How I Got Lost in a Random Domain

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Random Domain Problem

Let $D(\omega) = \{(x_1, x_2) \mid -1 \leq x_1 \leq 1,\ s(x_1, \omega) \leq x_2 \leq 1\}$

$$\langle s(x_1, \omega) \rangle = -1 \qquad C_s(z_1, z_2) = e^{-|z_1 - z_2|}$$

Find a stochastic solution $u(x_1, x_2, \omega)$ such that

$$\Delta u = 0, \qquad \text{in } D(\omega)$$

with boundary conditions $u = 1$ at $x_2 = 1$ and $u = 0$ elsewhere on the boundary.

Random Parameterization

A Karhunen-Loeve type expansion for $s(x_1,\omega)$:

$$s(x_1,\omega) = \underbrace{\langle s(x_1)\rangle}_{=-1} + \sigma \sum_{k=1}^{K} \sqrt{\lambda_k}\,\psi_k(x_1) Y_k(\omega)$$

where $\{Y_k\}$ are i.i.d. $U[-1,1]$ and $\{\lambda_k,\psi_k\}$ solves

$$\int C_s(z_1,z_2)\psi_k(z_2)\,dz_2 = \lambda_k \psi_k(z_1), \qquad k = 1,\ldots,K.$$

Eigenpairs

In particular, if ω_n and ω_n^* solve

$$1 - \omega_n \tan \omega_n = 0 \qquad \omega_n^* + \tan \omega_n^* = 0$$

then

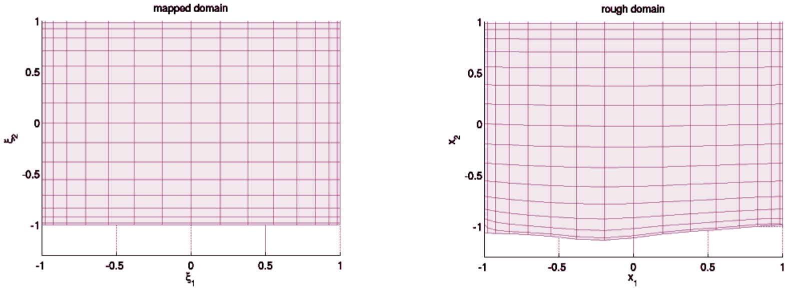
$$\psi_n(x_1) = \frac{\cos \omega_n x_1}{\sqrt{1 + \frac{\sin 2\omega_n}{2\omega_n}}} \qquad \psi_n^*(x_1) = \frac{\sin \omega_n^* x_1}{\sqrt{1 - \frac{\sin 2\omega_n^*}{2\omega_n^*}}}$$

for n even and odd, respectively, and

$$\lambda_n = \frac{2}{\omega_n^2 + 1} \qquad \lambda_n^* = \frac{2}{(\omega_n^*)^2 + 1}$$

Random Mapping

$$\xrightarrow{\xi(x_1, x_2)}$$



$$\overset{\Longleftarrow}{\mathbf{x}(\xi_1, \xi_2)}$$

Random Mapping

To compute $\mathbf{x}(\xi_1, \xi_2)$, solve

$$\frac{\partial^2 x_1}{\partial \xi_1^2} + \frac{\partial^2 x_1}{\partial \xi_2^2} = 0, \quad \frac{\partial^2 x_2}{\partial \xi_1^2} + \frac{\partial^2 x_2}{\partial \xi_2^2} = 0, \quad \text{in } E$$

subject to

$$x_1(-1, \xi_2) = -1, \quad x_1(1, \xi_2) = 1, \quad x_1(\xi_1, -1) = \xi_1, \quad x_1(\xi_1, 1) = \xi_1$$

and

$$x_2(-1, \xi_2) = \xi_2, \quad x_2(1, \xi_2) = \xi_2, \quad x_2(\xi_1, -1) = s(\xi_1, \omega), \quad x_2(\xi_1, 1) = 1$$

Random Mapping

In our case $x_1(\xi_1, \xi_2) = \xi_1$ and

$$x_2(\xi_1, \xi_2, \omega) = \sum_{k=0}^{K} \hat{x}_{2,k} Y_k(\omega)$$

where $\{\hat{x}_{2,k}\}$ solve

$$\frac{\partial^2 \hat{x}_{2,k}}{\partial \xi_1^2} + \frac{\partial^2 \hat{x}_{2,k}}{\partial \xi_2^2} = 0, \quad k = 0, \dots, K$$

subject to

$$\hat{x}_{2,k}(0, \xi_2) = \delta_{k0}\xi_2, \quad \hat{x}_{2,k}(\xi_1, 0) = \hat{s}_k(\xi_1)$$
$$\hat{x}_{2,k}(5, \xi_2) = \delta_{k0}\xi_2, \quad \hat{x}_{2,k}(\xi_1, 1) = \delta_{k0}1$$

Solve these with a Chebyshev pseudospectral method.

Stochastic Boundary Value Problem

Use the chain rule to get

$$\frac{\partial^2 u}{\partial \xi_1^2} + \underbrace{\left(\left(\frac{\partial \xi_2}{\partial x_1}\right)^2 + \left(\frac{\partial \xi_2}{\partial x_2}\right)^2\right)}_{\equiv C_2} \frac{\partial^2 u}{\partial \xi_2^2}$$

$$+ \underbrace{\left(\frac{\partial^2 \xi_2}{\partial x_1^2} + \frac{\partial^2 \xi_2}{\partial x_2^2}\right)}_{\equiv C_3} \frac{\partial u}{\partial \xi_2} + 2\underbrace{\frac{\partial \xi_2}{\partial x_1}}_{\equiv C_4} \left(\frac{\partial u}{\partial \xi_1 \partial \xi_2}\right) = 0$$

subject to

$$u(\xi_1, -1) = u(-1, \xi_2) = u(1, \xi_2) = 0, \qquad u(\xi_1, 1) = 1.$$

Stochastic Boundary Value Problem

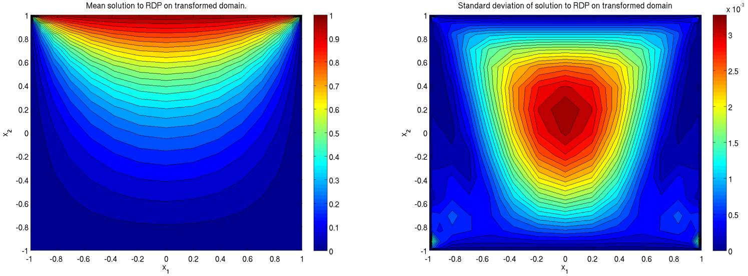
How can we solve the SBVP? Monte Carlo. . .

- Generate J random samples of $\{Y_k\}$.
- For each $j \leq J$, construct random mapping

$$x_2(\xi_1,\xi_2,\omega_j) = \sum_{k=0}^{K} \hat{x}_{2,k} Y_k(\omega_j).$$

- Evaluate the Jacobian terms and solve SBVP.
- Post process for mean and variance

Results from Monte Carlo. . .



Stochastic Boundary Value Problem

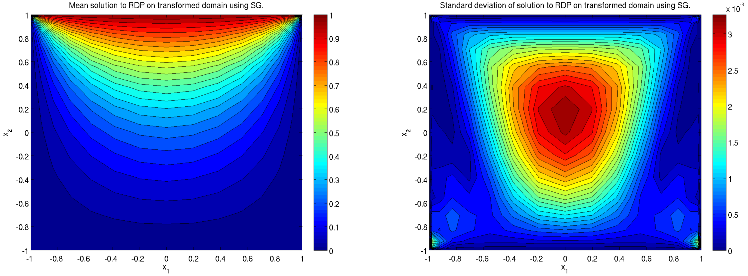
Or use a stochastic Galerkin technique...

- Expand the response u in a truncated PCE.

$$u(\xi_1, \xi_2, \omega) \approx \sum_{l=0}^{M} u_l(\xi_1, \xi_2)\Psi_l(\omega).$$

- Expand the random, variable coefficients in the transformed equation C_2, C_3, and C_4 in a PCE.
- Project the solution on the finite dimensional space spanned by $\{\Psi_l\}$ to obtain a coupled system of equations for $\{u_l\}$.
- Discretize in space. Solve the linear system. Compute statistics.

Results from stochastic Galerkin. . .



Summary & Remarks

- What's the catch?
  - Error analysis?
  - Well-posedness?
  - Cost & scalability?
- Summary:
  - Represent rough geometries with the proper stochastic representation.
  - Compute a mapping from random domain to deterministic domain.
  - Solve the SBVP on the deterministic domain.
  - Compute statistics (mean and variance).