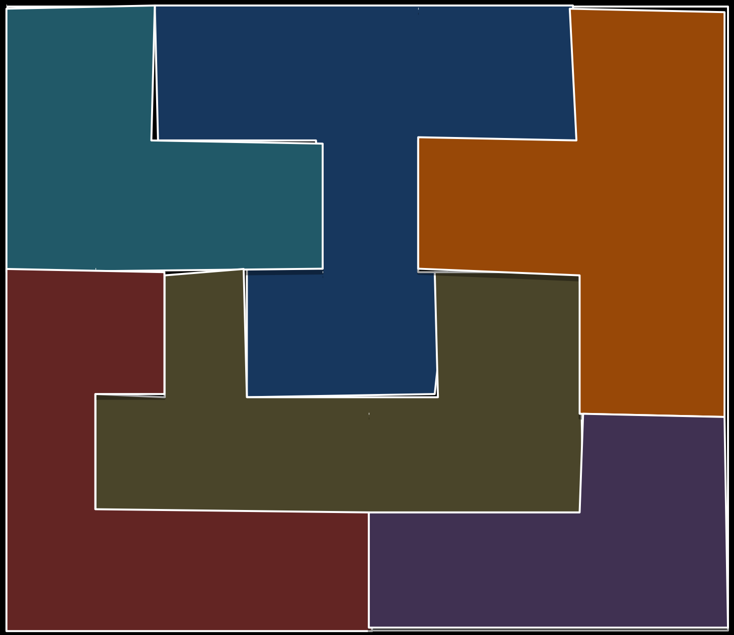
Symmetric Graph Properties Have Independent Edges



Paris Siminelakis

PhD Student, Stanford EE

Joint work with
Dimitris Achlioptas,
UCSC CS

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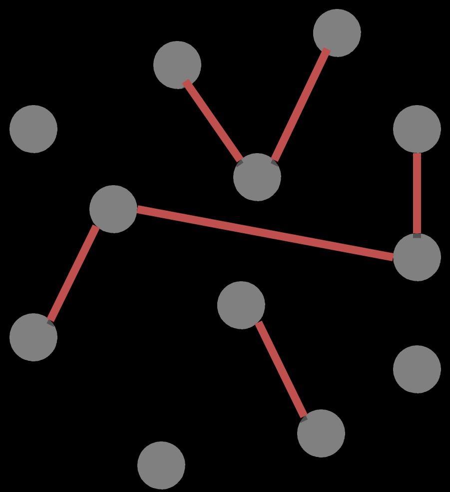
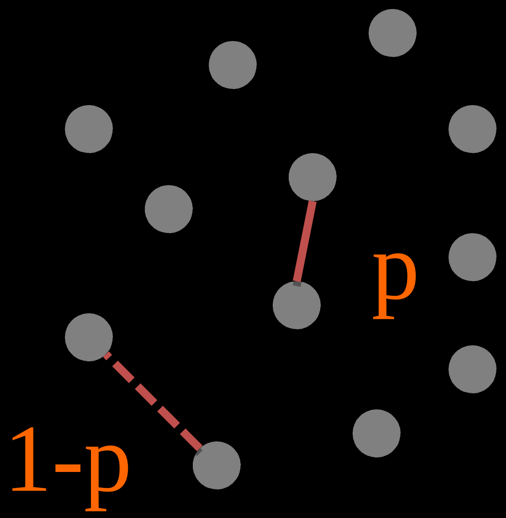
Random Graphs

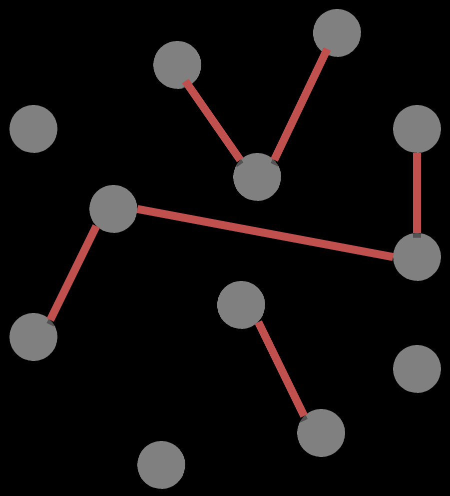


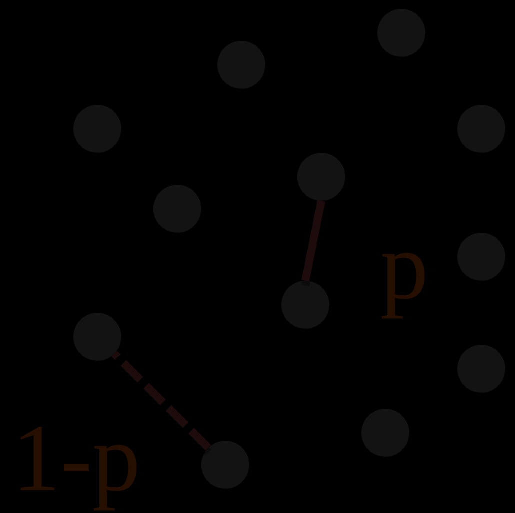
G(n,m)



G(n,p)

Random Graphs

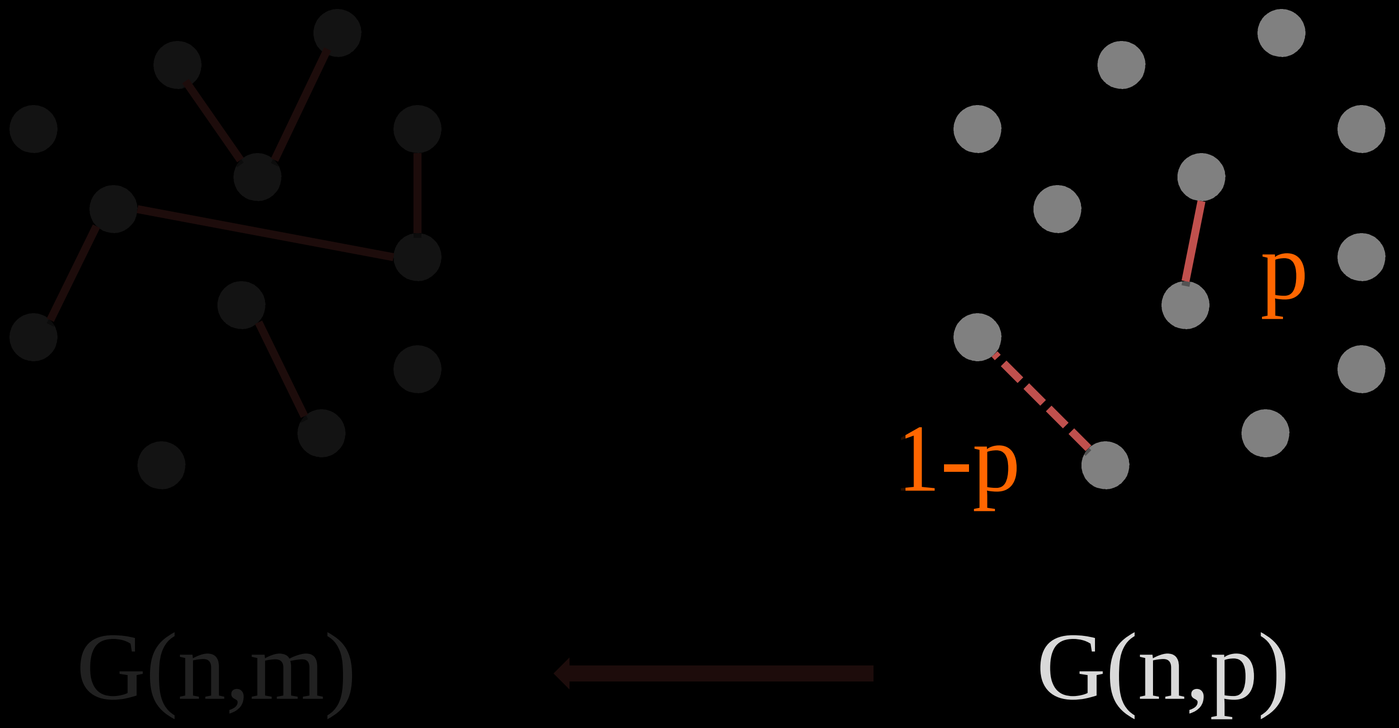




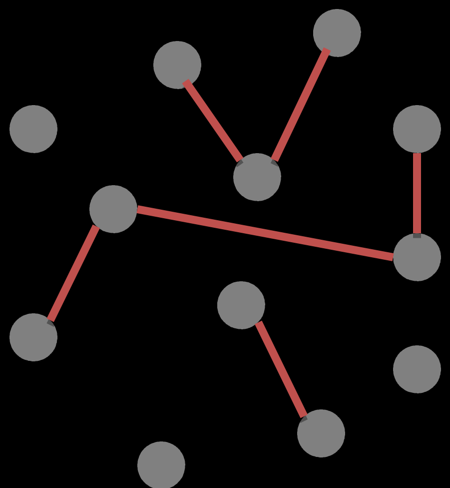
G(n,m)

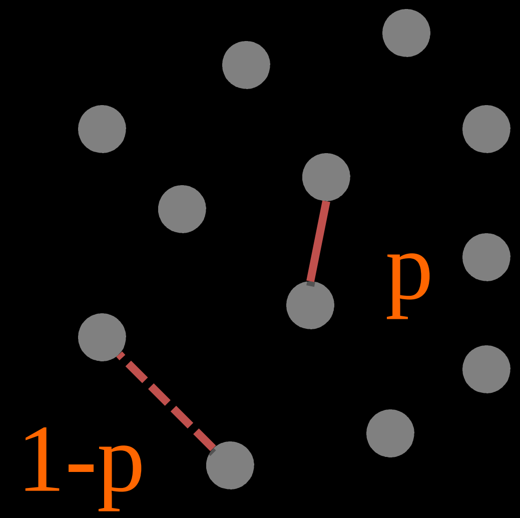
G(n,p)

Random Graphs



Random Graphs





G(n,m) ← G(n,p)

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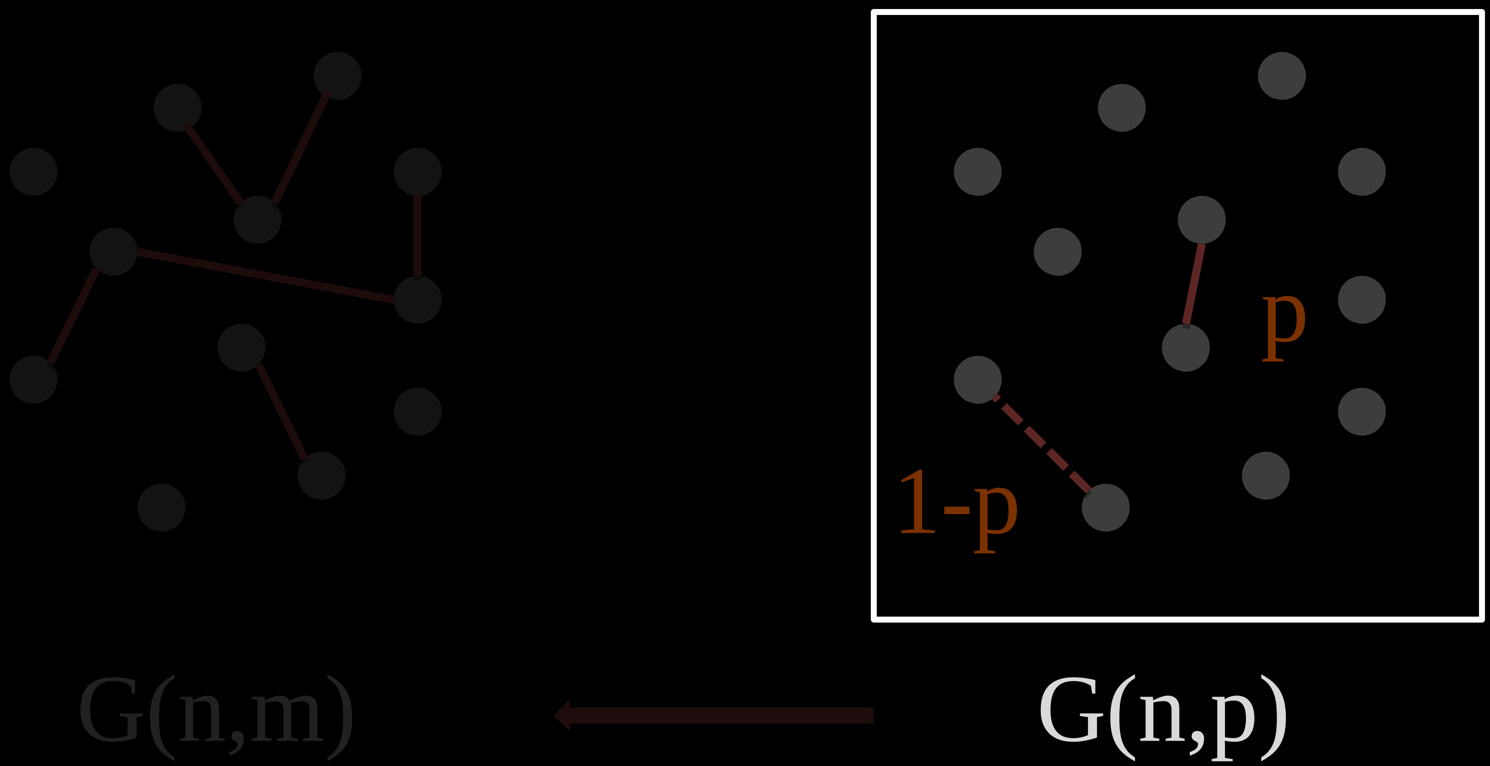
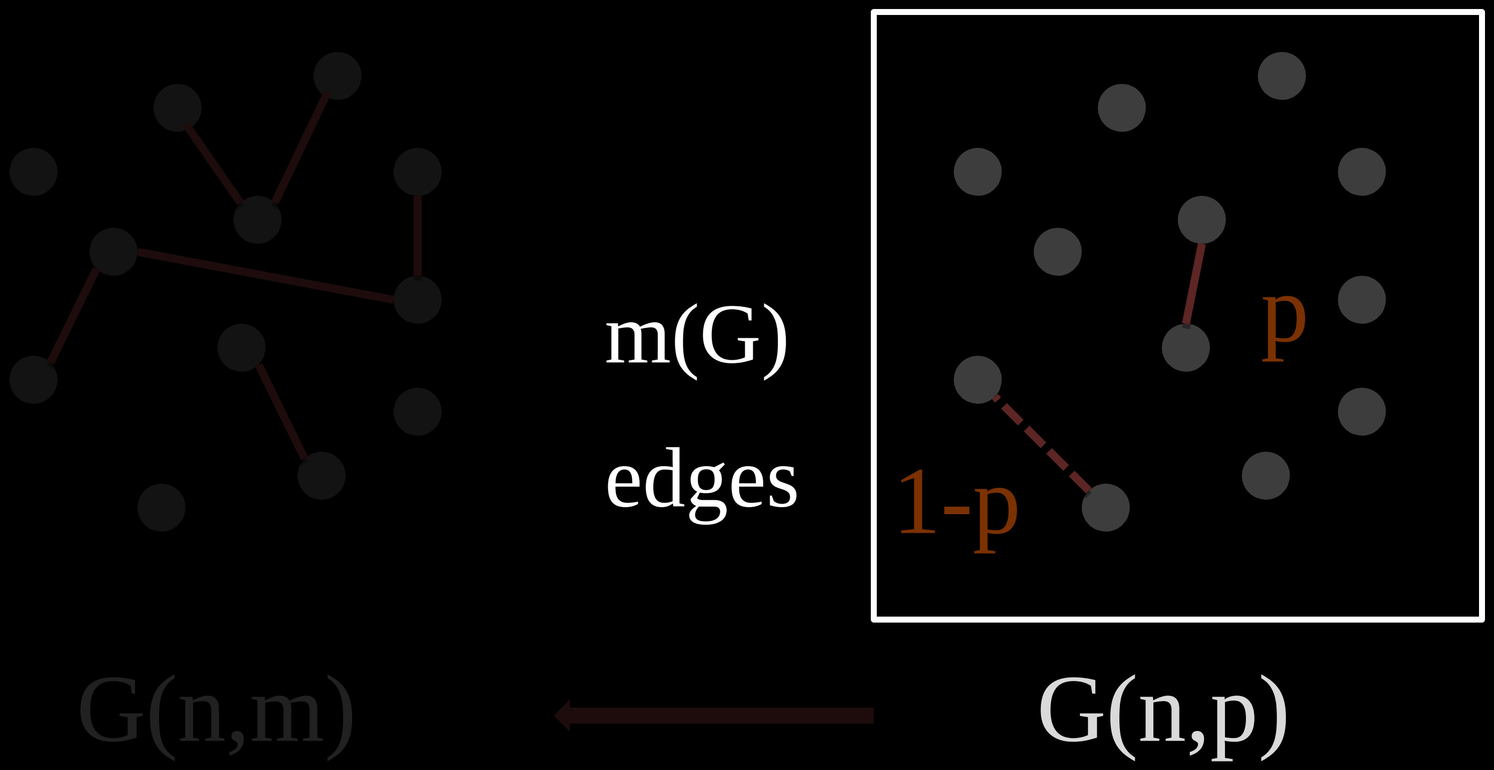
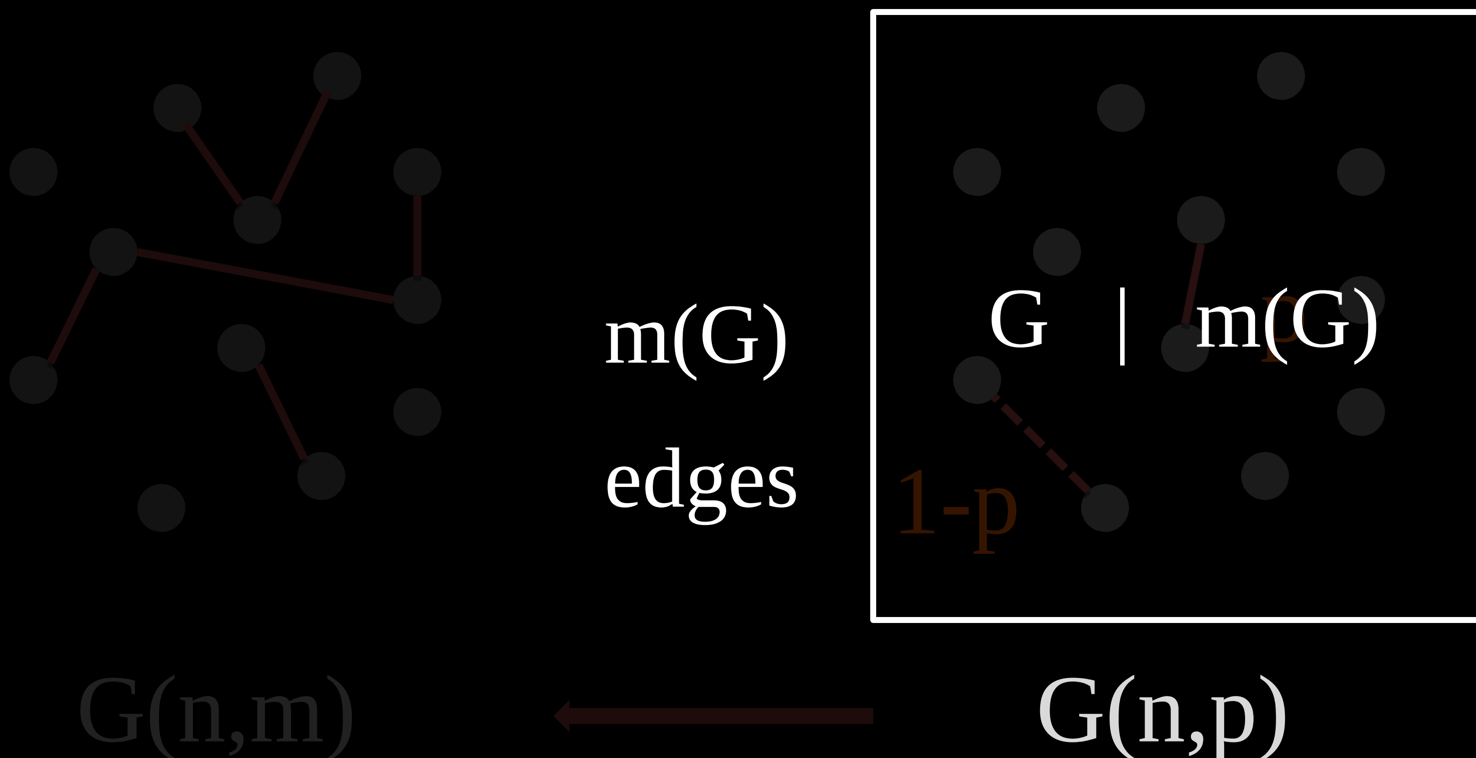
Coupling as Simulation



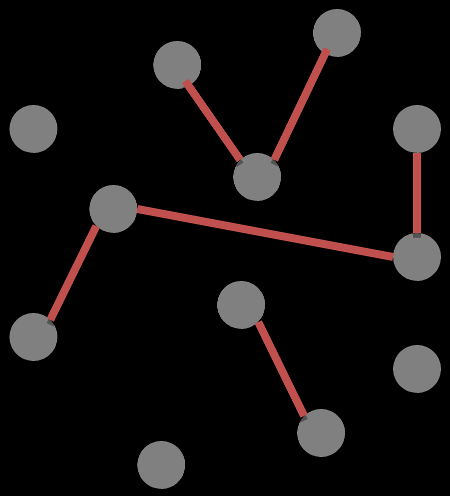
Coupling as Simulation



Coupling as Simulation

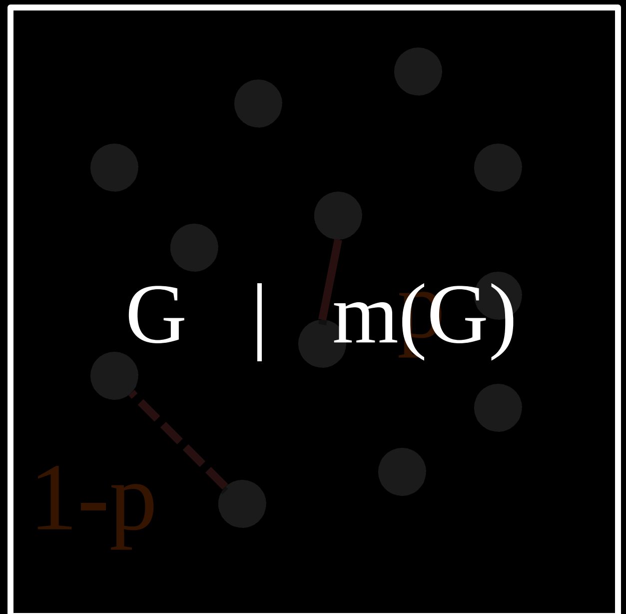


Coupling as Simulation



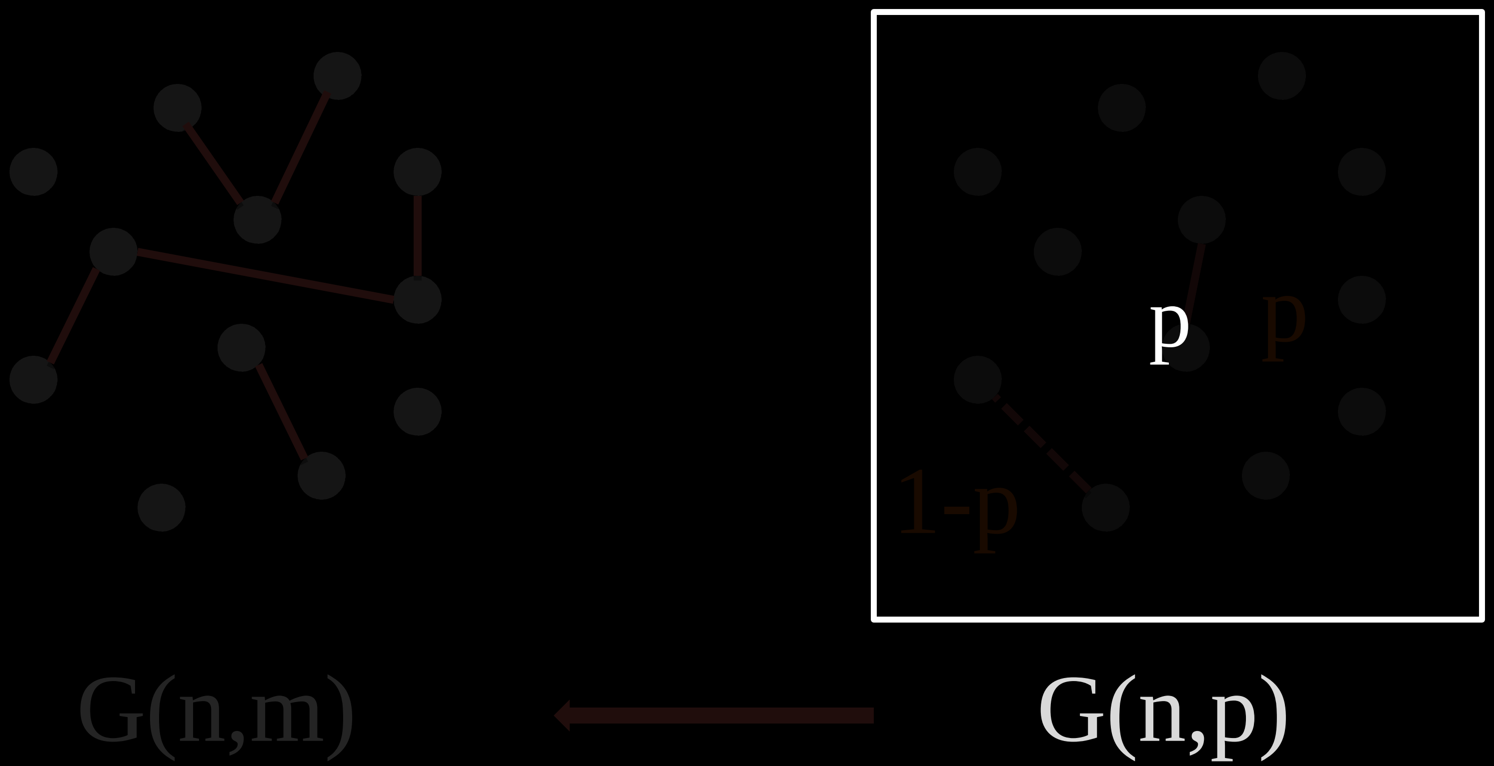
m(G)

edges

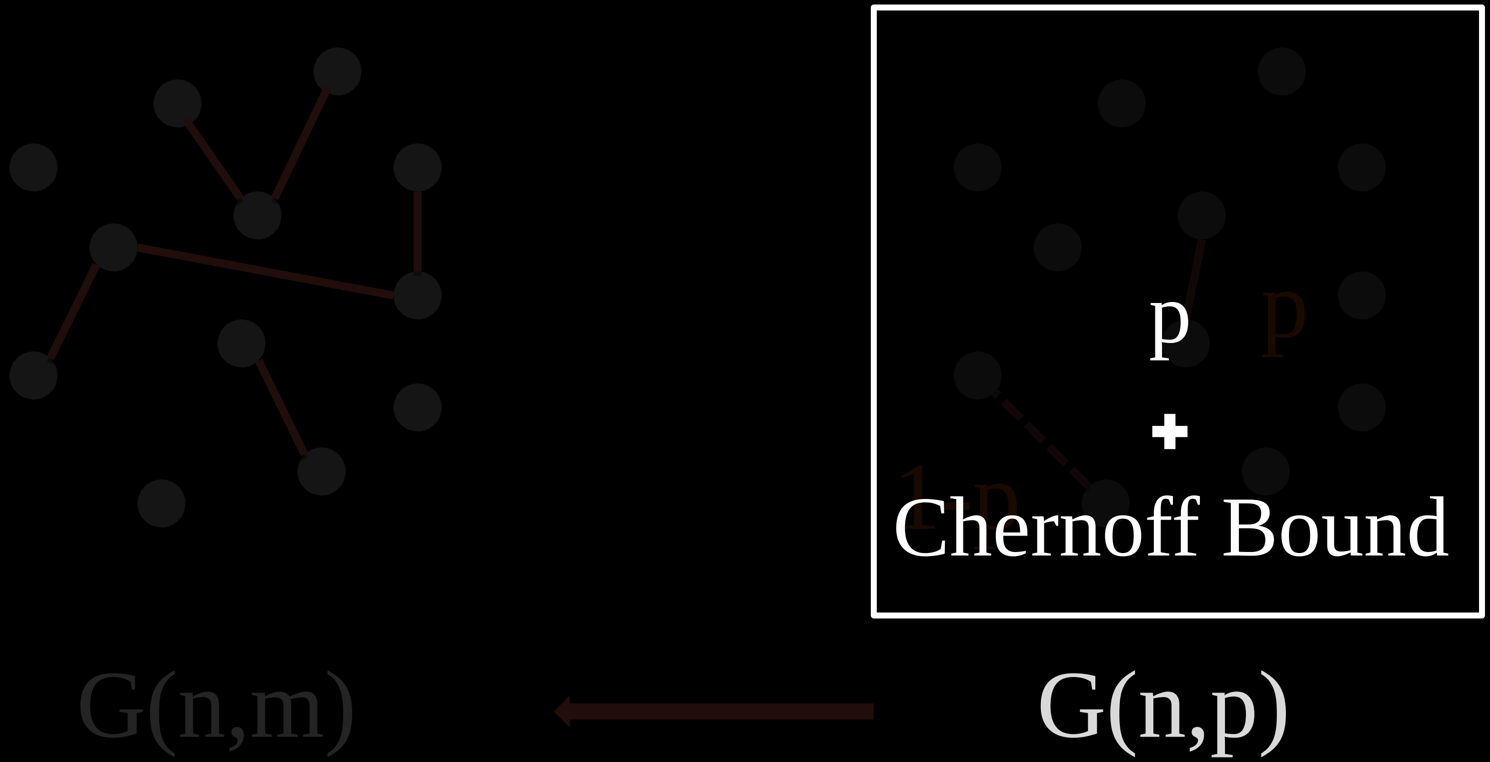


G(n,m) ← G(n,p)

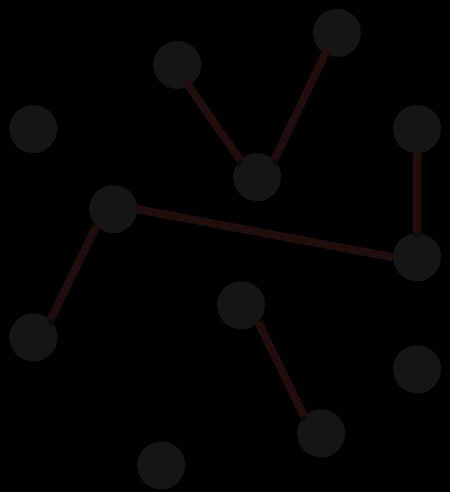
Coupling as Simulation



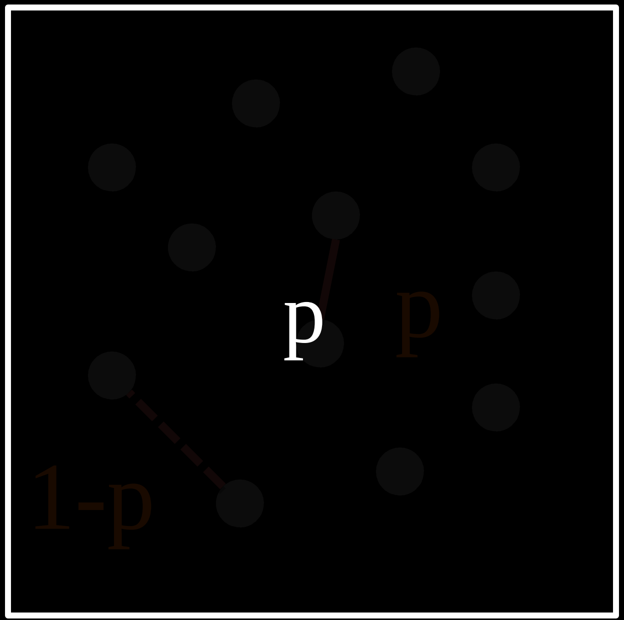
Coupling as Simulation



Concentration

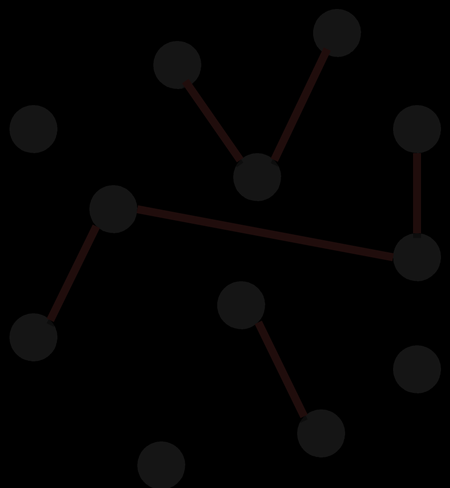


$$m(G) \approx (1\pm\varepsilon)\, n^2 p/2$$



G(n,m) ← G(n,p)

Concentration



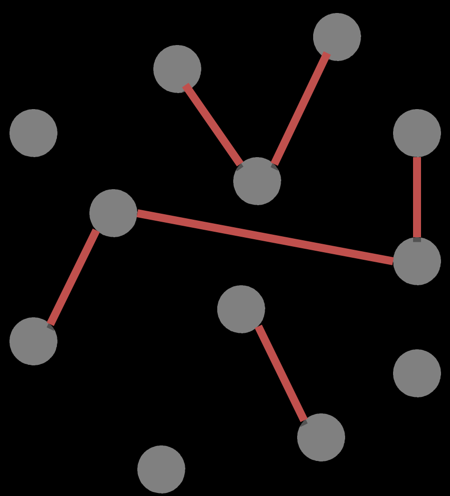
$m(G) \approx (1\pm\varepsilon)\, n^2p/2$

$G \mid m(G)$

1-p

G(n,m) ← G(n,p)

Concentration + Conditioning



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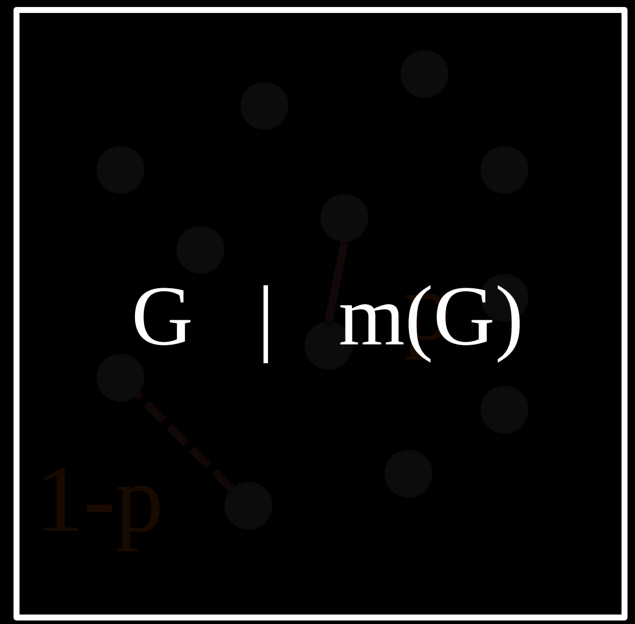
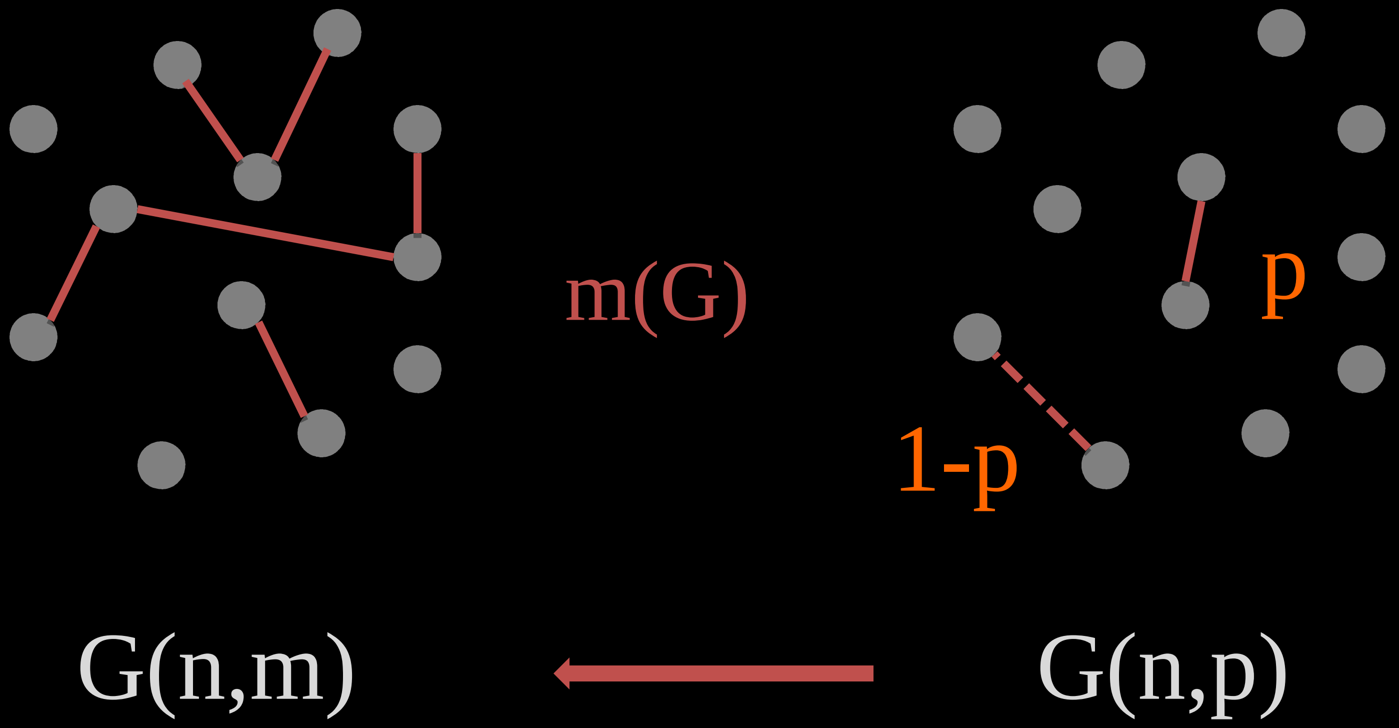
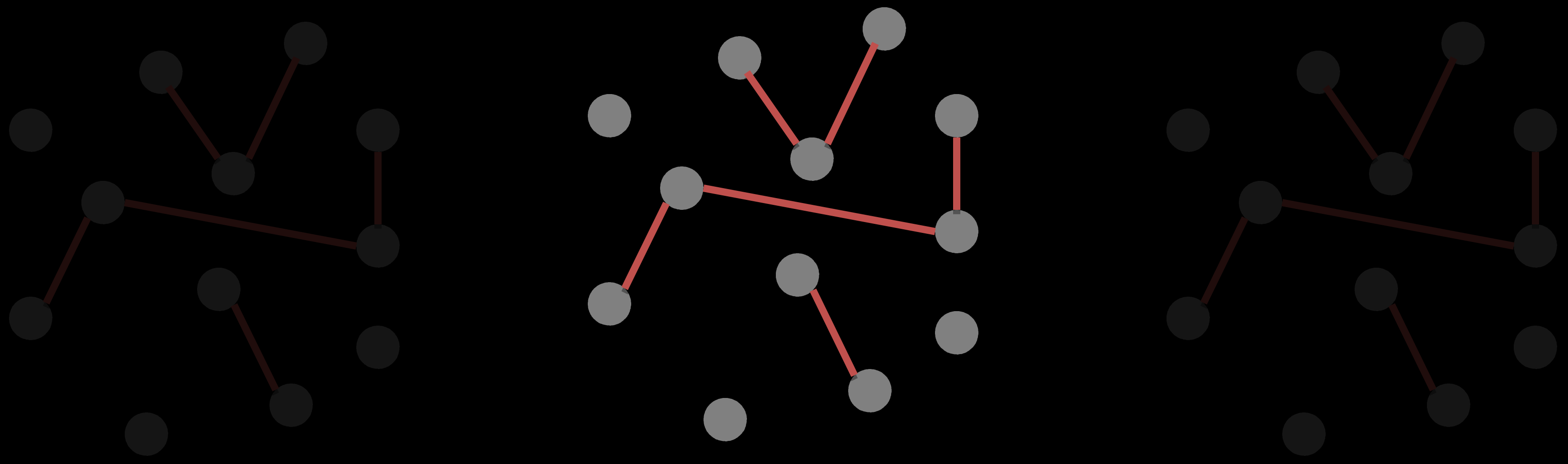


G(n,m) ← G(n,p)

Concentration + Conditioning → Coupling

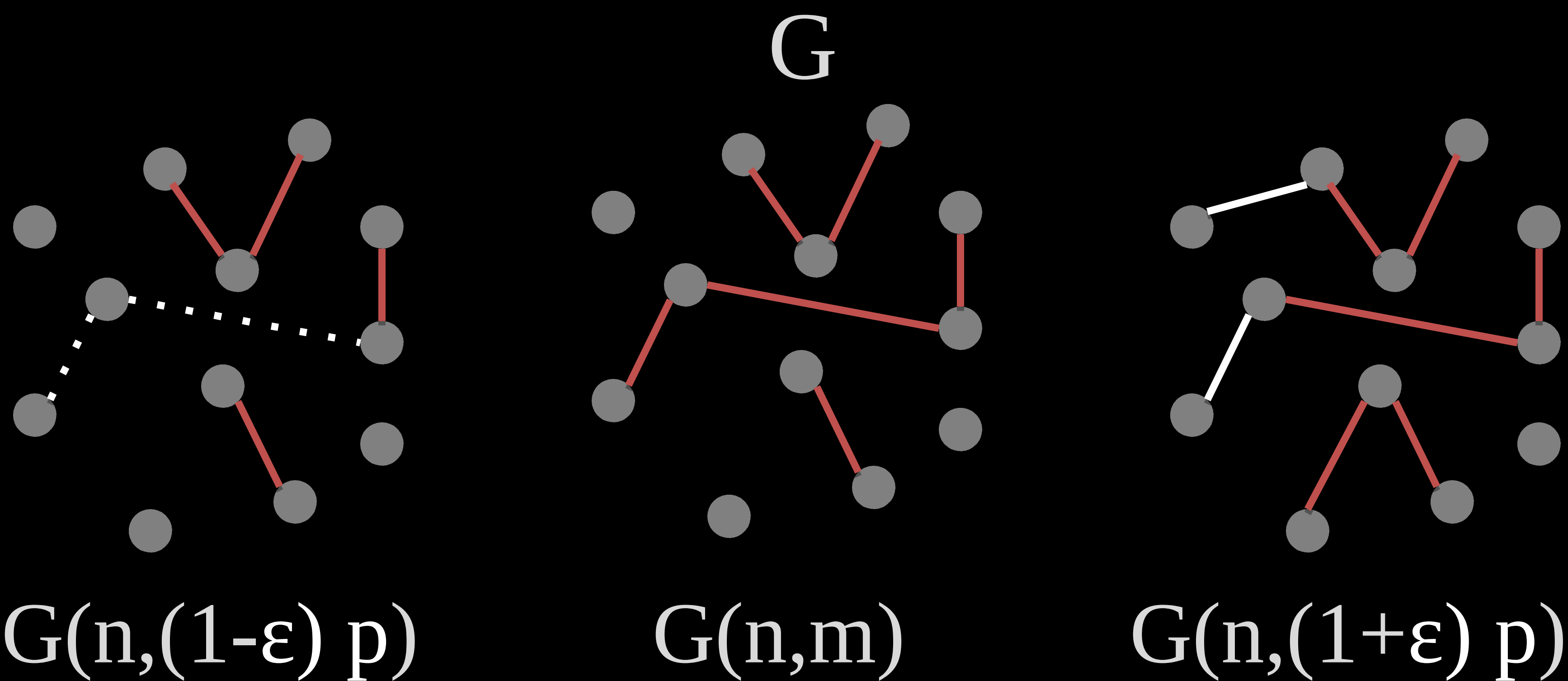


Coupling G(n,m) by G(n,p) with $p(m) = 2m / n^2$

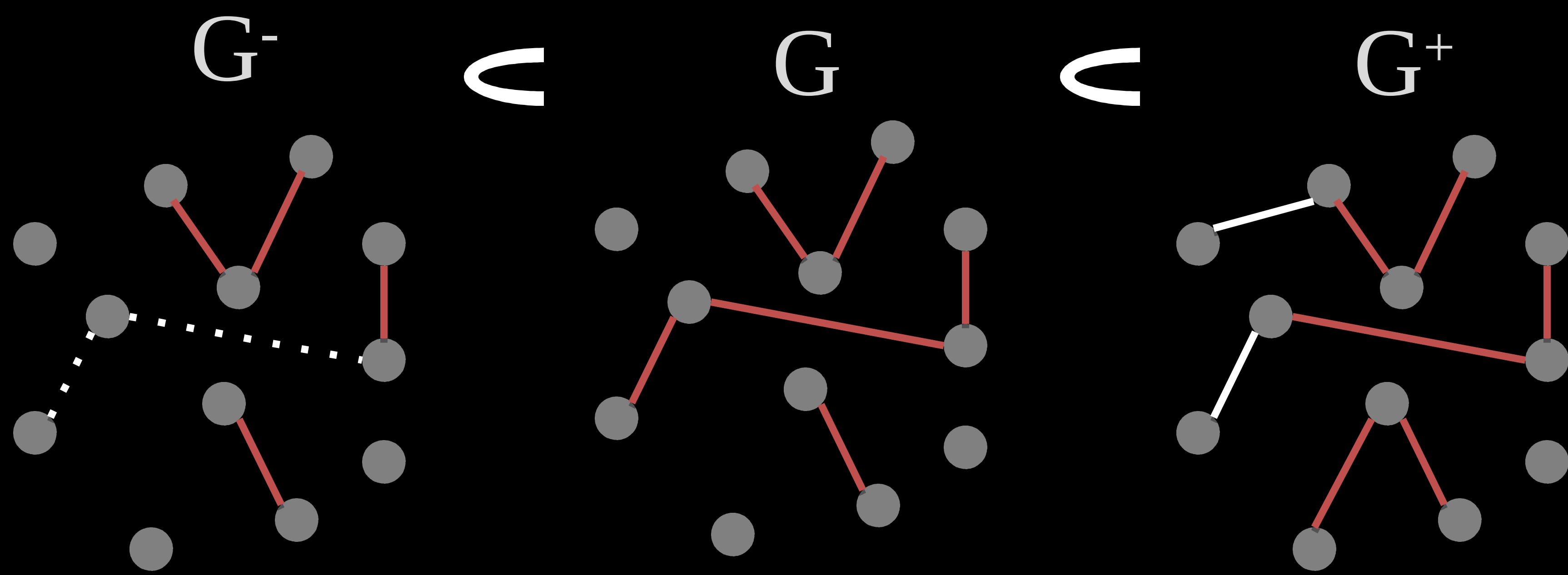


G(n,(1-ε) p) G(n,m) G(n,(1+ε) p)

Coupling G(n,m) by G(n,p) with $p(m) = 2m/n^2$



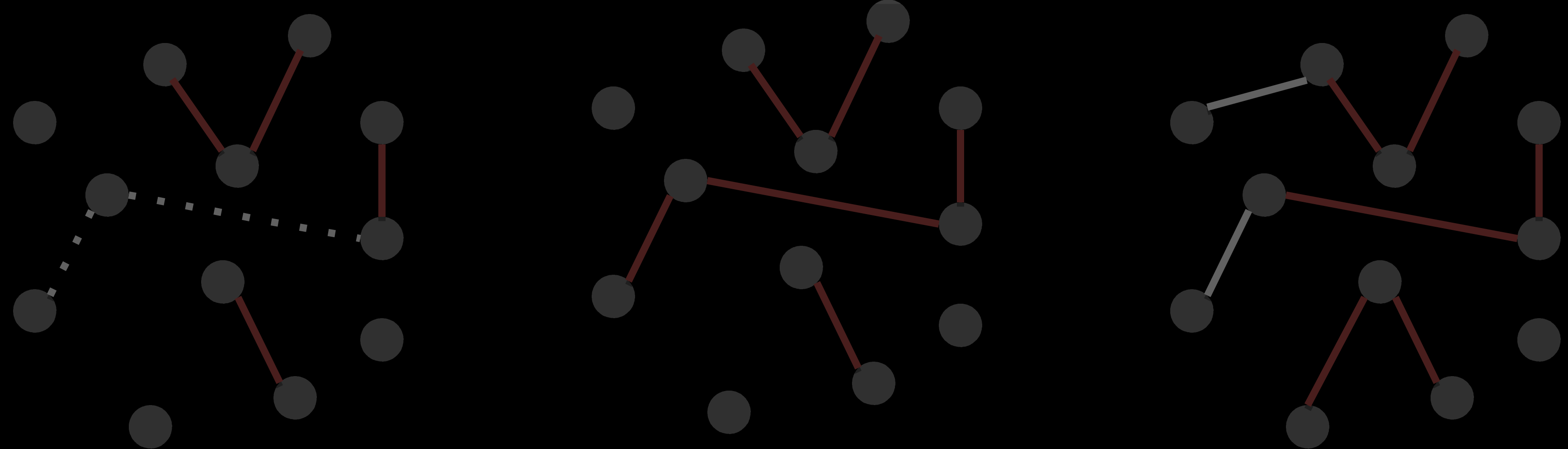
Coupling G(n,m) by G(n,p) with $p(m) = 2m/n^2$



$G(n,(1-\varepsilon)\,p)$ $G(n,m)$ $G(n,(1+\varepsilon)\,p)$

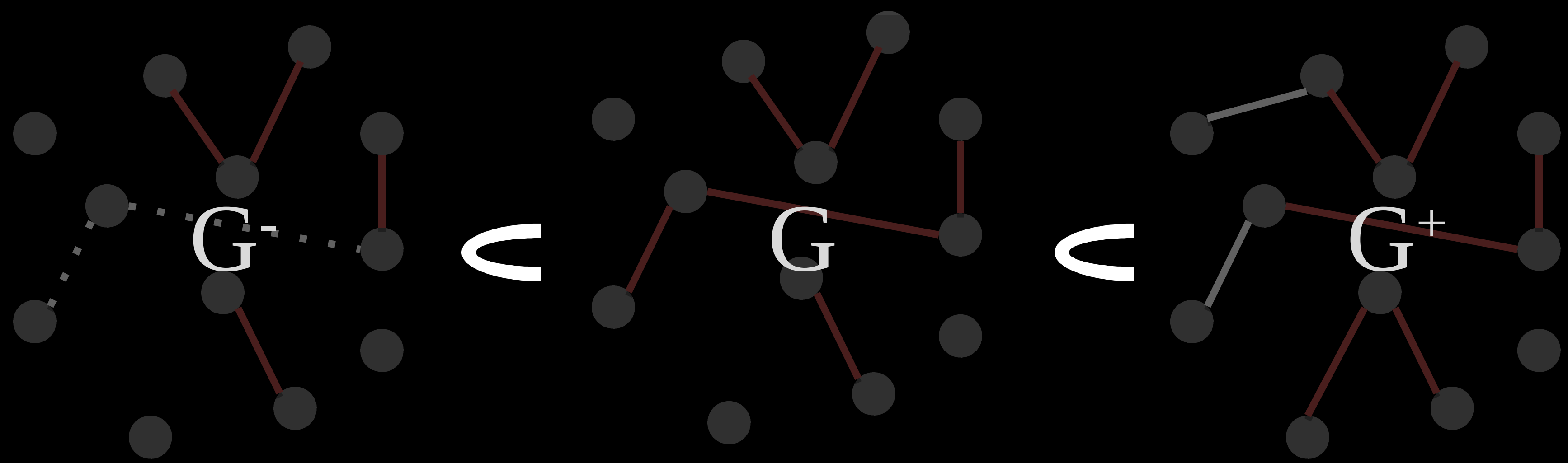
Sanwiching G(n,m) by G(n,p)

$$G^- \subset G \subset G^+$$



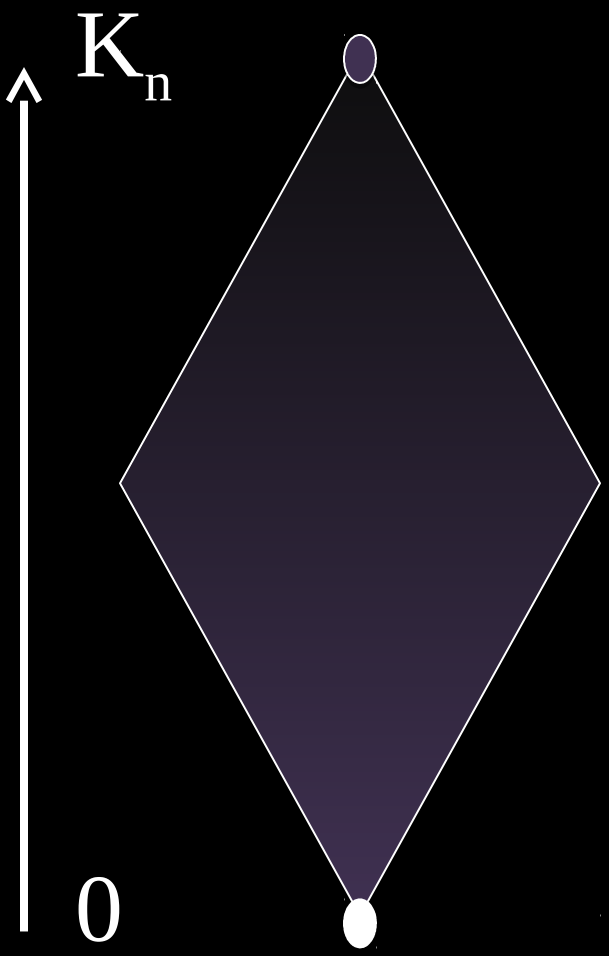
$$G(n,(1-\varepsilon)\,p) \qquad G(n,m) \qquad G(n,(1+\varepsilon)\,p)$$

Sandwichability of G(n,m)



$G(n,(1-\varepsilon)\,p)$ $\quad$ $G(n,m)$ $\quad$ $G(n,(1+\varepsilon)\,p)$

Sandwichability of G(n,m)



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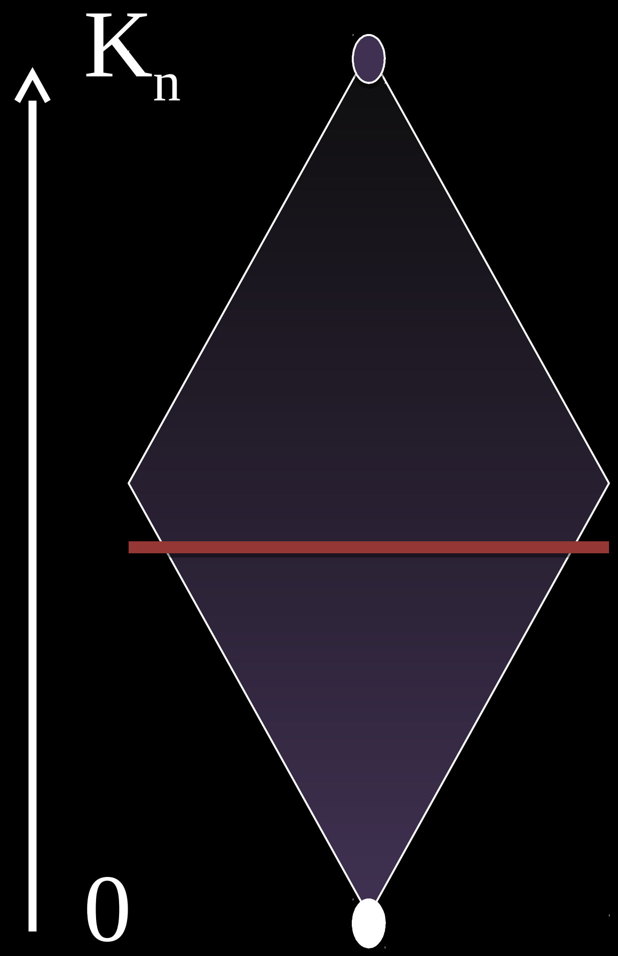
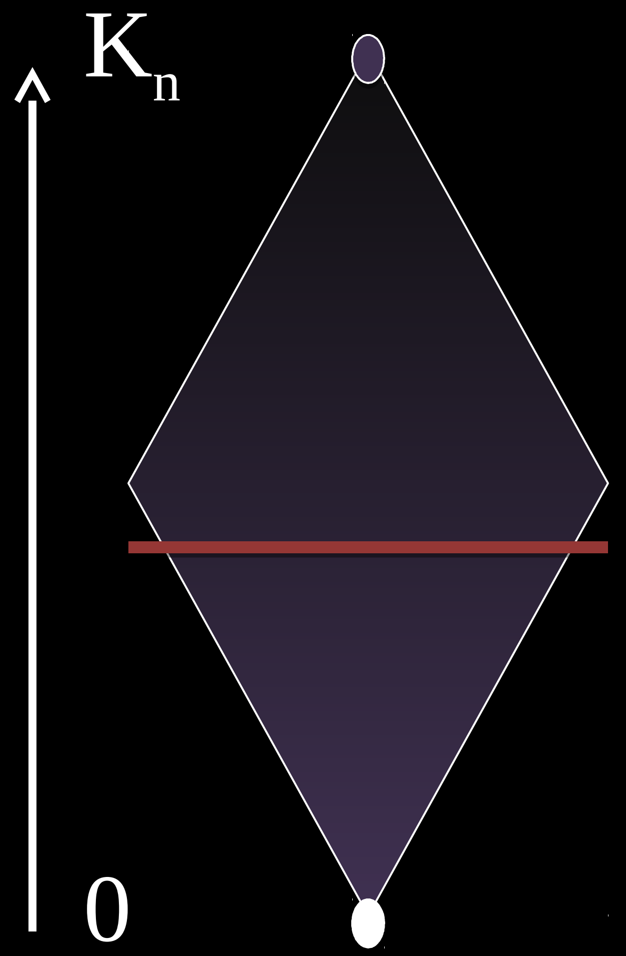


G(n,m)

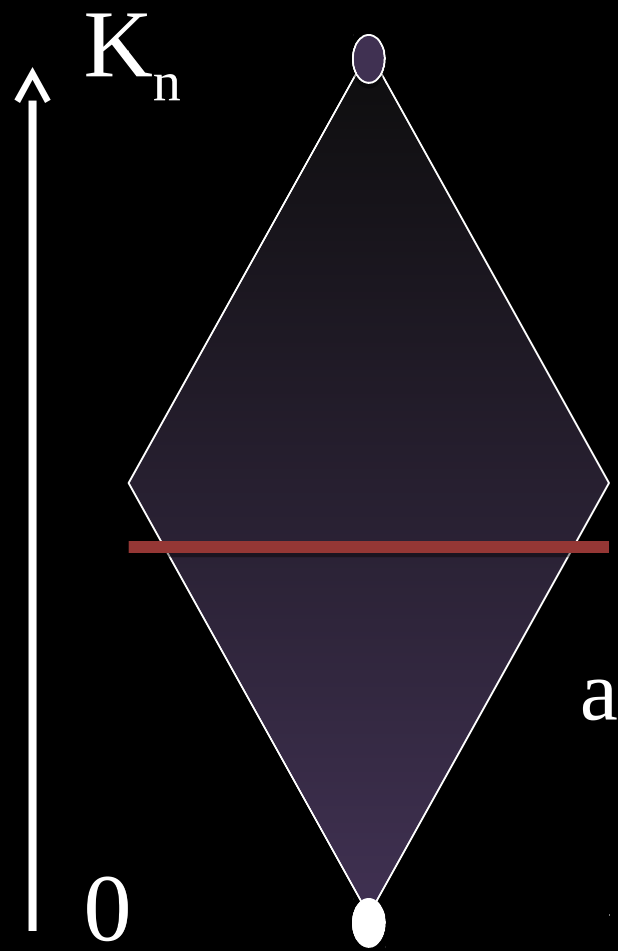
Sandwichability of G(n,m)



Uniform Measure

S={ G: |E(G)| = m}

Sandwichability of G(n,m)



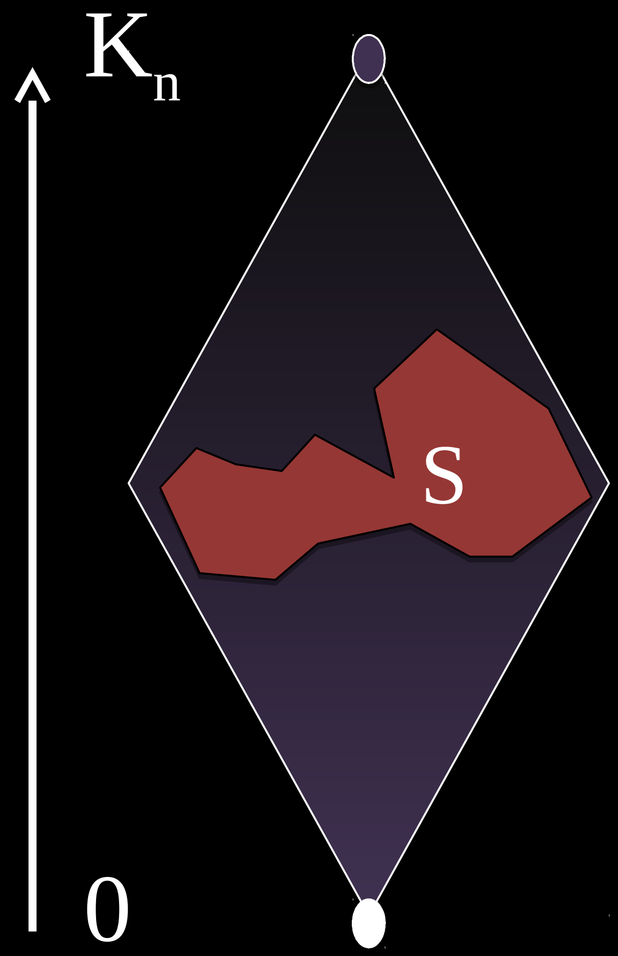
Uniform Measure

$S = \{ G : |E(G)| = m \}$

approximated by

G(n,p)

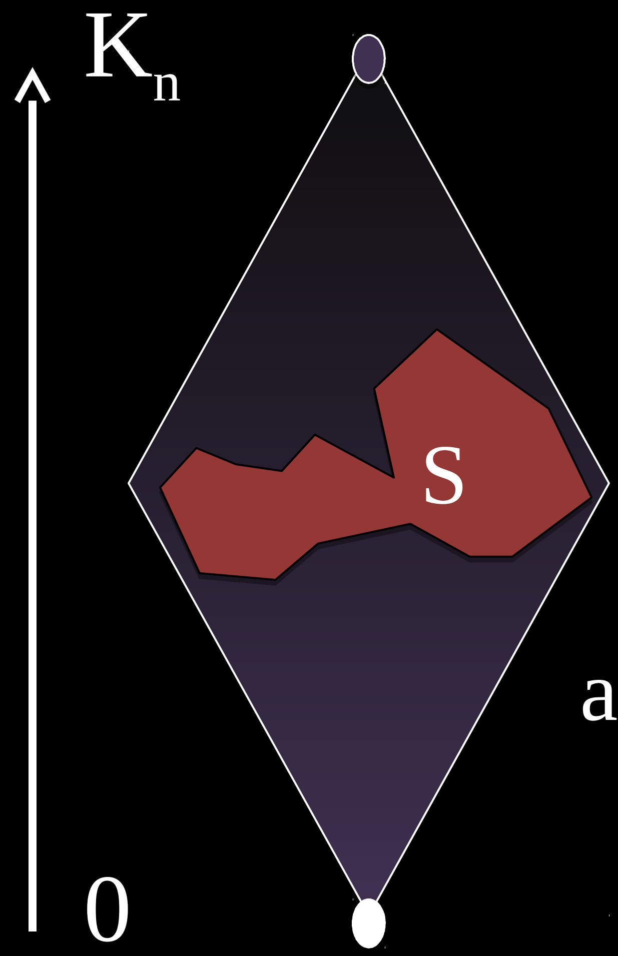
Sandwichability of Graph Properties



Uniform Measure

S subset of graphs

Sandwichability of Graph Properties



Uniform Measure

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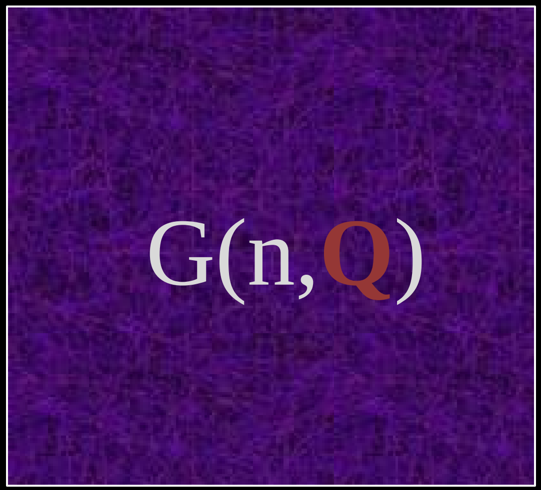
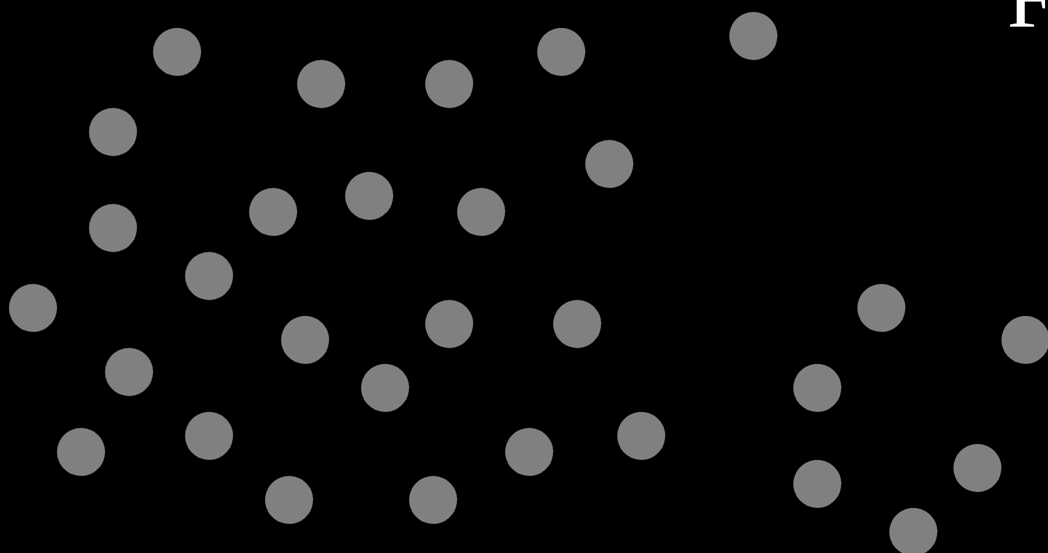


Motivating Example [D.Achlioptas]

n arbitrary points

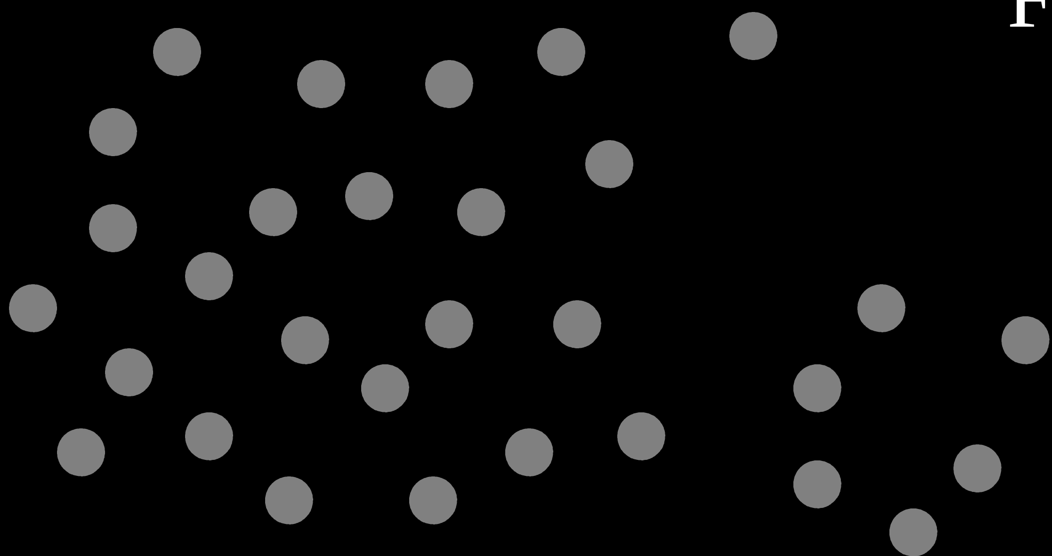


Finite spool of wire B

Motivating Example [D.Achlioptas]

Random Graph with given wire?

n arbitrary points



Finite spool of wire B

Motivating Example [D.Achlioptas]

Random Graph with given wire?

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Finite spool of wire B

$$P(r) = \frac{1}{1 + \exp(\lambda \times r)}$$

Our Approach

Random Graphs subject to Constraints

Geometric Analogues of Erdos-Renyi Graphs

No independence assumptions

Maximally Agnostic aka Uniform Measure

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Random Graphs subject to Constraints

Geometric Analogues of Erdos-Renyi Graphs

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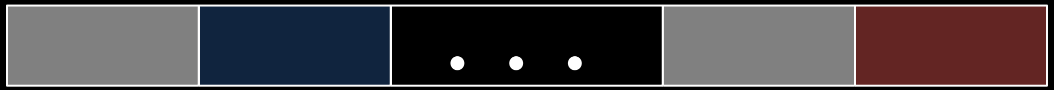
Tools to Analyze such objects as if Edges were Independent!!

Partition Symmetry

$P = \{P_1, . . . , P_k\}$ of all edges



G in $\{0,1\}^{|E|}$

Edge Profile of **x**

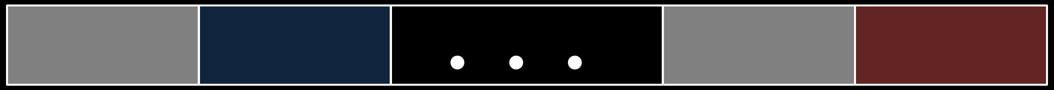
$\mathbf{m}(\mathbf{x}) = (m_1(\mathbf{x}), . . . , m_k(\mathbf{x}))$

$m_i(\mathbf{x}) = | \mathbf{x} \wedge I_{P_i} |$

$m_i(G) = | E(G) \wedge P_i |$

Partition Symmetry

$\mathcal{P} = \{P_1, \ldots, P_k\}$ of all edges



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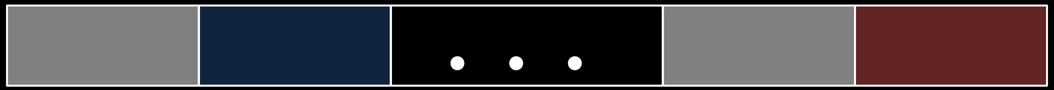

G in $\{0,1\}^{|E|}$

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$\mathbf{m}(G) = (m_1(G), \ldots, m_k(G))$

S={ G: **m**(G) in C }

$m_i(G) = | E(G) \wedge P_i |$

Definition: *P*-symmetric set

Partition Symmetry

S={ G: **m**(G) in C }

Partition Symmetry

$$S=\{\ G:\ \mathbf{m}(G)\ \text{in}\ C\ \}$$

Full symmetry
k=1

G(n,m)

Partition Symmetry

S={ G: **m**(G) in C }

Full symmetry
k=1

C = {m}

Partition Symmetry

$$S=\{\ G: \mathbf{m}(G) \text{ in } C\ \}$$

Full symmetry
k=1

C=[a , b]

Partition Symmetry

$$S=\{\ G:\ \mathbf{m}(G)\ \text{in}\ C\ \}$$

Full symmetry	No symmetry
$k=1$	$k= n^2 / 2$
$G(n,m)$	arbitrary S

Stochastic Block Model

p_{11}	p_{12}	p_{13}
p_{21}	p_{22}	p_{23}
p_{31}	p_{32}	p_{33}

Vertex partition $V=\{V_1, . . . , V_q\}$

$$k = q^2 / 2$$

Szemeredi Regularity Lemma

Approximation in Cut norm

Graph sequence G_0 , G_1 , … , G_T , …

SBM sequence SBM_0 , SBM_1 , … , SBM_T , …

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Stochastic Block Model

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Graph Limits [LS’06]

SBM sequence SBM_0 , SBM_1 , … , SBM_T , …

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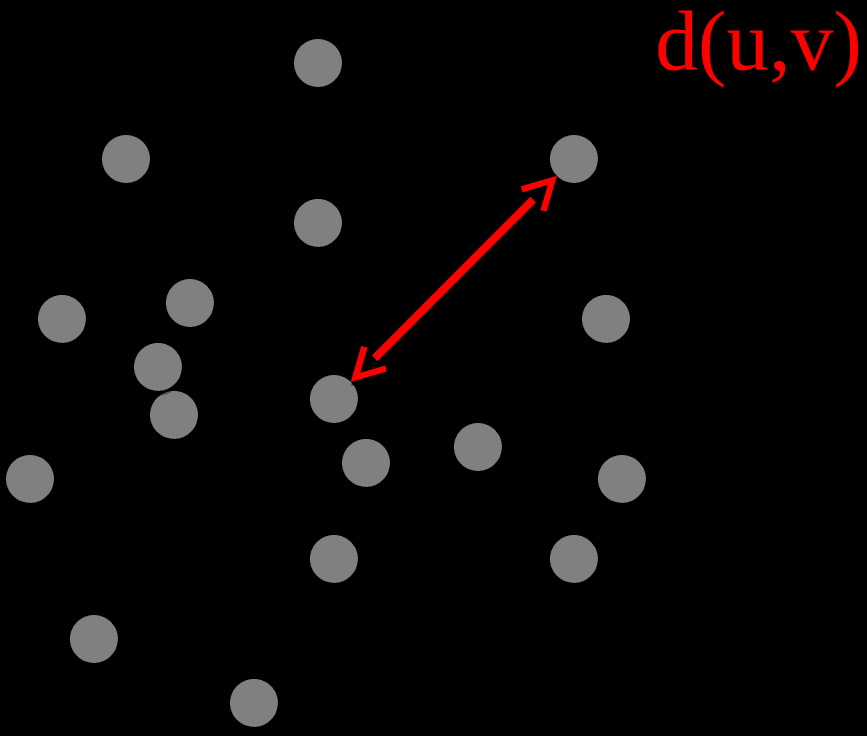
Graph Limits [LS'06]

Geometric Partitions

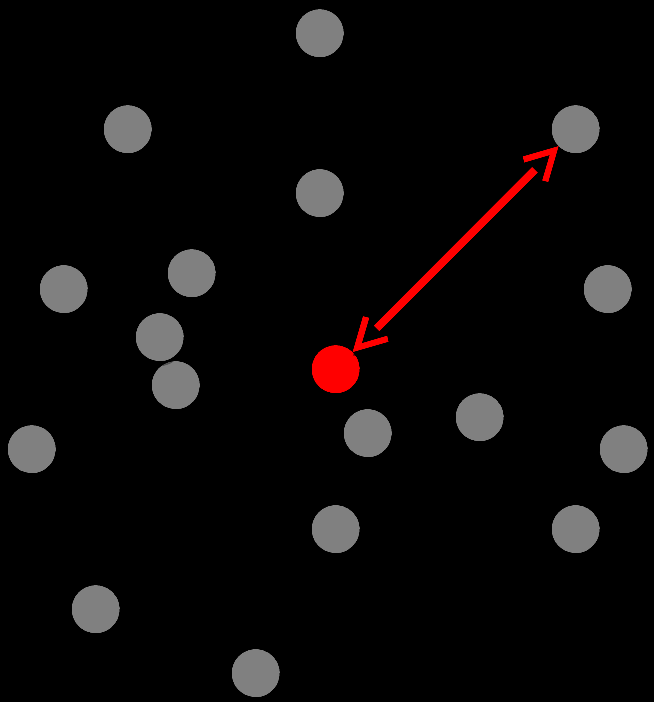
Distance function



Geometric Partitions

Distance function

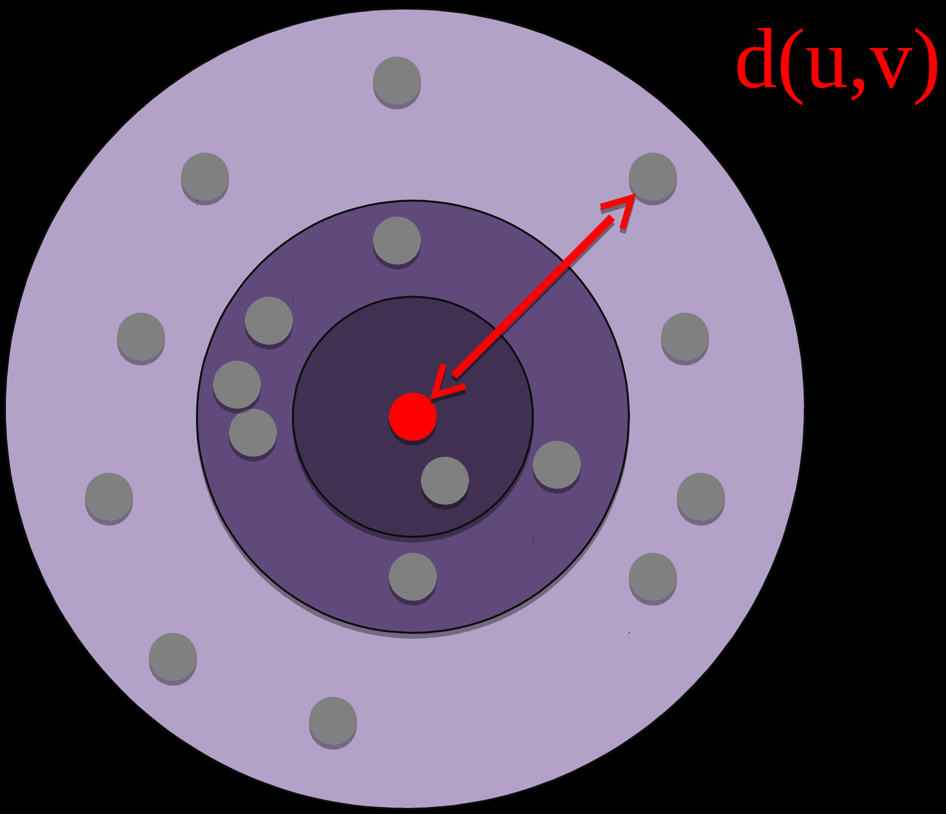
d(u,v)



$$R_i = [\ (1+\delta)^{i-1}, (1+\delta)^{i}\ )$$

Geometric Partitions

Distance function



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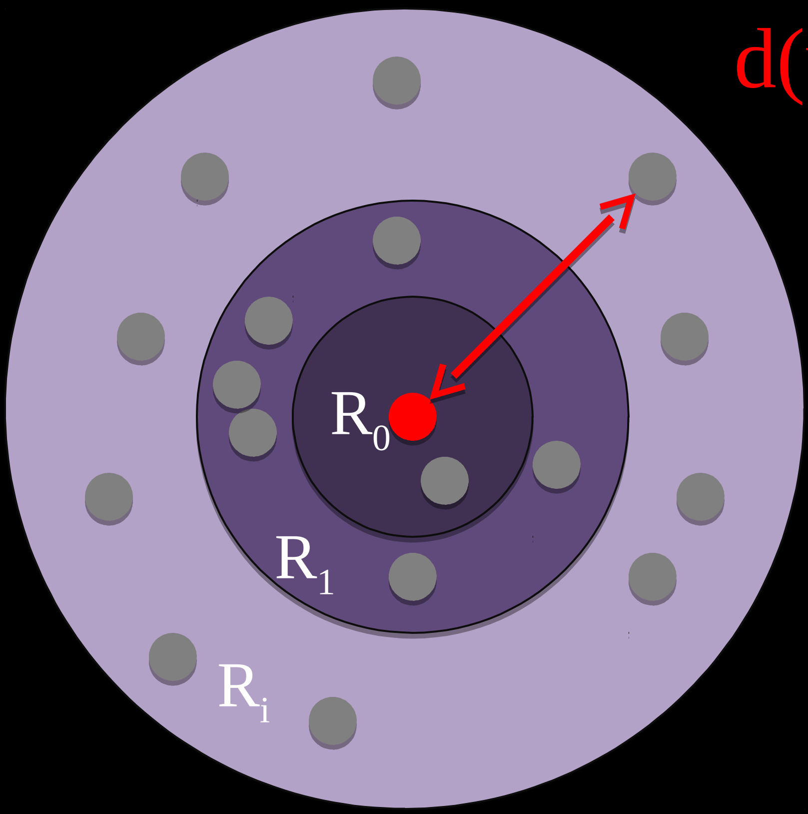

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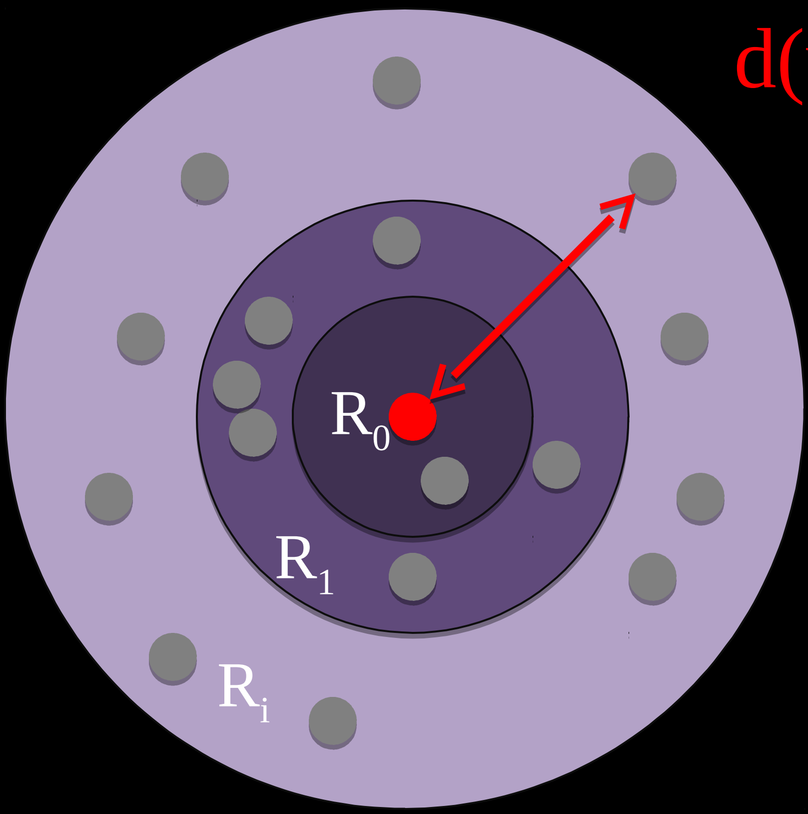

$d(u,v)$

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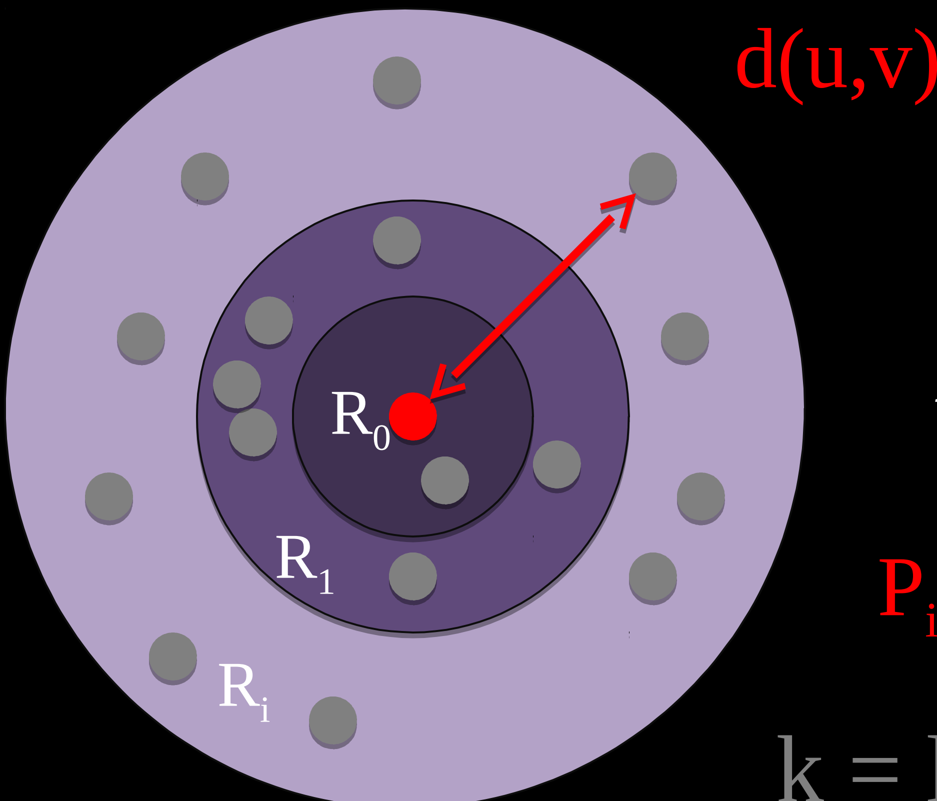
Geometric Partitions

Distance function



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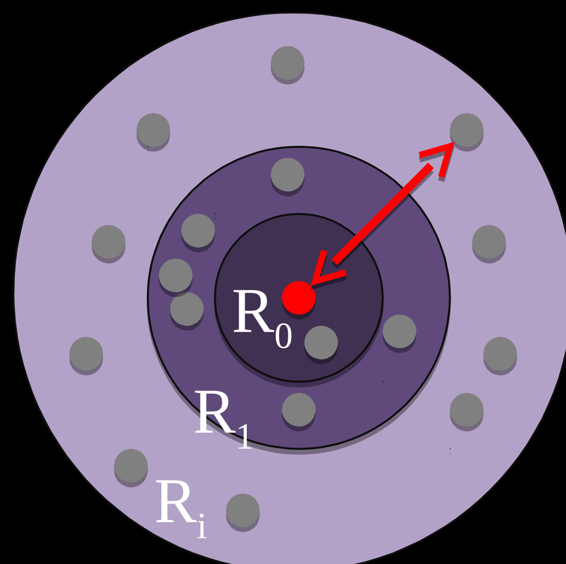
$k = \log(\ Dmax\ /\ Dmin\ )$

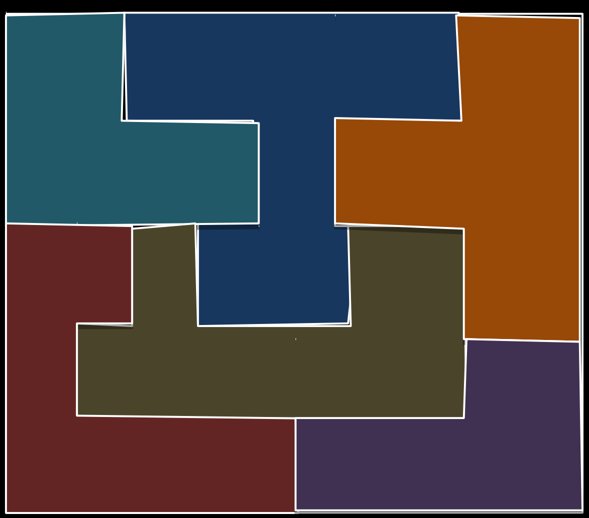
Language of Partitions

Block models

Geometry

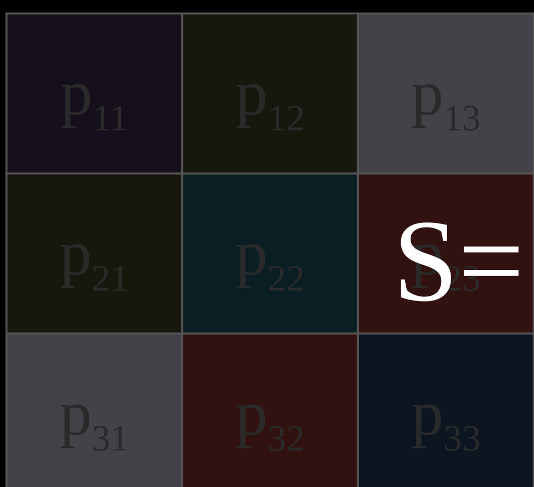
Abstract Geometry

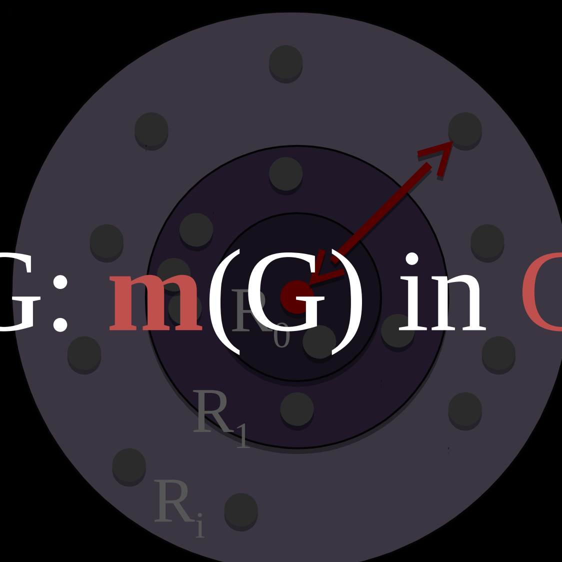




Language of Partitions

Block models | Geometry | Abstract Geometry

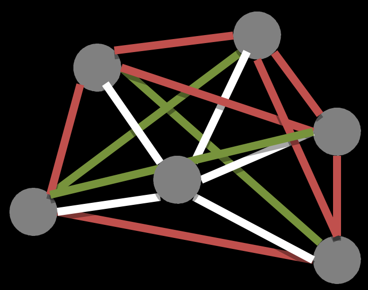




$$S = \{\, G : \mathbf{m}(G) \text{ in } C \,\}$$

Uniform Measure and *P*-symmetry

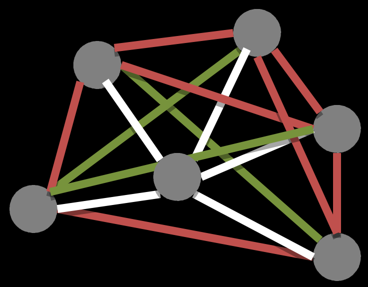
S={ G: **m**(G) in C }



$P = \{ P_1, P_2, P_3 \}$ $n = 6$, $k = 3$

Uniform Measure and *P*-symmetry

$$S=\{\ G: \mathbf{m}(G) \text{ in } \{\mathbf{m}^*\}\ \}$$



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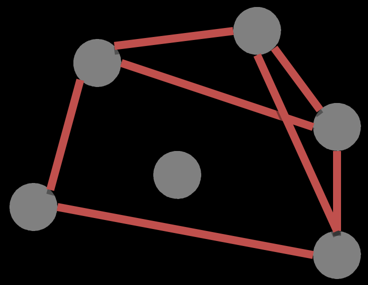
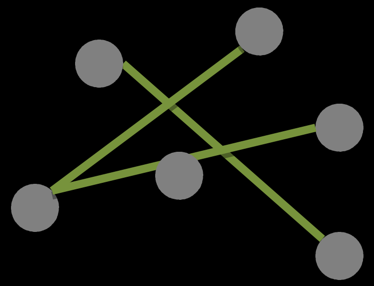
Uniform Measure and *P*-symmetry

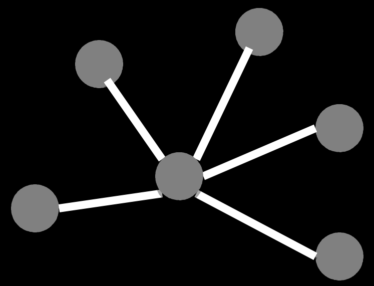
$$S=\{\ G: \mathbf{m}(G) \text{ in } \{\mathbf{m}^*\}\ \}$$



$m_1^* = 4$



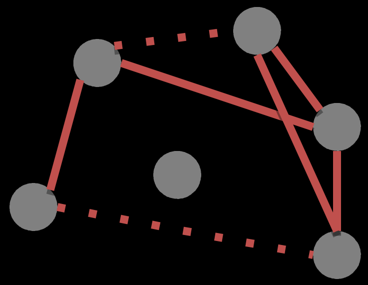
$m_2^* = 2$



$m_3^* = 2$

Uniform Measure and *P*-symmetry

$$S=\{\, G: \mathbf{m}(G) \text{ in } \{\mathbf{m}^*\} \,\}$$



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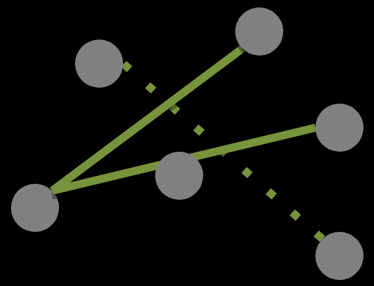

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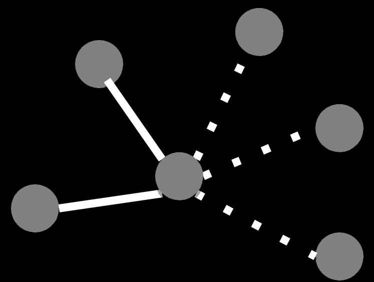

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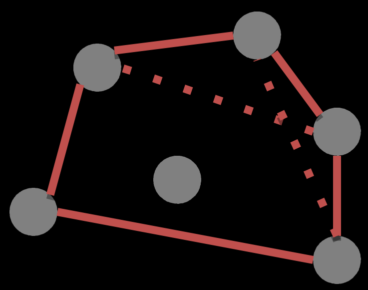

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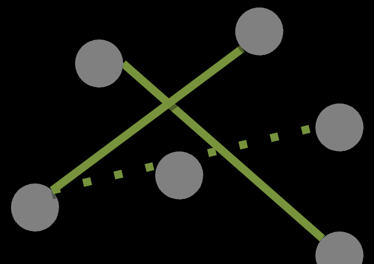

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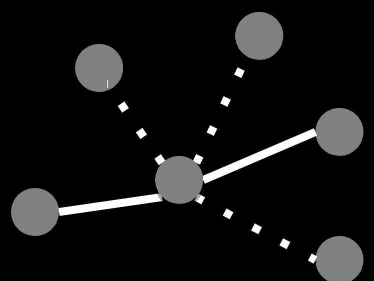

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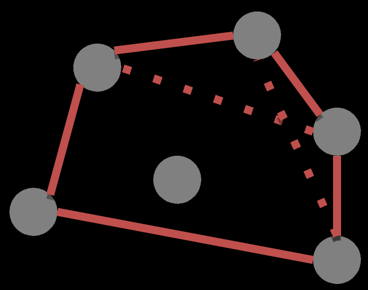
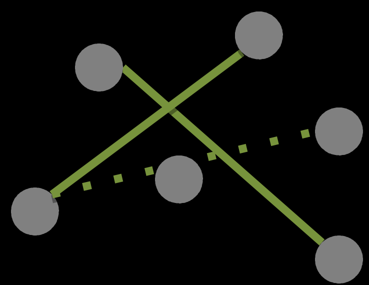
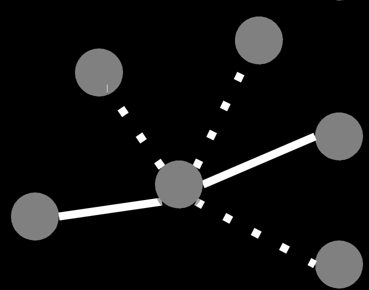


$m_1^* = 4$ $G(P_1, m_1^*)$



$m_2^* = 2$ $G(P_2, m_2^*)$



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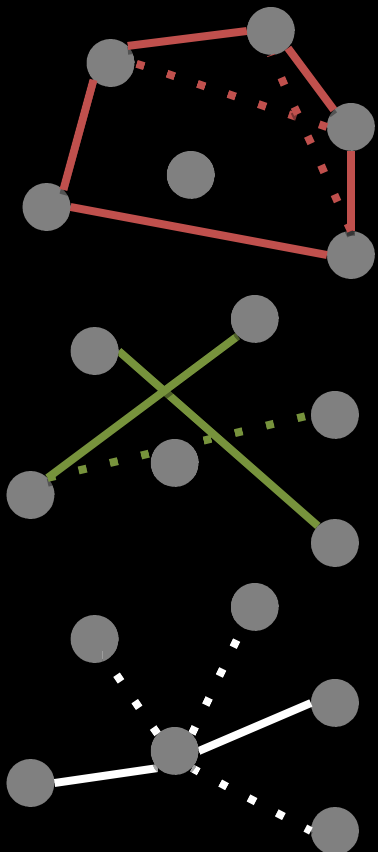
Coupling



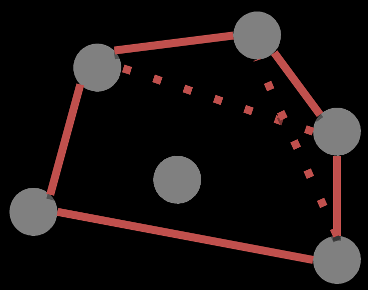
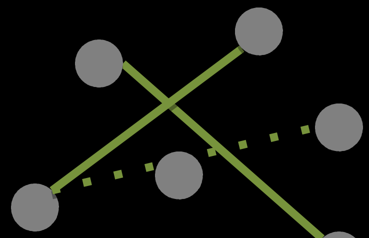
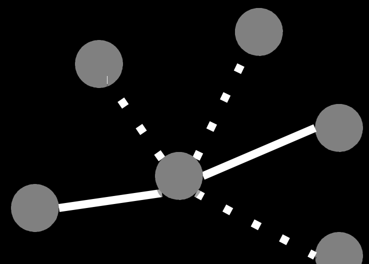
$m_1^* = 4$ $G(P_1, m_1^*)$ q_1

$m_2^* = 2$ $G(P_2, m_2^*)$ q_2

$m_3^* = 2$ $G(P_3, m_3^*)$ q_3

Uniform Measure and *P*-symmetry

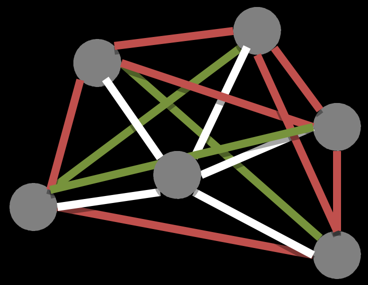
$$S=\{\, G: \mathbf{m}(G) \text{ in } \{\mathbf{m}^*\}\,\}$$

			Coupling
	$m_1^* = 4$	$G(P_1, m_1^*)$	q_1
	$m_2^* = 2$	$G(P_2, m_2^*)$	q_2
	$m_3^* = 2$	$G(P_3, m_3^*)$	q_3

Union Bound

Uniform Measure and *P*-symmetry

$$S = \{ G : \mathbf{m}(G) \text{ in } \{\mathbf{m}^*\} \}$$



$$\mathbf{m}^* = (m_1^*, m_2^*, m_3^*\}$$

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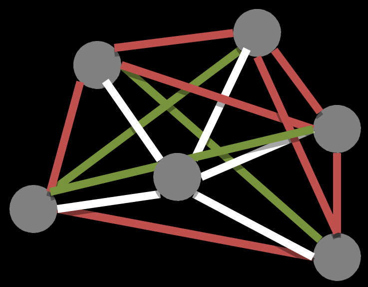
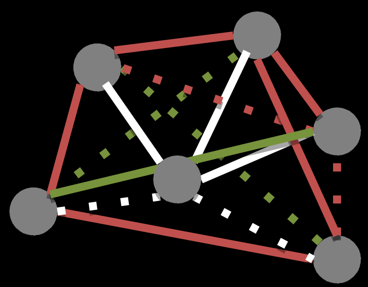


$$\mathbf{m}^* = (m_1^*,\ m_2^*,\ m_3^*\}$$

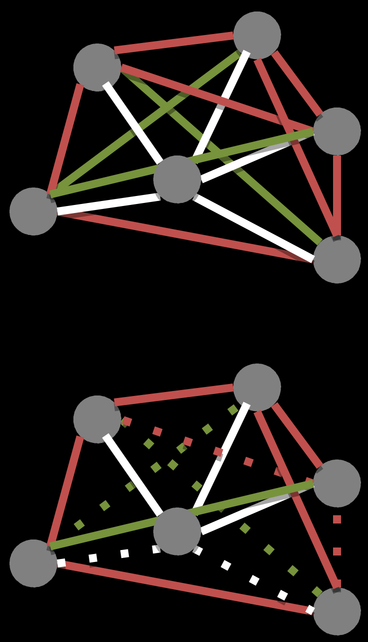
Coupling



$$\mathbf{q}^* = (q_1^*,\ q_2^*,\ q_3^*\}$$

Uniform Measure and *P*-symmetry

$$S=\{\, G: \mathbf{m}(G) \text{ in } C \,\}$$



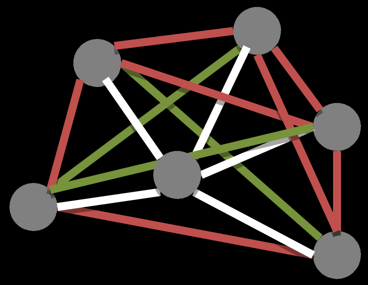
m in C

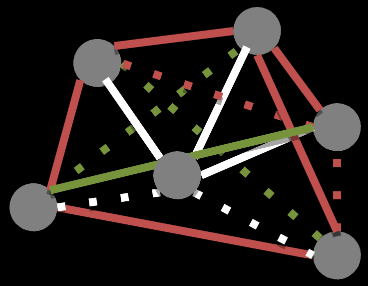
Coupling ?

$$q^* = (q_1^*, q_2^*, q_3^*\}$$

Uniform Measure and *P*-symmetry

S={ G: **m**(G) in C}





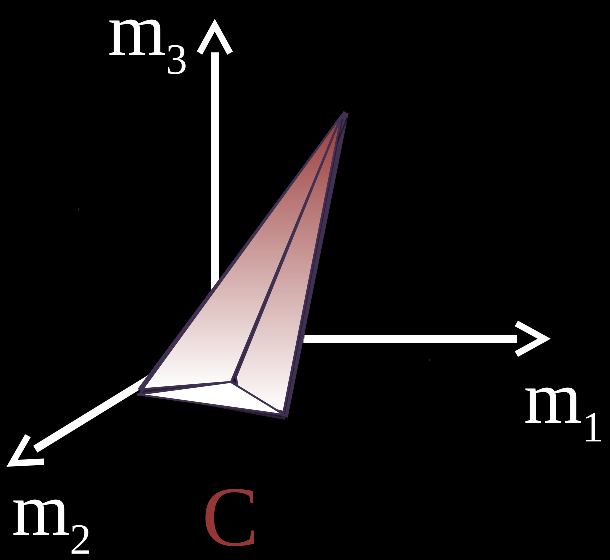
m in C

Coupling ?

$q^* = (q_1^*, q_2^*, q_3^*)$

Concentration of **m**(G)!

Concentration of Edge-profiles



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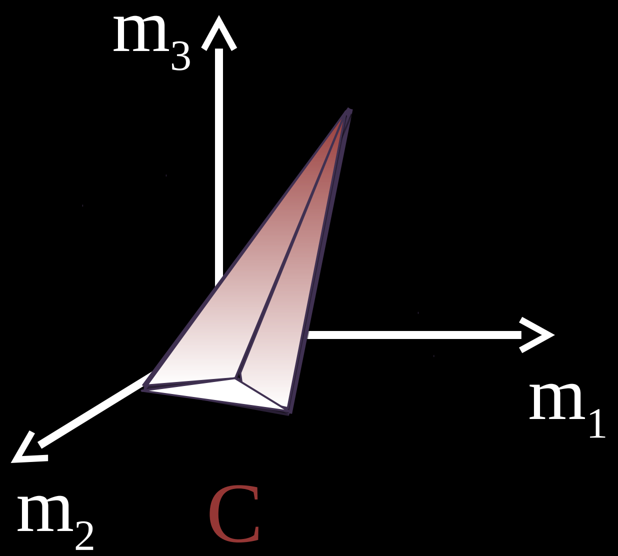
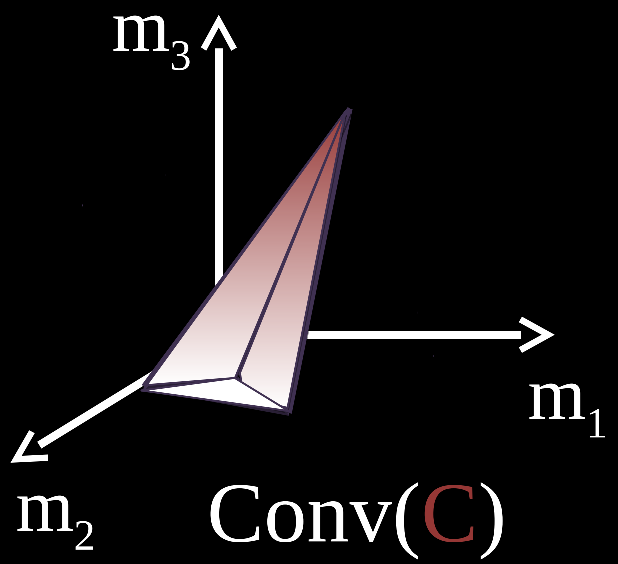


max Ent(**m**)

s. t. **m** in C

Concentration of Edge-profiles



$$\max \quad \text{Ent}(\mathbf{m})$$
$$\text{s. t.} \quad \mathbf{m} \text{ in } \text{Conv}(C)$$

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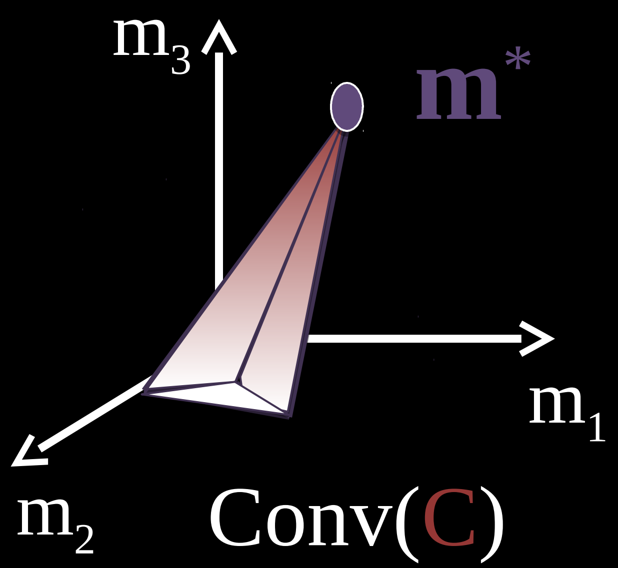

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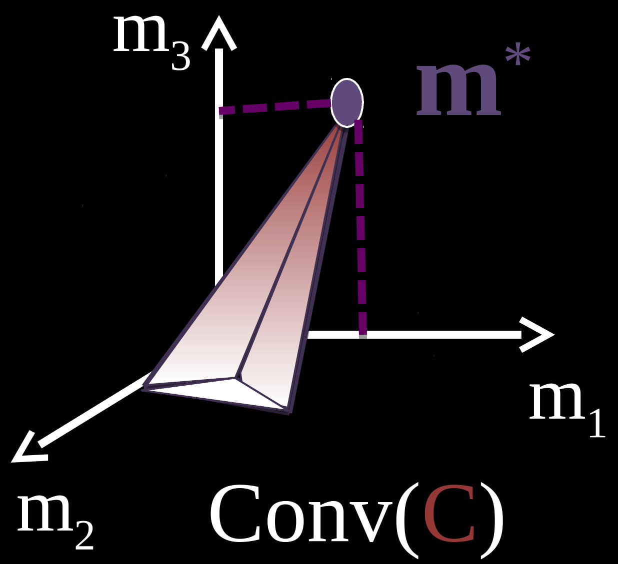

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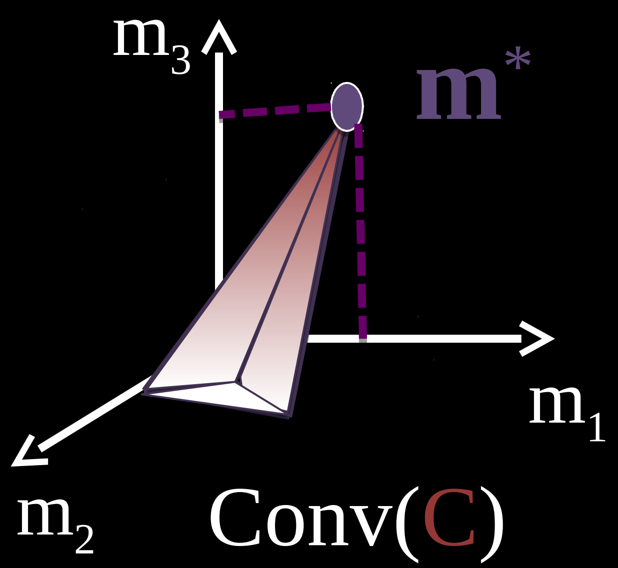
$$\boldsymbol{\mu}(\mathbf{S}) = \min \{m_i^*, P_i - m_i^*\}$$

Concentration of Edge-profiles



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Theorem [Achlioptas, S', 15]

For a convex P-symmetric set S and for every $\varepsilon > r(S)$

$$P(\,|\,m_i(G) - m_i^*\,| > \varepsilon\, m_i^*\,) < \exp(-\mu(S)[\varepsilon^2 - \lambda(S)])$$

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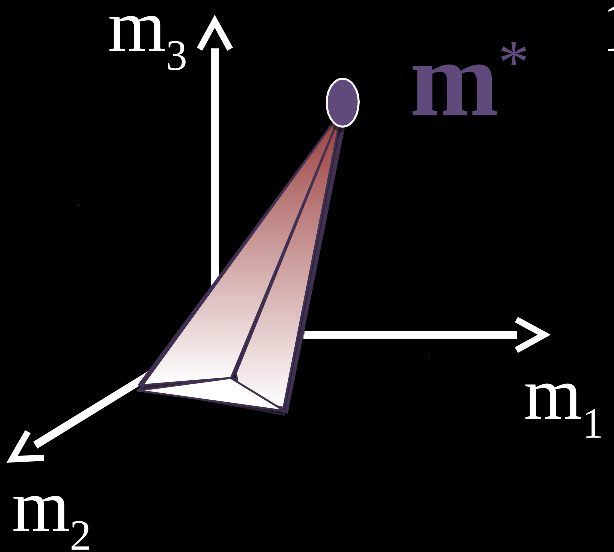
Applications



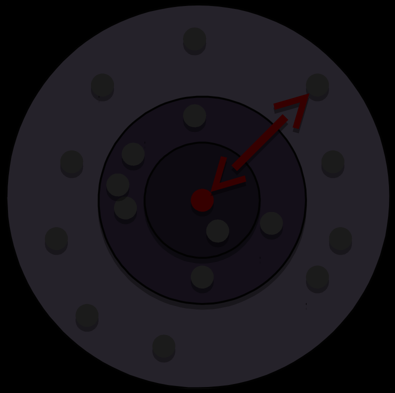
1. Graphs defined by Linear Programs

$$S = \{ G: \; \mathbf{X}\, m(G) < \mathbf{b} \}$$

$$\mathbf{X} \text{ in } \mathbf{R}^{m \times k}$$

$$P(e_i) = \frac{1}{1 + \exp(\mathbf{X}_i^T \boldsymbol{\lambda})}$$

2. Kleinberg's Navigability for Set Systems [Achlioptas, S'2015].



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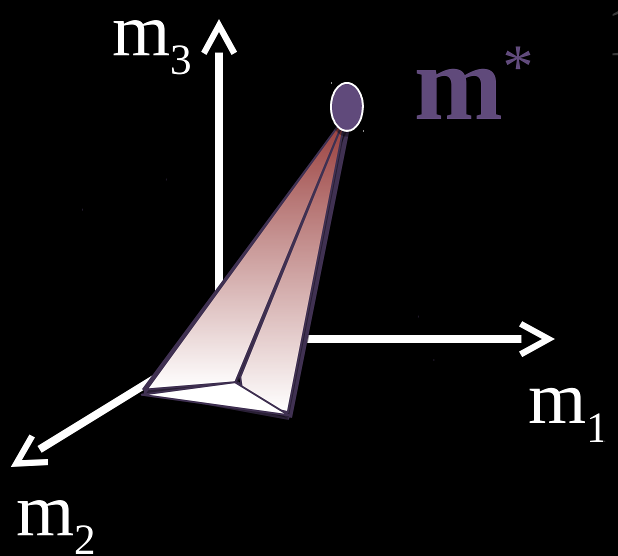

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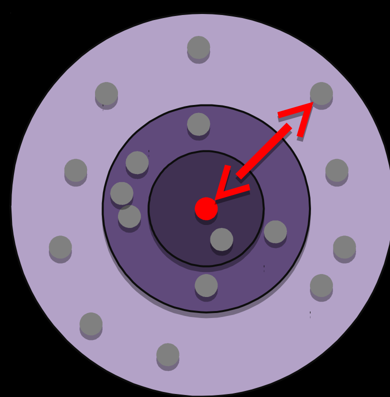


Take Home Message

Random Graphs subject to Constraints

Geometric Analogues of Erdos-Renyi Graphs

Independence = Symmetry + Concentration

P-symmetry concept to be further exploited

Future Work

Generalize notion of symmetry and extend beyond uniform measure

Geometric coding of Edge Partitions for Inference and Sampling.

Applications in Learning Theory, Approximation Algorithms, Average Case Analysis.

Maximum Entropy in Computer Science

Max-Min Fair Allocation of Indivisible Goods
Asadpour, Saberi STOC 2007

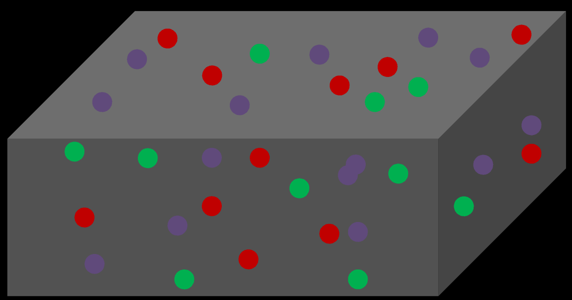
An O(log n/ loglog n)- approx Algorithm for the Asymmetric TSP
Asadpour, Goemans, Madry, Oveis Gharan, Saberi. SODA 2010

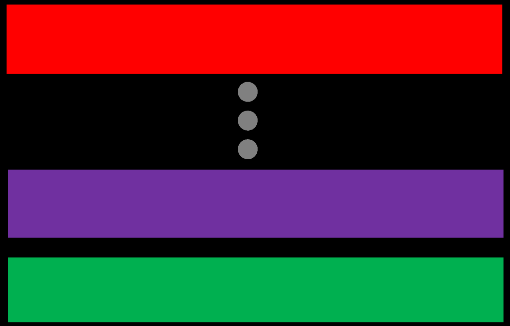
Randomized Rounding Approach to the Traveling Salesman Problem
Oveis Gharan, Saberi, Singh FOCS 2011

The Entropy Rounding Method in Approximation Algorithms
Rothvoß SODA 2012

Entropy, Optimization and Counting
Singh, Vishnoi STOC 2014

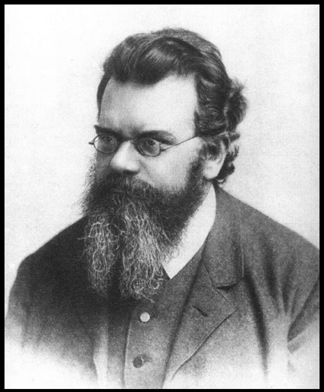
Boltzmann's Heuristic





Fixed Energy

Uniform Measure



Maximum Entropy

Realistic?

Rigorous?