ldots(A1)$$

$$\beta_1(N)x_{1,TN} + \beta_2(N)x_{2,TN} = 0, \qquad \ldots(A2)$$

where x_j is a two-dimensional vector:

$$x_j = \begin{bmatrix} x_{1,j} \\ x_{2,j} \end{bmatrix}, \qquad \ldots(A3)$$

I is a 2×2 identity matrix, and $A(N)$ is a 2×2 matrix with the properties

$$\lim_{N\to\infty} A(N) = A = \begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \ldots(A4)$$

$$a_{12}a_{21} > 0, \qquad \ldots(A5)$$

$$a_{22} > 0. \qquad \ldots(A6)$$

Suppose further that

$$\lim_{N\to\infty} \beta_1(N) = \beta_1, \qquad \ldots(A7)$$

$$\lim_{N\to\infty} \beta_2(N) = \beta_2, \qquad \ldots(A8)$$

and that one of the two functions $\beta_1(N)$ and $\beta_2(N)$ is nonzero for all N. Then

$$\lim_{N\to\infty}\lim_{T\to\infty} x_{[Nt]} = e^{\lambda_2 t}\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} x_{2,0} \qquad \ldots(A9)$$

for all $t \geqq 0$ and all $x_{2,0}$.

$[Nt]$ is the greatest integer less than Nt and λ_2 is the negative characteristic root of A.

Proof. We note first that the characteristic roots of A are real and opposite in sign: $\lambda_1 > 0$ and $\lambda_2 < 0$, and that the corresponding characteristic vectors can be written

$$\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}. \qquad \ldots(A10)$$

Now let $L(N)$ be the diagonal matrix of characteristic roots of $I+\frac{1}{N}A(N)$:

$$L(N)=\begin{bmatrix}1+\frac{1}{N}\lambda_1(N) & 0\\ 0 & 1+\frac{1}{N}\lambda_2(N)\end{bmatrix} \qquad \ldots(A11)$$

and let $H(N)$ be a matrix of characteristic vectors of $I+\frac{1}{N}A(N)$ (these are also characteristic vectors of $A(N)$ itself). It is easily seen by direct calculation that

$$\lim_{N\to\infty}\lambda_1(N)=\lambda_1, \qquad \ldots(A12)$$

$$\lim_{N\to\infty}\lambda_2(N)=\lambda_2, \qquad \ldots(A13)$$

and that there is a sequence of normalizations of $H(N)$ such that

$$\lim_{N\to\infty}H(N)=\begin{bmatrix}\lambda_2 & \lambda_1\\ 1 & 1\end{bmatrix}. \qquad \ldots(A14)$$

At least for suitably large values of N, $\lambda_1(N)>0$, $\lambda_2(N)<0$, and $H(N)$ admits the normalization

$$H(N)=\begin{bmatrix}h_1(N) & h_2(N)\\ 1 & 1\end{bmatrix}. \qquad \ldots(A15)$$

Henceforward we restrict our attention to values of N large enough for these to hold.

The solution to the difference equation stated in terms of these matrices is

$$x_j=H(N)(L(N))^j(H(N))^{-1}x_0. \qquad \ldots(A16)$$

Substituting the terminal conditions and writing the equation in terms of the matrix elements, we find

$$\begin{aligned}0=\frac{1}{h_1(N)-h_2(N)}\Bigg\{&\Bigg[\beta_1(N)\left\{h_1(N)\left(1+\frac{\lambda_1(N)}{N}\right)^{TN}-h_2(N)\left(1+\frac{\lambda_1(N)}{N}\right)^{TN}\right\}\\ &+\beta_2(N)\left\{\left(1+\frac{\lambda_1(N)}{N}\right)^{TN}-\left(1+\frac{\lambda_2(N)}{N}\right)^{TN}\right\}\Bigg]x_{1,0}\\ &+\Bigg[\beta_1(N)h_1(N)h_2(N)\left\{-\left(1+\frac{\lambda_1(N)}{N}\right)^{TN}+\left(1+\frac{\lambda_2(N)}{N}\right)^{TN}\right\}\\ &+\beta_2(N)\left\{-h_2(N)\left(1+\frac{\lambda_1(N)}{N}\right)^{TN}+h_1(N)\left(1+\frac{\lambda_2(N)}{N}\right)^{TN}\right\}\Bigg]x_{2,0}\Bigg\}.\end{aligned} \qquad \ldots(A17)$$

Solving this for $x_{1,0}$ and taking the limit as $T\to\infty$, we find that the terms with

$$\left(1+\frac{\lambda_1(N)}{N}\right)^{TN}$$

predominate, giving

$$\lim_{T\to\infty}x_{1,0}=h_2(N)x_{2,0}. \qquad \ldots(A18)$$

Thus for an arbitrary period, j, the solution has the property

$$\lim_{T\to\infty}x_j=\left(1+\frac{\lambda_2(N)}{N}\right)^j\begin{bmatrix}h_2(N)\\ 1\end{bmatrix}x_{2,0}. \qquad \ldots(A19)$$

Finally, we consider $j = [Nt]$ for an arbitrary t:

$$\left(1+\frac{\lambda_2(N)}{N}\right)^{[Nt]} = \left(1+\frac{\lambda_2(N)}{N}\right)^{Nt}\left(1+\frac{\lambda_2(N)}{N}\right)^{[Nt]-Nt}. \qquad ...(A20)$$

Now

$$\lim_{N\to\infty}\left(1+\frac{\lambda_2(N)}{N}\right)^{[Nt]} = \left(\lim_{N\to\infty}\left(1+\frac{\lambda_2(N)}{N}\right)^{Nt}\right)\left(\lim_{N\to\infty}\left(1+\frac{\lambda_2(N)}{N}\right)^{[Nt]-Nt}\right)$$

$$= e^{\lambda_2 t}. \qquad ...(A21)$$

We conclude that

$$\lim_{N\to\infty}\lim_{T\to\infty} x_{[Nt]} = e^{\lambda_2 t}\begin{bmatrix}\lambda_1\\ 1\end{bmatrix} x_{2,0} \qquad ...(A22)$$

as required.

Theorem. *Suppose*

$$x_{j+1} = \left(I+\frac{1}{N}A(N)\right)x_j, \quad j = -TN, ..., -1, \qquad ...(A23)$$

$$x_1 = \left(I+\frac{1}{N}A(N)\right)x_0 + \frac{1}{N}b(N)+S(N), \qquad ...(A24)$$

$$x_{j+1} = \left(I+\frac{1}{N}A(N)\right)x_j + \frac{1}{N}b(N), \quad j = 1, ..., TN-1, \qquad ...(A25)$$

$$\frac{1}{N}x_{1,-TN}+x_{2,-TN} = 0, \qquad ...(A26)$$

$$x_{2,TN} = 0, \qquad ...(A27)$$

where x_j, I, and $A(N)$ are as defined in the Lemma, and $b(N)$ and $S(N)$ are two-dimensional vectors with limits as follows:

$$\lim_{N\to\infty} b(N) = b, \qquad ...(A28)$$

$$\lim_{N\to\infty} S(N) = S. \qquad ...(A29)$$

Then

$$\lim_{N\to\infty}\lim_{T\to\infty} x_{[Nt]} = M_0 e^{\lambda_1 t}\begin{bmatrix}\lambda_2\\ 1\end{bmatrix}, \quad t \leqq 0,$$

$$= M_1 e^{\lambda_2 t}\begin{bmatrix}\lambda_1\\ 1\end{bmatrix} + \begin{bmatrix}m_1\\ m_2\end{bmatrix}, \quad t>0, \qquad ...(A30)$$

where M_0, M_1, m_1, and m_2 are constants depending on A, b, and S in the following way:

$$\begin{bmatrix}m_1\\ m_2\end{bmatrix} = -A^{-1}\,b, \qquad ...(A31)$$

$$M_0 = \frac{S_1-\lambda_1 S_2+\lambda_1 m_2-m_1}{\lambda_1-\lambda_2}, \qquad ...(A32)$$

$$M_1 = \frac{S_1-\lambda_1 S_2+\lambda_2 m_2-m_1}{\lambda_1-\lambda_2}. \qquad ...(A33)$$

Proof. The proof consists of a double application of the Lemma. For $j<0$, we can write the difference equation with time reversed:

$$x_{k+1} = \left(I + \frac{1}{N} A(N)\right)^{-1} x_k, \quad k>0$$
$$= \left(I + \frac{1}{N} B(N)\right) x_k. \qquad ...(A34)$$

The matrix $B(N)$ so defined is easily seen to have the following property:

$$\lim_{N \to \infty} B(N) = -A. \qquad ...(A35)$$

Further, its characteristic roots are $-\lambda_1$ and $-\lambda_2$, and its matrix of characteristic vectors is H. Thus we have, as $N \to \infty$ and $T \to \infty$

$$x_{1,0} = \lambda_2 x_{2,0} \qquad ...(A36)$$

by the previous lemma.

For $j>0$, we need to take account of the inhomogeneity of the difference equation. For this purpose, we define

$$y_j = x_j + \frac{1}{N} (A(N))^{-1} b(n). \qquad ...(A37)$$

Then

$$y_{j+1} = \left(I + \frac{1}{N} A(N)\right) y_j, \qquad ...(A38)$$

and a suitable application of our earlier result gives, as $N \to \infty$ and $T \to \infty$

$$y_{1,1} = \lambda_1 y_{2,1}. \qquad ...(A39)$$

In terms of $x_{1,1}$ and $x_{2,1}$, this can be written

$$x_{1,1} = (\lambda_1 \alpha_{21} - \alpha_{11}) b_1 + (\lambda_1 \alpha_{22} - \alpha_{12}) b_2 + \lambda_1 x_{2,1}, \qquad ...(A40)$$

where

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = A^{-1}. \qquad ...(A41)$$

At $j = 0$, we have, as $N \to \infty$

$$x_{1,1} = x_{1,0} + S_1, \qquad ...(A42)$$
$$x_{2,1} = x_{2,0} + S_2. \qquad ...(A43)$$

This gives a linear system of four equations in the four variables $x_{1,0}$, $x_{2,0}$, $x_{1,1}$, and $x_{2,1}$. If we let

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = -A^{-1} b \qquad ...(A44)$$

we can write its solution as follows:

$$x_{1,0} = \lambda_2 \frac{S_1 - \lambda_1 S_2 + \lambda_1 m_2 - m_1}{\lambda_1 - \lambda_2}, \qquad ...(A45)$$

$$x_{2,0} = \frac{x_{1,0}}{\lambda_2}, \qquad ...(A46)$$

$$x_{1,1} = \lambda_1 \frac{S_1 - \lambda_2 S_2 + \lambda_2 m_2 - m_1}{\lambda_1 - \lambda_2} + m_1, \qquad ...(A47)$$

$$x_{2,1} = \frac{S_1 - \lambda_2 S_2 + \lambda_2 m_2 - m_1}{\lambda_1 - \lambda_2} + m_2. \qquad ...(A48)$$

Finally, the expression for the limiting solution for arbitrary t given in the Lemma can be applied to derive the expression for the solution appearing in the statement of the theorem.

REFERENCES

[1] Arrow, K. J. and Hurwicz, L. " On the Stability of Competitive Equilibrium, I ", *Econometrica* **26** (1958), 522-552.

[2] Cass, D. " Optimum Growth in an Aggregative Model of Capital Accumulation: A Turnpike Theorem ", *Econometrica* **34** (1966), 833-850.

[3] Debreu, G. *The Theory of Value*, Cowles Foundation Monograph 17 (New York, Wiley, 1959).