Model

- Finite set A of nodes, with partial order "$\succeq$".
  For $a, b \in A$, "$a \succ b$" means "b is a downstream node for a."

- Some nodes are the "suppliers of basic inputs," i.e., nodes a such that there is no $a' \succ a$. Some nodes are the "consumers of final outputs," i.e., nodes z such that there is no $z' \prec z$. The rest are "intermediaries."

The basic unit of analysis is a *contract*.

Each contract $c = (s, b, l, p)$ consists of four variables:

- Seller $s \in A$, buyer $b \in A$, $s \succ b$;

- "Unit identifier" / "serial number" $l \in \mathbb{N}$;

  Nodes s and b can trade multiple units of the same good or service, units of different types of goods or services, or both. Each unit has its own "unit identifier."

- Price $p \in \mathbb{R}$.

The set of available contracts, C, is finite.

- Each node a has a utility function over the *sets of contracts involving it*. E.g., the utility can be quasilinear:

$$V_a(X) = W_a\left(\{(s_c, b_c, l_c) | c \in X\}\right) + \sum_{c \in C_1} p_c - \sum_{c \in C_2} p_c,$$

  where $C_1 = \{c \in X | a = s_c\}$ and $C_2 = \{c \in X | a = b_c\}$, i.e., C_1 is the set of contracts in X in which a is the seller and C_2 is the set of contracts in which a is the buyer.

- Choice function $Ch_a(X)$ returns node a's most preferred subset of X, i.e., $X' \subset X$ that maximizes $V_a(X')$:

$$Ch_a(X) = \operatorname*{argmax}_{X' \subset X} \{V_a(X')\}.$$

Restrictions on preferences

- Preferences of agent a are *same-side substitutable* if, choosing from a bigger set of contracts on one side, the agent does not accept any contracts *on that side* that he rejected when he was choosing from the smaller set.

- Preferences of agent a are *cross-side complementary* if, facing a bigger set of contracts on one side, an agent does not reject any contract *on the other side* that he accepted when he was choosing from the smaller set.

- A *network* is a set of contracts. Network μ is *individually rational* if no node wants to drop any of its contracts.

- A *chain* is a sequence of contracts, $(c_1, \ldots, c_n)$, such that $b_i = s_{i+1}$, i.e., the buyer of c_i is the seller of c_{i+1}.

- A *chain block* of network μ is a chain $C = (c_1, \ldots, c_n)$ such that $\mu \cap C = \emptyset$ and all agents in the chain would like to add their contracts in C to those in μ: $c_1 \in Ch_{s_1}(\mu(s_1) \cup c_1)$; $c_n \in Ch_{b_n}(\mu(b_n) \cup c_n)$; and $\forall i < n$, $\{c_i, c_{i+1}\} \subset Ch_{b_i}(\mu(b_i) \cup c_i \cup c_{i+1})$.

- A network is *chain stable* if it is individually rational and has no chain blocks. If there are no intermediaries in the market, chain stability is equivalent to pairwise stability.

- Each node treats its links independently of one another.

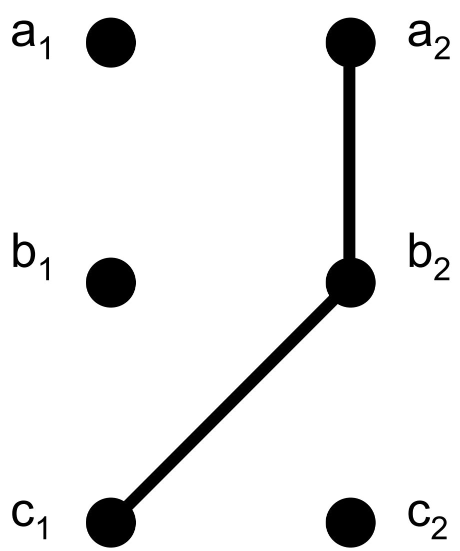
Example. Two suppliers of basic inputs (a_1, a_2), two intermediaries (b_1, b_2), two consumers of final outputs (c_1, c_2).

Suppliers cannot trade directly with consumers: trade flows have to go through intermediaries. All agents have unit capacities: each supplier can supply one unit of the good; each consumer needs one unit; each intermediary can process one unit. There are no prices in the market (e.g., they are fixed by regulation).

Each supplier is willing to sell to any intermediary. Each consumer is willing to buy from any intermediary. An intermediary only wants to trade with a consumer if he also trades with a supplier, and vice versa.

Each agent x_i prefers to sell to an agent with the same index i, but prefers to buy from an agent with the opposite index, $3-i$.

Unstable Networks - 1



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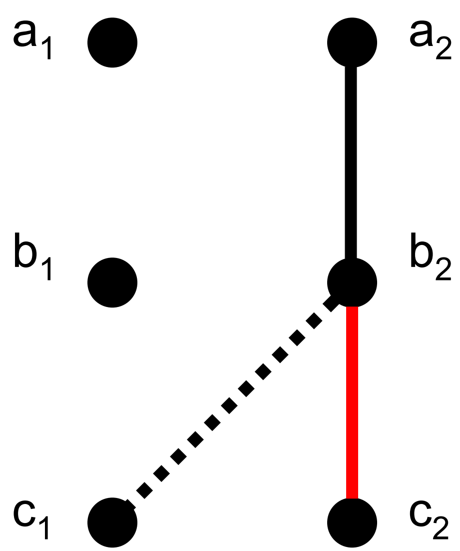
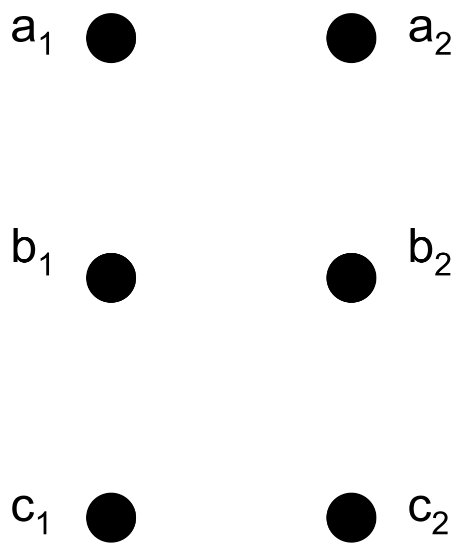


Unstable Networks - 2



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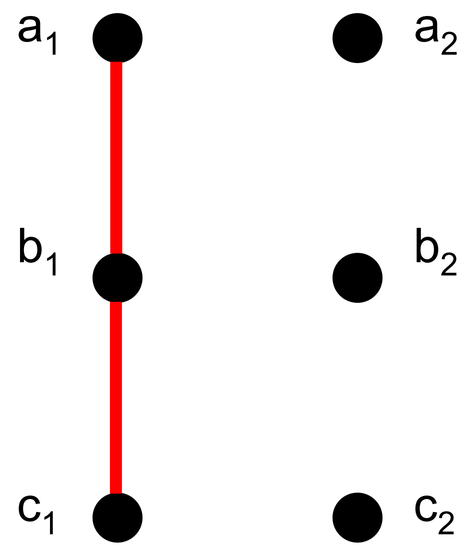


Stable Networks

Theorem. *There exists a chain stable network.*

Proof.

A *pre-network* is a set of arrows ("offers") from nodes in A to other nodes. Each arrow has a contract attached to it.

For pre-networks ν_1 and ν_2, say that $\nu_1 \leq \nu_2$ if the set of downstream arrows in ν_1 is a subset of the set of downstream arrows in ν_2 and the set of upstream arrows in ν_1 is a superset of the set of upstream arrows in ν_2.

The smallest pre-network, ν_{min}, includes all possible upstream arrows and no downstream arrows. The largest pre-network, ν_{max}, includes all possible downstream arrows and no upstream arrows.

Mapping T from the set of pre-networks to itself considers the "offers" that each node has (i.e., the contracts attached to the arrows pointing to that node), and constructs all "offers" that the node would like to make (i.e., arrows going from that node) given its options.

That is, for pre-network ν, node a, set of arrows $\nu(a)$ pointing <u>to</u> a in ν, and arrow r with contract c attached going <u>from</u> node a, $r \in T(\nu)$ if and only if

$$c \in Ch_a\left(\nu(a) \cup c\right).$$

Lemma. *If* $\nu_1 \leq \nu_2$*, then* $T(\nu_1) \leq T(\nu_2)$*.*

By definition, $\nu_{min} \leq T(\nu_{min})$.

Therefore, $T(\nu_{min}) \leq T^2(\nu_{min})$, $T^2(\nu_{min}) \leq T^3(\nu_{min})$, etc., and so $\{\nu_{min}, T(\nu_{min}), T^2(\nu_{min}), T^3(\nu_{min}), \ldots\}$ is an increasing sequence, converging after a finite number of steps to a fixed point, ν^*_{min}, such that $T(\nu^*_{min}) = \nu^*_{min}$.

Similarly, sequence $\{\nu_{max}, T(\nu_{max}), T^2(\nu_{max}), T^3(\nu_{max}), \ldots\}$ also converges to a fixed point, ν^*_{max}.

Define mapping F from the set of pre-networks to the set of networks as follows. Take any pre-network ν and contract c. Contract c belongs to $\mu = F(\nu)$ if and only if ν contains both arrows with contract c attached. In other words, mapping F removes all one-directional links from a pre-network, and replaces all two-directional links with the corresponding contracts. I.e.,

$$F(\nu) = \{c | (s_c, b_c, c) \in \nu \text{ and } (b_c, s_c, c) \in \nu\}$$

Lemma. *For any pre-network ν^* such that $T(\nu^*) = \nu^*$, network $\mu^* = T(\nu^*)$ is chain stable. Moreover, for any chain-stable network μ^*, there exists exactly one fixed-point pre-network ν^* such that $\mu^* = F(\nu^*)$.*

Proving this lemma will complete the proof of the main theorem: networks $\mu^*_{min} = F(\nu^*_{min})$ and $\mu^*_{max} = F(\nu^*_{max})$ are chain stable.

Proof of the first claim of the lemma

Let ν be a fixed point of mapping T, and let $\mu = F(\nu)$. Let us show that μ is chain stable.

1. Network μ is individually rational, because for any node a,

$$Ch_a(\mu(a)) = Ch_a(Ch_a(\nu(a))) = Ch_a(\nu(a)) = \mu(a),$$

and so node a cannot improve its payoff by dropping any of its contracts in μ.

2. No chain blocks. Suppose $(c_1, c_2 \dots c_n)$ is a chain block of μ. Let s_i and b_i be the seller and the buyer involved in contract c_i.

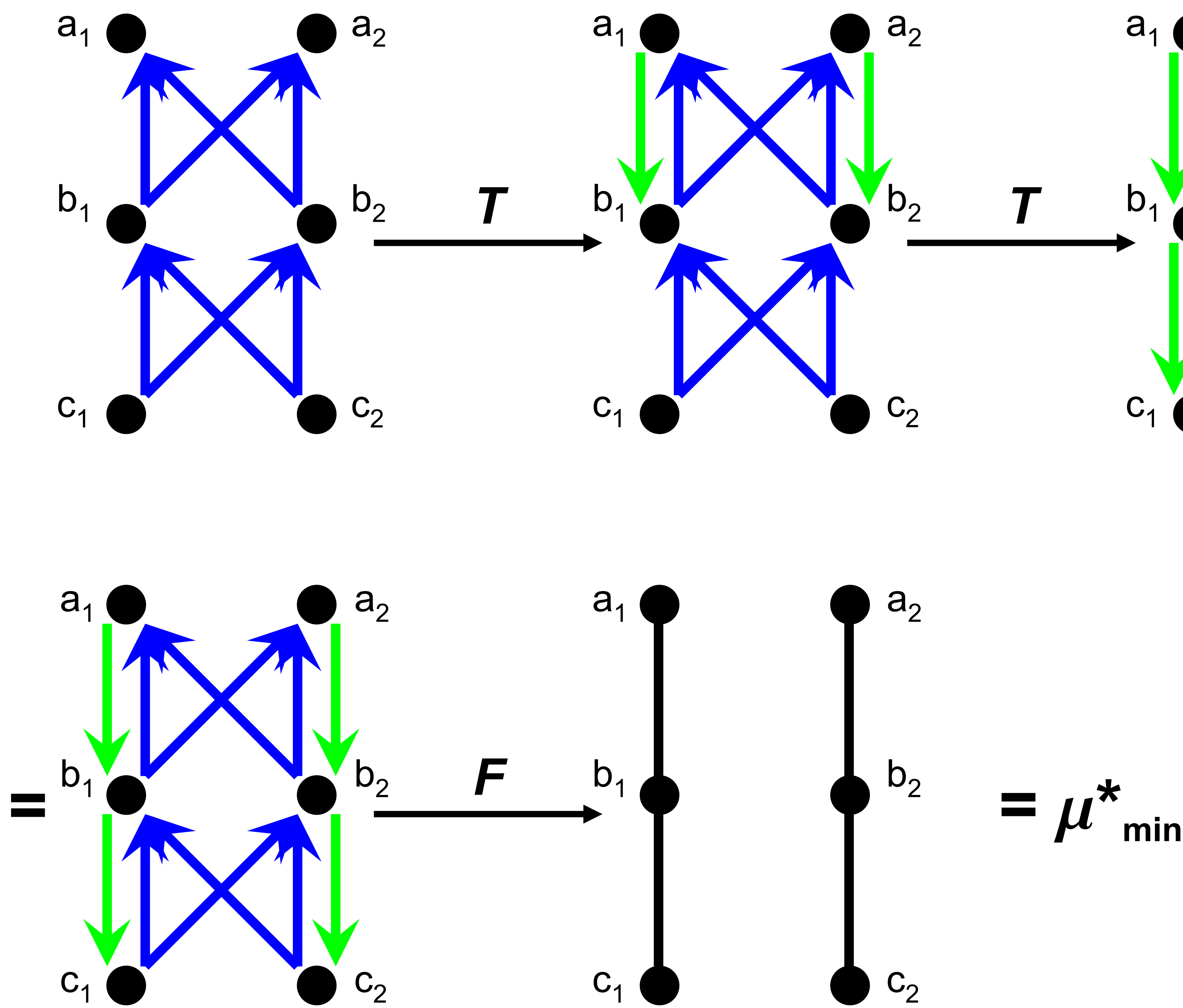
Since $c_1 \in Ch_{s_1}(\mu(s_1) \cup c_1)$, we have $c_1 \in Ch_{s_1}(\nu(s_1) \cup c_1)$, and so there is an arrow from s_1 to b_1 with c_1 attached in $T\nu = \nu$.

Since $\{c_1, c_2\} \subset Ch_{s_2}(\mu(s_2) \cup c_1 \cup c_2)$ and $c_1 \in \nu(s_2)$, we have $c_2 \in Ch_{s_2}(\nu(s_2) \cup c_2)$, and so there is an arrow from s_2 to b_2 with c_2 attached in $T\nu = \nu$.

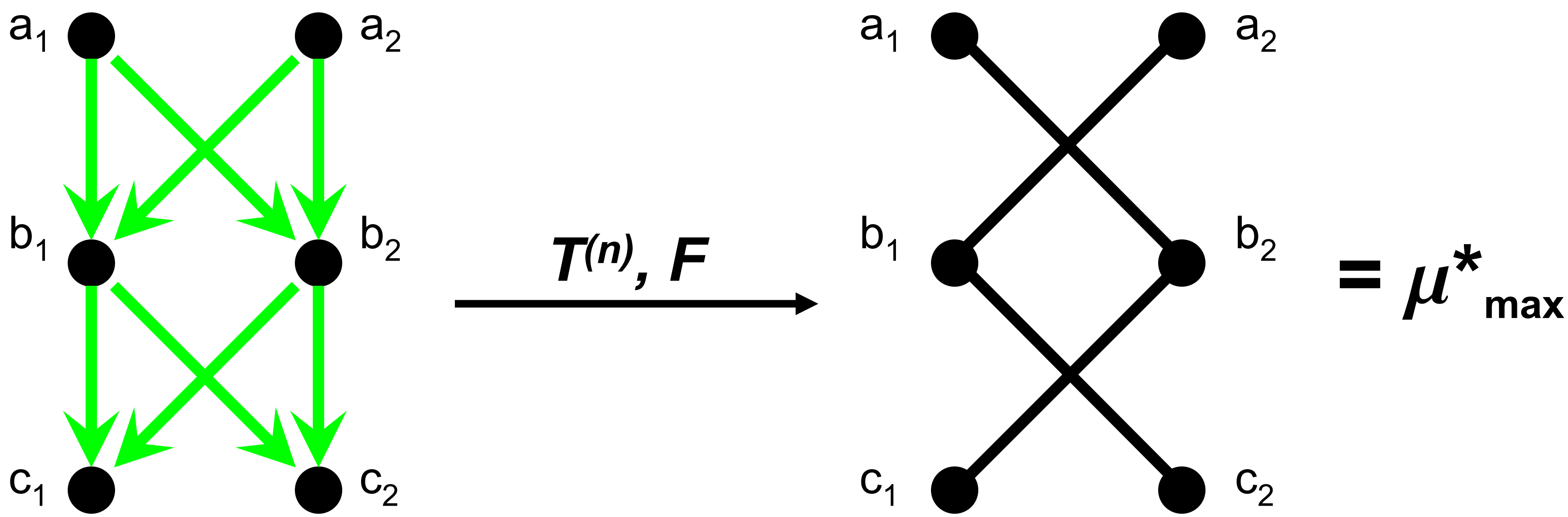
Proceeding by induction, there is an arrow from s_i to s_{i+1} with c_i attached in ν for any $i < n$.

Similarly, we could have started from node b_n, and so there must be an arrow going from b_n to $b_{n-1} = s_n$ with c_n attached in ν, which implies that $c_n \in \mu$—contradiction.

Therefore, for any $\nu = T\nu$, $F(\nu)$ is a chain stable network.



a1
a2
b1
b2
c1
c2
T
T
=
=
F
= μ*min



a1
a2
b1
b2
c1
c2
T(n), F
a1
a2
b1
b2
c1
c2
= μ*max

Theorem. *(Corollary of Tarski's theorem) The set of chain stable networks is a lattice with extreme elements* μ^*_{min} *and* μ^*_{max}.

Theorem. *Network* μ^*_{min} *is the best chain stable network for the suppliers of basic inputs and the worst chain stable network for the consumers of final outputs. Symmetrically, network* μ^*_{max} *is the worst chain stable network for the suppliers of basic inputs and the best chain stable network for the consumers of final outputs.*

An intermediate agent's most preferred chain stable network may be neither μ^*_{min} nor μ^*_{max}. Different intermediate agents may have different most preferred chain stable networks.

Theorem. *Adding a supplier of basic inputs to the market makes other such suppliers weakly worse off, and makes the consumers of final outputs weakly better off, at side-optimal chain stable networks. Symmetrically, adding a consumer of final outputs to the market makes other such consumers weakly worse off, and makes the suppliers of basic inputs weakly better off.*

The change in the welfare of intermediate agents is ambiguous—it can go either way. Adding new intermediate nodes can also have opposite effects on different extreme nodes (e.g., some suppliers may become better off and other suppliers may become worse off), as well as on other intermediate nodes.