Navigability is a robust property

Paris Siminelakis
Stanford

Dimitris Achlioptas
UC Santa Cruz

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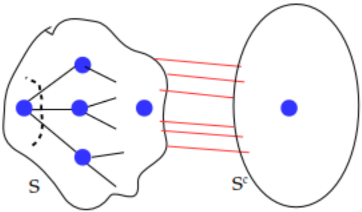
Outline

Small World Phenomenon

Small World Phenomenon existence of short paths between any two people using personal acquaintances.



Interpretation: Small Diameter

$$D = O(\log n) \ w.h.p$$

Reason: Expansion

Branching process - Renormalization

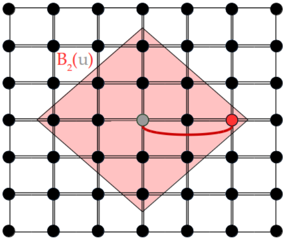
How to find such paths efficiently? $\Rightarrow$ Geometry

Navigability

Millgram's Experiment given occupation and location of a random individual, send a message to the person most likely to know that person.

[Kleinberg'99] Algorithmic interpretation to Milgram's experiment.

Navigability: greedy decentralized search in $\text{poly}\log(n)$ *time.*



Rank Based Augmentation (RBA)

$$P(u, v) \propto \frac{1}{\mathrm{Vol}(B_u(r))^{\alpha}}$$

Uniform over distance scales for $\alpha = 1$

Distance is cut in half every $\log(n)$ steps.

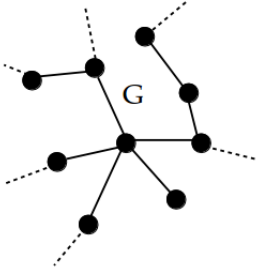
How general is this phenomenon?

Geometric Requirements

Geometric Requirements

Graph Augmentation

Can any graph be turned into a small world? [Duchon et al. DISC'05]



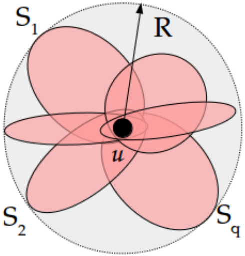
[Local Connectivity]
- Connected Graph G .

[Geometry]
- Shortest path metric

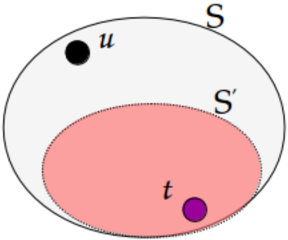
[Augmentation]
- RBA using SP-metric.

- Navigable for $O(\log\log n)$ Doubling Dimension [Slivkins PODC'05]
- Necessary for RBA-type augmentation [Fraigniaud'10]..

(λ, β)-**set system** $\Sigma = \{S_1, \ldots, S_m\}$ with $S_i \subset V$ and $\cup S_i = V$.



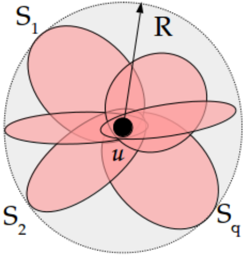
Bounded Growth:
$|S_i| \leq R$ and $u \in S_i \Rightarrow$
$|\cup S_i| \leq \beta R$.



Descent Property:
$\forall S \in \Sigma$ and $\forall u, t \in S \Rightarrow$
$\exists S' \in \Sigma, t \in S', |S'| \geq \lambda |S|$.

Navigability in Set Systems

(λ, β)-**set system** $\Sigma = \{S_1, \ldots, S_m\}$ with $S_i \subset V$ and $\cup S_i = V$.



[Local Connectivity]
- Adding $\Omega(\log^2 n)$ links.

[Geometry]
- $d_\Sigma(u, v) := \min_{S \in \Sigma}\{|S| - 1 | u, v \in S\}$.

[Augmentation]
- RBA with d_Σ semi-metric.

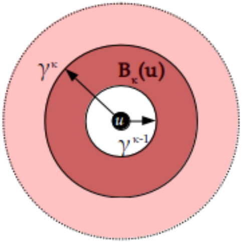
Navigability only for 1-RBA [Kleinberg NIPS'03]

Unified framework for Navigability

Unified Framework for Navigability

Unified Framework

Geometry (V, D): a set V and a semi-metric $d : V \times V \to R_+$.



[Local Connectivity]

- substrate E_0.

[Geometry]

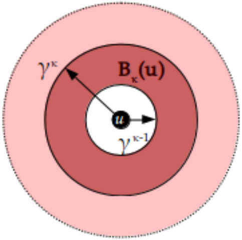
- γ-coherent semi-metric d.

[Augmentation]

- Uniformly Rich product measure.

Coherent Geometries

Geometry (V, D): a set V and a semi-metric $d : V \times V \to R_+$.



- **[Coarsening]** $\gamma > 1$
  $d(u, v) \in I_k := [\gamma^{k-1}, \gamma^k)$
- **[Bounded Growth]**
  $Vol(B_k(u)) \propto \gamma^k, \forall u \in V$
- **[Coherence]**
  $Vol(G(u, v)) \geq \phi \cdot \gamma^k$.

"Bounded" Density Fluctuations for all non-trivial scales.

Example: n random points in $[0, 1]$ are $(1 + \frac{c}{n})$-coherent.

Set Systems are Coherent Geometries

Theorem[Achlioptas, S'15]

For every (λ, β)- **set system** Σ, there exists $\gamma(\lambda, \beta)$ such that (V, d_Σ) is a γ-coherent geometry, where

$$d_\Sigma(u, v) = \min_{S \in \Sigma}\{|S| - 1 | u, v \in S\}$$

Proof: main quantity to control $Vol(B_k(u))$. $B_{[k]}(u) = \cup_{i \leq k} B_i(u)$.

$$\mathrm{Vol}(B_k(u)) = \mathrm{Vol}(B_{[k]}(u)) - \mathrm{Vol}(B_{[k-1]}(u))$$

- **Upper bound:** easy follows from bounded growth.
- **Lower bound:** uses the descent property.
  a. $\exists S \in \Sigma$ such that $u \in \Sigma$ and $|S| \in [\gamma^{k-1}, \gamma^k]$ for each k.
  b. coarsening of geometry.

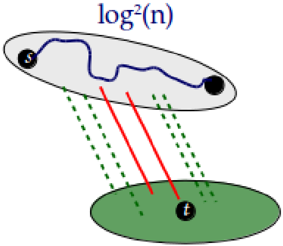
Uniform Richness

A product measure $Q \in [0,1]^{n \times n}$ such that: $(u,v) \overset{ind}{\sim} \mathrm{Ber}(Q_{ij})$, is called uniformly rich $d(u,v) \in [\gamma^{k-1}, \gamma^k) \Rightarrow Q(u,v) \geq \frac{c}{\log^\theta n}\gamma^k$.

Theorem[Achlioptas, S'15]

Coherent geometry (V,d) with substrate E_0, let $E_Q \sim Ber(\mathbf{Q})$ where Q a uniformly rich product measure, then

$$G(V, E_0 \cup E_Q) \text{ is navigable.}$$



Proof sketch:

- Substrate $\Rightarrow \exists$ local path $P(s,t) \subset E_0$.
- Coherence $\Rightarrow$ enough good pairs.
- Uniform Richness $\Rightarrow \exists$ good edge w.h.p.

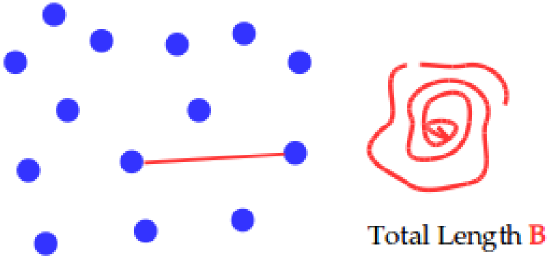
Robustness of Navigability

Robustness of Navigability

Questions about Navigability

- **Independence:** where does it come from? Is it necessary?
- **Evolvability:** of RBA or Uniform richness.
- **Robustness:** Kleinberg showed unique exponent for augmentation through RBA.

Graph G uniformly at random such that total length being $\leq B$?



Graphs of Bounded Cost

Agnostic approach: uniformly at random subject to constraints.
Cost-geometry $\Gamma = (V, d, c)$, with $c : V \times V \to R_+$ symmetric.

$$G_\Gamma(B) = \left\{ E \subset V \times V \middle| \sum_{e \in E} c_e \leq B \right\}$$

- c= 1: essentially $G(n, m)$ with $m = B$ edges.
- c = 0: essentially $G(n, 1/2)$ uniform at random.
- $c = \begin{cases} 1 & d = 1 \\ \infty & d > 1 \end{cases}$ nearest neighbor percolation

No independence assumption, geometry drives edge formation!

Product measure approximation

Theorem[Achlioptas, S ICALP'15]

For all sufficiently symmetric sets S of graphs, $\exists$ product measures $\mathbf{Q}^{\pm}$ such that if $G \sim U(S)$, $G^{\pm} \sim \mathrm{Ber}(\mathbf{Q}^{\pm})$ then w.h.p.

$$G_{-} \subseteq G \subseteq G_{+}$$

Intuition:

- Uniform measure $\Rightarrow$ counting.
- Symmetry $\Rightarrow$ counting possible.
- Concentration around **MaxEnt** solution (product measure).

For Bounded cost graphs the product measure has an explicit form!

$$Q_{ij} = \frac{1}{1 + \exp(\lambda(B) c_{ij})}$$

Inevitabilty

A cost function c is γ-consistent if it is constant in $[\gamma^{k-1}, \gamma^k)$.

Theorem[Achlioptas, S'15]

A $\Gamma = (V, d, c)$ coherent cost geometry:

- Navigability: for every $B \geq B_+(\Gamma)$ almost all elements of $G_\Gamma(B)$ are navigable.
- Sparsity: for every $B_0 \leq B \leq B^-$ almost all elements of $G_\Gamma(B)$ have at most poly-log density.

Proof:

- Coupling with product measure.
- Proof for product measure.
- Use **monotonicity** and coupling to translate back to uniform measure.

RBA through Indexing

Corollary [Achlioptas, S '15]

Let $\bar{B}_k = \frac{|\cup_{u \in V} B_k(u)|}{n}$, and $c = \frac{1}{\beta} \log \bar{B}_k$, for $B_-(\Gamma) \le B \le B_+(\Gamma)$:

1. Navigability with poly-log density.
2. Robustness: $B_+/B^- = \Omega(\log^\theta n)$.
3. RBA for B^*, $P(u,v) \propto \frac{1}{\mathrm{Vol}(B_k(u))}$.

Kleinberg's Phase transition $\Rightarrow$ artifact of λ parametrization.

Rank Based Augmentation $\Rightarrow$ cost of indexing.

Optimal adaptation of technology (cost) to geometry!

Historical Emergence of Navigability

Technology: cost function c (cost communicating at distance d).
Economic activity: total budget B
Consider the cost function (King-Plosser-Rebelo preferences):

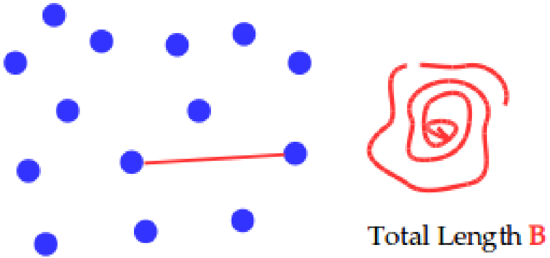
$$c_k(\beta) := \begin{cases} \frac{\bar{B}_k^{\beta}-1}{\beta}, & \beta > 0 \\ \log \bar{B}_k, & \beta = 0 \end{cases}$$

Navigability and Sparsity for $\beta = 0 \Rightarrow$ **Technological breakthrough**!

Summary

- Why? ⇒ Coherent Geometry
- How? ⇒ Uniform richness
- When?⇒ Technology becomes as good as Indexing!

Robustness: almost all feasible graphs are Navigable!



Open questions

Inhomogeneous random graph: generate a sample in time proportional to the density?

Questions?

Thank You!

psiminelakis@gmail.com

"How can i link, with three, four, or at most five links of the chain, trivial, everyday things of life. How can I link one phenomenon to another? How can i join the relative and the ephemeral with steady, permanent things - how can I tie up the part with the whole"

Frigyes Karinthy 1929, *Chains*.