Models of Kidney Exchange (a lightning overview…)

Market Design 285

October 17, 2012

Al Roth

Outline

- Kidney exchange
- What it is
- How we initially modeled it
- Initial kidney exchange institutions--pairwise
- How the game changed as kidney exchange grew
- An idealized model of large kidney exchange
- Why it doesn't work that way
- A model with highly sensitized patients—why chains work so well

Kidney exchange--background

- There are more than **90,000 patients on the waiting list for cadaver kidneys in the U.S.** today (94,005 this morning…)
- In 2011 33,581 patients were added to the kidney waiting list, and 28,625 patients were removed from the list.
- In 2011 there were 11,043 transplants of cadaver kidneys performed in the U.S.
- In the same year, 4,697 patients died while on the waiting list (and 2,466 others were removed from the list as "Too Sick to Transplant".
- In 2011 there were also 5,771 transplants of kidneys from *living* donors in the US.
- Sometimes donors are incompatible with their intended recipient.
- This opens the possibility of *exchange* .

Two Pair Kidney Exchange

Donor 1
Blood type A

Recipient 1
Blood type B

Donor 2
Blood type B

Recipient 2
Blood type A
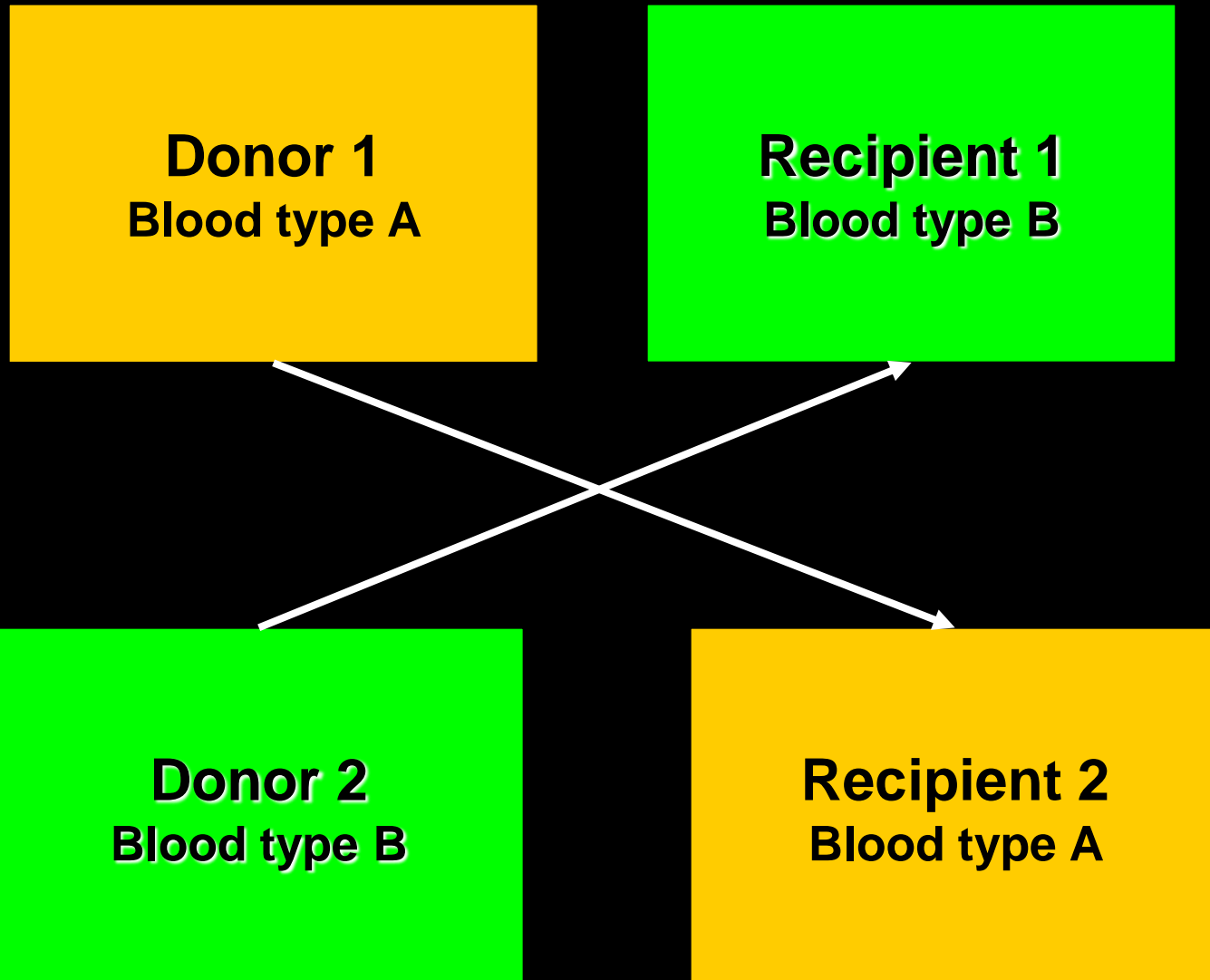
A classic economic problem: Coincidence of wants

(Money and the Mechanism of Exchange, Jevons 1876)

Chapter 1: "The first difficulty in barter is to find two persons whose disposable possessions mutually suit each other's wants. …to allow of an act of barter, there must be a **double coincidence**, which will rarely happen. ... the owner of a house may find it unsuitable, and may have his eye upon another house exactly fitted to his needs. But even if the owner of this second house wishes to part with it at all, it is exceedingly unlikely that he will exactly reciprocate the feelings of the first owner, and wish to barter houses. Sellers and purchasers can only be made to fit by the use of some commodity... which all are willing to receive **for a time**, so that what is obtained by sale in one case, may be used in purchase in another. This common commodity is called a **medium, of exchange**..."

Section 301,National Organ Transplant Act (NOTA), 42 U.S.C. 274e 1984:

"it shall be unlawful for any person to knowingly acquire, receive or otherwise transfer any human organ for valuable consideration for use in human transplantation".

Charlie W. Norwood Living Organ Donation Act

Public Law 110-144, 110th Congress, Dec. 21, **2007**

- Section 301 of the National Organ Transplant Act (42 U.S.C. 274e) is amended-- (1) in subsection (a), by adding at the end the following:

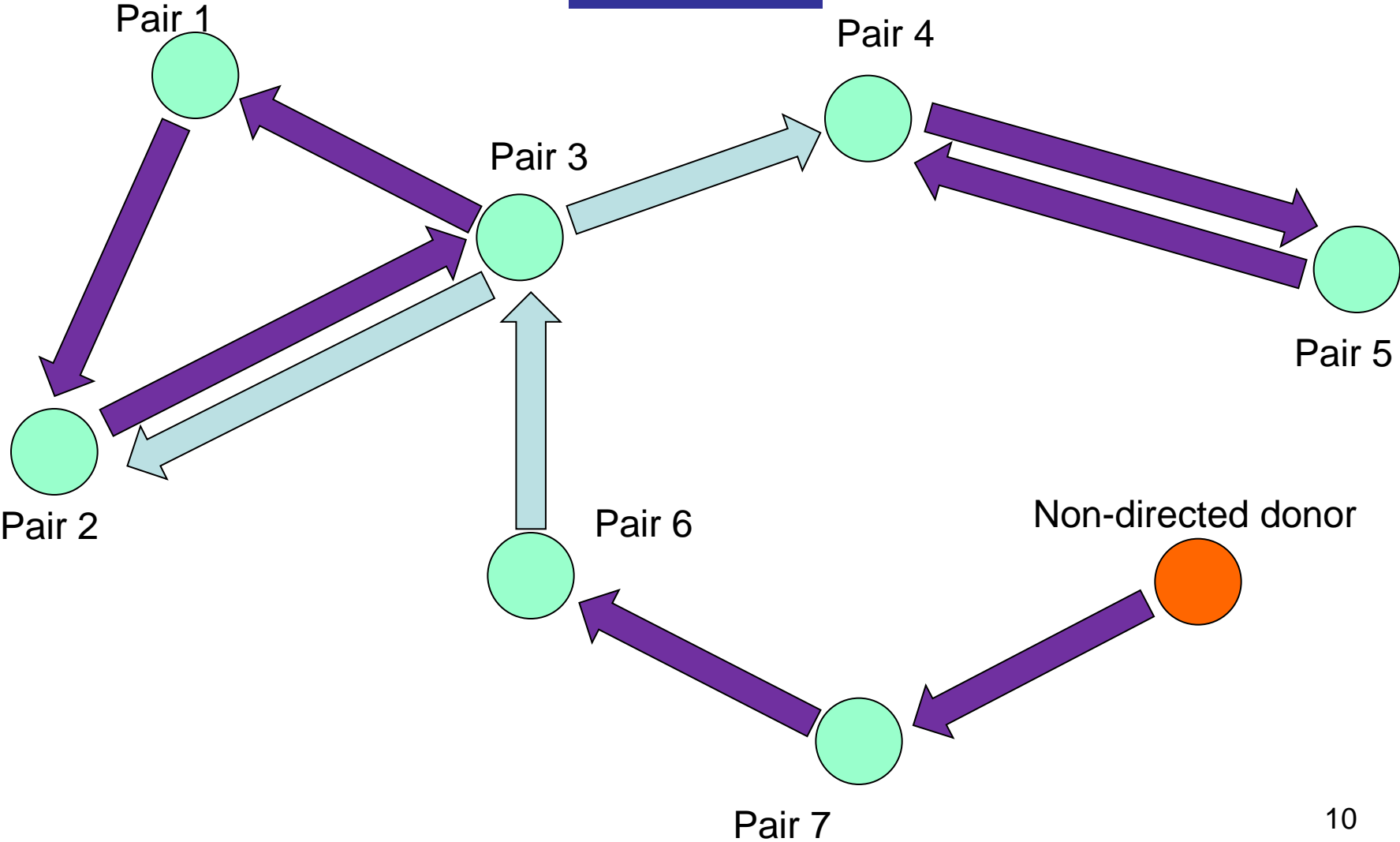- **``The preceding sentence does not apply with respect to human organ paired donation."**

Incentive Constraint: 2-way exchange involves 4 *simultaneous* surgeries.

3-pair exchange (6 simultaneous surgeries)
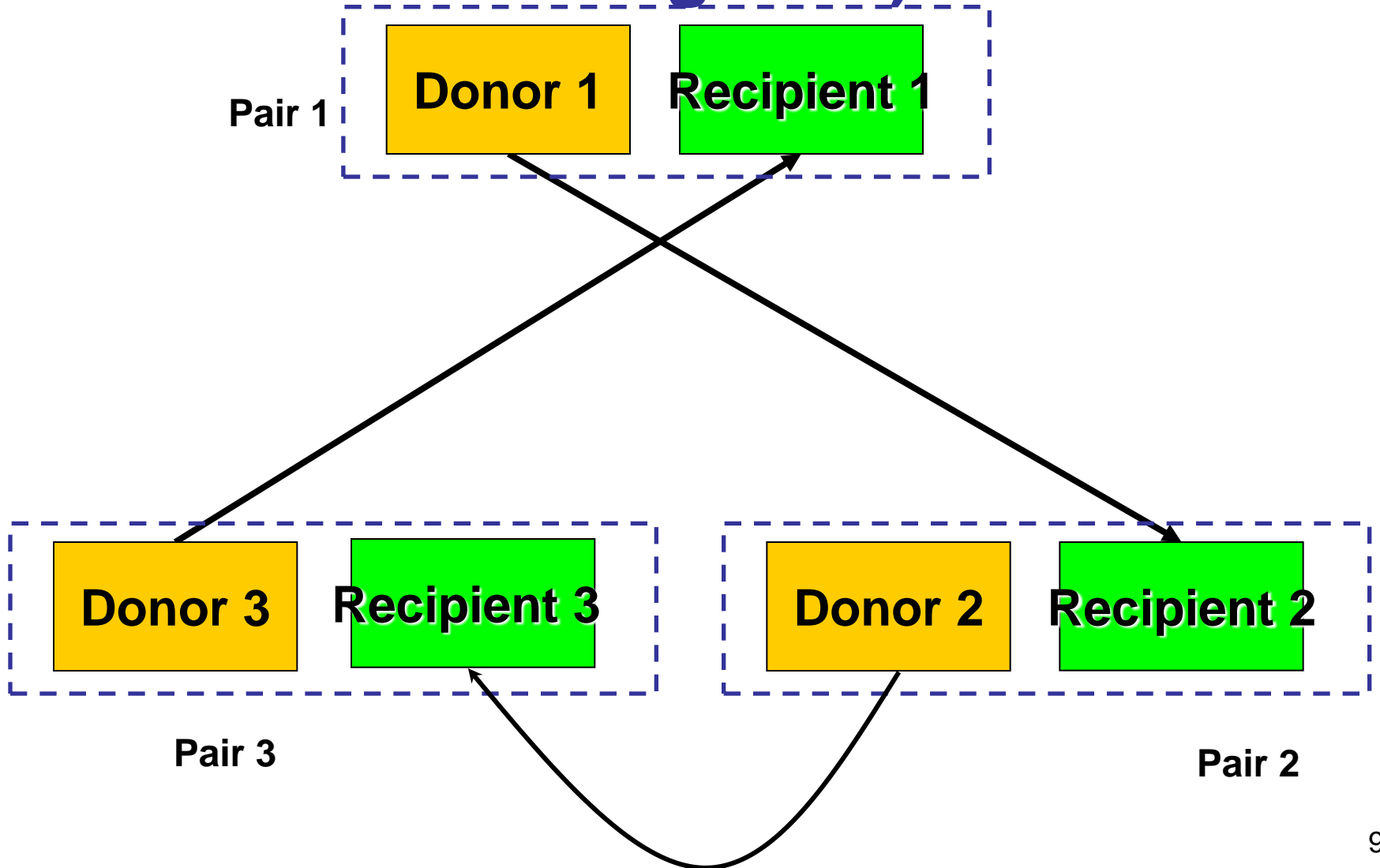
Non-directed donors: cycles plus <u>chains</u>

Pair 1

Pair 4

Pair 3

Pair 5

Pair 2

Pair 6

Non-directed donor

Pair 7

Kidney exchange clearinghouse design

Roth, Alvin E., Tayfun Sönmez, and M. Utku Ünver, "**Kidney Exchange**," *Quarterly Journal of Economics*, 119, 2, May, 2004, 457-488.

_____started talking to docs_____

____ "**Pairwise Kidney Exchange**," *Journal of Economic Theory*, 125, 2, 2005, 151-188.

___ "A Kidney Exchange Clearinghouse in New England**,**" *American Economic Review,* Papers and Proceedings, 95,2, May, 2005, 376-380.

_____ "**Efficient Kidney Exchange**: Coincidence of Wants in Markets with Compatibility-Based Preferences," *American Economic Review*, June 2007, 97, 3, June 2007, 828-851

___multi-hospital exchanges become common—hospitals become players in a new "kidney game"_____

Ashlagi, Itai and Alvin E. Roth "Individual rationality and participation in large scale, multi-hospital kidney exchange," revised June 2012.

Ashlagi, Itai, David Gamarnik and Alvin E. Roth, The Need for (long) Chains in Kidney Exchange, May 2012

And in the medical literature

Saidman, Susan L., Alvin E. Roth, Tayfun Sönmez, M. Utku Ünver, and Francis L. Delmonico, "Increasing the Opportunity of Live Kidney Donation By Matching for **Two and Three Way Exchanges**," *Transplantation*, 81, 5, March 15, 2006, 773-782.

Roth, Alvin E., Tayfun Sönmez, M. Utku Ünver, Francis L. Delmonico, and Susan L. Saidman, "Utilizing **List Exchange and Undirected Donation through "Chain" Paired Kidney Donations**," *American Journal of Transplantation*, 6, 11, November 2006, 2694-2705.

Rees, Michael A., Jonathan E. Kopke, Ronald P. Pelletier, Dorry L. Segev, Matthew E. Rutter, Alfredo J. Fabrega, Jeffrey Rogers, Oleh G. Pankewycz, Janet Hiller, Alvin E. Roth, Tuomas Sandholm, Utku Ünver, and Robert A. Montgomery, "**A Non-Simultaneous Extended Altruistic Donor Chain**," *New England Journal of Medicine* , 360;11, March 12, 2009, 1096-1101.

Ashlagi, Itai, Duncan S. Gilchrist, Alvin E. Roth, and Michael A. Rees, "**Nonsimultaneous Chains** and Dominos in Kidney Paired Donation – Revisited," *American Journal of Transplantation*, 11, 5, May 2011, 984-994

Ashlagi, Itai, Duncan S. Gilchrist, Alvin E. Roth, and Michael A. Rees, "**NEAD Chains** in Transplantation," *American Journal of Transplantation*, December 2011; 11: 2780–2781.

First pass (2004 QJE paper)

- Shapley & Scarf [1974] housing market model: n agents each endowed with an indivisible good, a "house".

- Each agent has preferences over all the houses and there is no money, trade is feasible only in houses.

- Gale's top trading cycles (TTC) algorithm: Each agent points to her most preferred house (and each house points to its owner). There is at least one cycle in the resulting directed graph (a cycle may consist of an agent pointing to her own house.) In each such cycle, the corresponding trades are carried out and these agents are removed from the market together with their assignments.

- The process continues (with each agent pointing to her most preferred house that remains on the market) until no agents and houses remain.
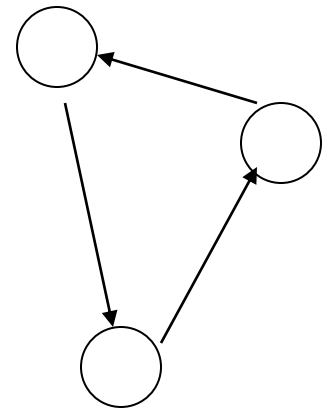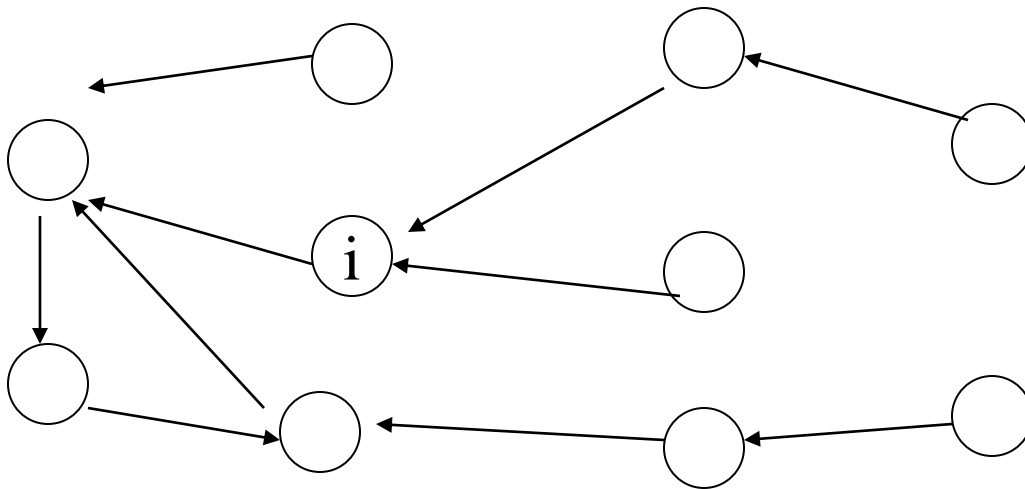
Theorem (Shapley and Scarf): the allocation x produced by the top trading cycle algorithm is in the core (no set of agents can all do better than to participate)

- We'll see that contemporary kidney exchange algorithms don't have this property, and we're starting to suffer from it—but it wasn't so important when we were the only game in town…

- When preferences are strict, Gale's TTC algorithm yields the *unique* allocation in the core (Roth and Postlewaite 1977).

Theorem (Roth '82): if the top trading cycle procedure is used, **it is a dominant strategy for every agent to state his true preferences**.

- The *idea* of the proof is simple, but it takes some work to make precise.
- When the preferences of the players are given by the vector P, let $N_t(P)$ be the set of players still in the market at stage t of the top trading cycle procedure.
- A *chain* in a set N_t is a list of agents/houses a_1, a_2, …a_k such that a_i's first choice in the set N_t is a_{i+1}. (A cycle is a chain such that $a_k=a_1$.)
- At any stage t, the graph of people pointing to their first choice consists of cycles and chains (with the 'head' of every chain pointing to a cycle…).

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Cycles and chains

The cycles leave the system (regardless of where i points), but i's choice set (the chains pointing to i) remains, and can only grow

Chains that integrate exchange with the waiting list

- Paired exchange and list exchange (deceased donors are non-directed…)

P on waiting list ← P1-D1 ← Deceased donor

P on waiting list ← P2-D2 ← P1-D1 ← Deceased donor
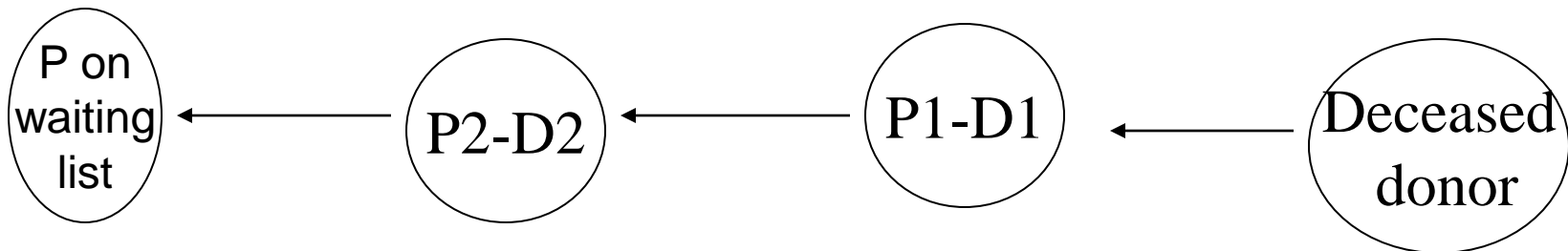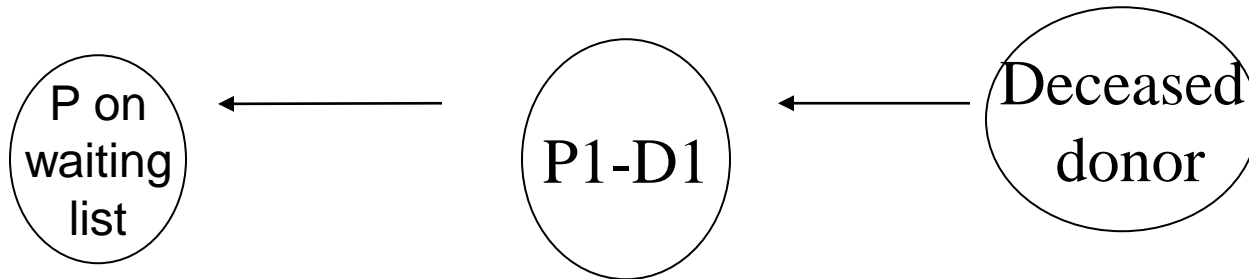
Top trading cycles and chains

- Unlike cycles, chains can intersect, so a kidney or patient can be part of several chains, so an algorithm will have choices to make.

Suppose exchanges involving more than two pairs are impractical?

- Our New England surgical colleagues had (as a first approximation) 0-1 (feasible/infeasible) preferences over kidneys.
    - (see also Bogomolnaia and Moulin (2004) for the case of two sided matching with 0-1 prefs)
- Initially, exchanges were restricted to pairs.

    - This involves a substantial welfare loss compared to the unconstrained case
    - But it allows us to tap into some elegant graph theory for constrained efficient and incentive compatible mechanisms.

Pairwise matchings and matroids

- Let (V,E) be the graph whose vertices are incompatible patient-donor pairs, with mutually compatible pairs connected by edges.

- A *matching* M is a collection of edges such that no vertex is covered more than once.

- Let S ={S} be the collection of subsets of V such that, for any S in S, there is a matching M that covers the vertices in S

- Then (V, S) is a matroid:
  - If S is in S, so is any subset of S.
  - If S and S' are in S, and |S'|>|S|, then there is a point in S' that can be added to S to get a set in S.

Pairwise matching with **0-1** preferences
(December 2005 *JET* paper)

- All maximal matchings match the same number of couples.
- If patients (nodes) have priorities, then a "greedy" priority algorithm produces the efficient (maximal) matching with highest priorities (or edge weights, etc.)
- Any priority matching mechanism makes it a dominant strategy for all couples to
  - accept all feasible kidneys
  - reveal all available donors
- So, there are efficient, incentive compatible mechanisms in the constrained case also.
  - Hatfield 2005: these results extend to a wide variety of possible constraints (not just pairwise)

Gallai-Edmonds Decomposition
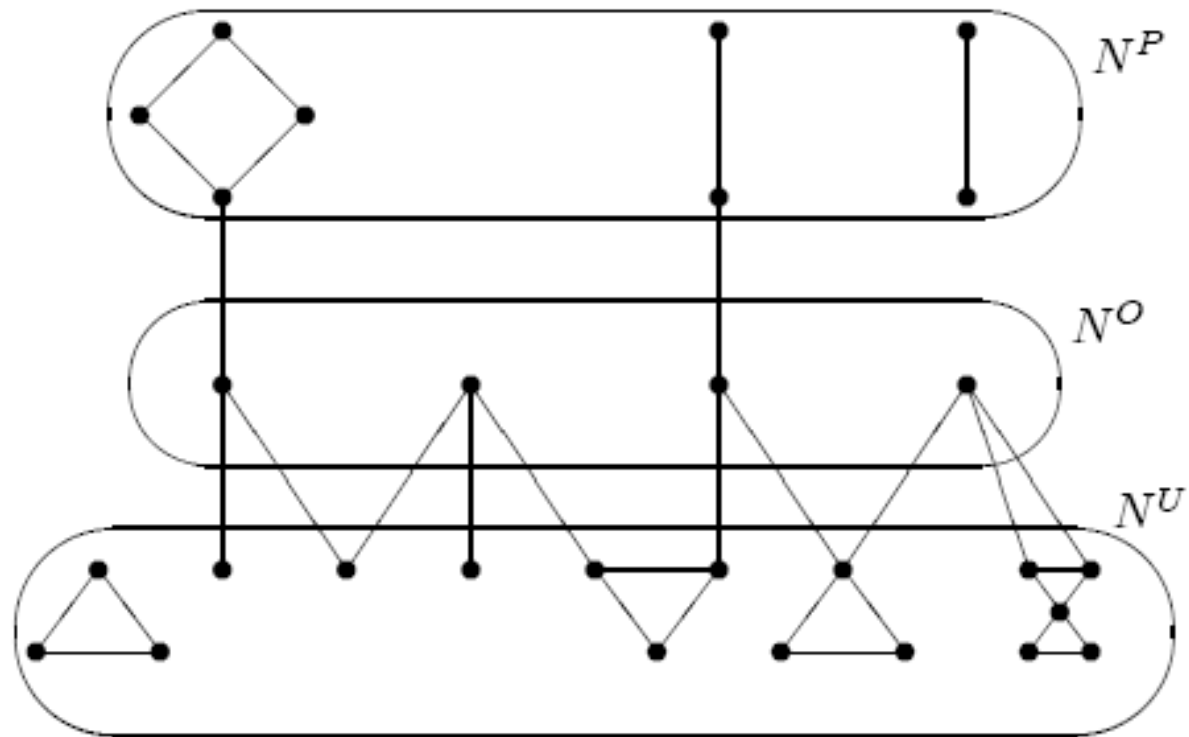
Factors determining transplant opportunity

• **Blood compatibility**
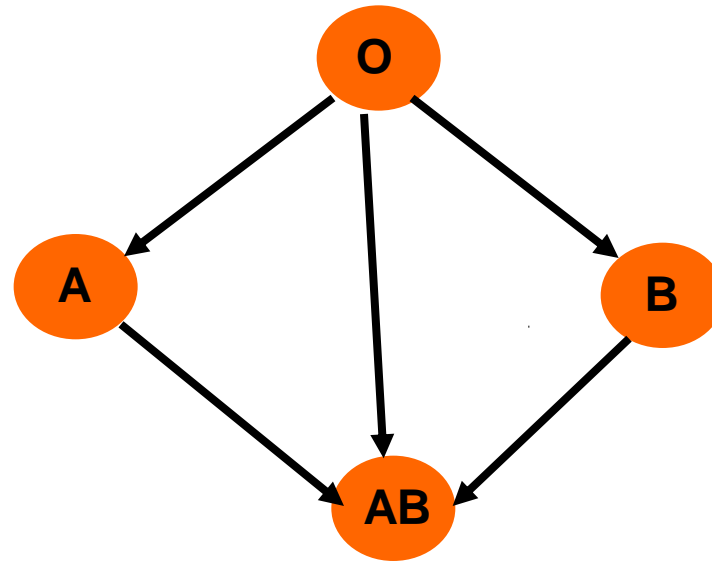
So type O *patients* are at a disadvantage in finding compatible kidneys—they can only receive O kidneys.
And type O *donors* will be in *short* supply.

• **Tissue type compatibility.** Percentage reactive antibodies (PRA)

❑ Low sensitivity patients (PRA < 79)
❑ High sensitivity patients (80 < PRA < 100)

A. Patient ABO Blood Type	Frequency
O	48.14%
A	33.73%
B	14.28%
AB	3.85%
B. Patient Gender	Frequency
Female	40.90%
Male	59.10%
C. Unrelated Living Donors	Frequency
Spouse	48.97%
Other	51.03%
D. PRA Distribution	Frequency
Low PRA	70.19%
Medium PRA	20.00%
High PRA	9.81%

Random Compatibility Graphs

n hospitals, each of a size c>0

D(n) - random compatibility graph:

1. n pairs/nodes are randomized –compatible pairs are disregarded
2. Edges (crossmatches) are randomized

Random graphs will allow us to ask two related questions:

What would efficient matches look like in an "ideal" large world?

What is the efficiency loss from requiring the outcome to be individually rational for hospitals?

(Large) Random Graphs

G(n,p) – n nodes and each two nodes have a non directed edge with probability p

Closely related model: G(n,M): n nodes and M edges—the M edges are distributed randomly between the nodes

Erdos-Renyi: For any p(n)$\geq$(1+ϵ)(ln n)/n *almost every large* graph G(n,p(n)) has a perfect matching, i.e. as **n→∞** the probability that a perfect matching exists converges to 1.

A natural case for kidneys is p(n) = p, a constant (maybe different for different kinds of patients), hence always above the threshold.

"**Giant connected component**"

Similar lemma for a random bipartite graph G(n,n,p).

Can extend also for r-partite graphs, directed graphs…

"Ideally" Efficient Allocations: if we were seeing all the patients in sufficiently large markets

Theorem 4.1 (Ashlagi and Roth (2011)). *In almost every large (limit) graph without nondirected donors there exists an efficient allocation with cycles of size at most 3 whose structure is as described in Figure 1.*
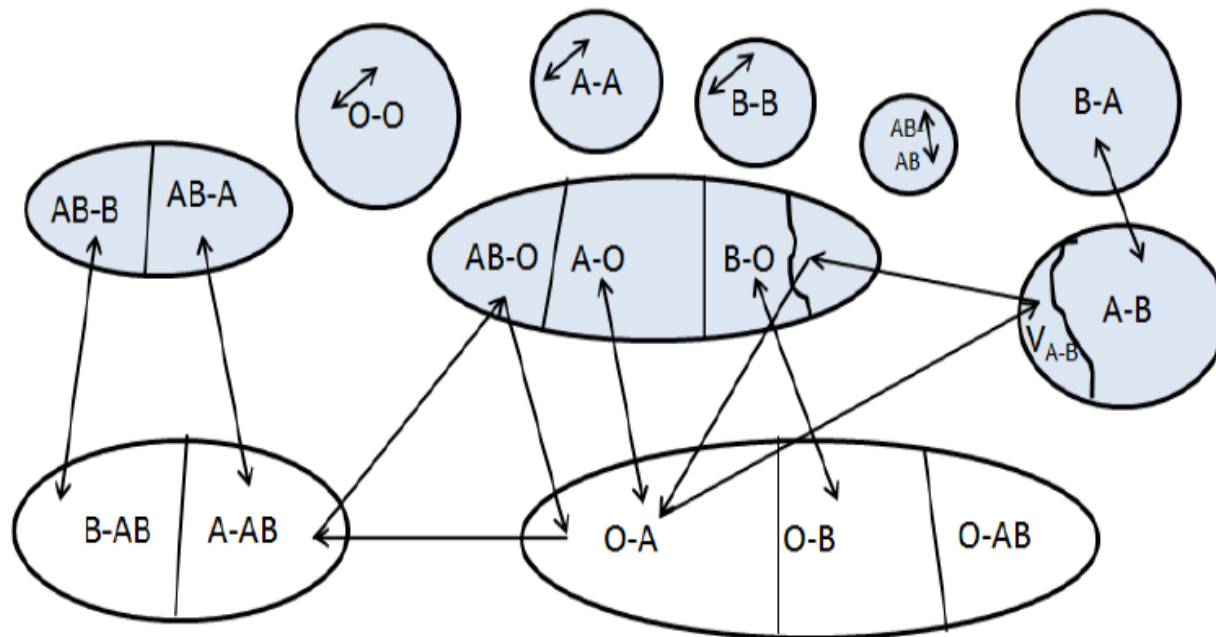
Over-demanded (shaded) pairs are all matched.

How about when hospitals become players?

- We are seeing some hospitals withhold internal matches, and contribute only hard-to-match pairs to a centralized clearinghouse.

- Mike Rees (APD director) writes us: "As you predicted, competing matches at home centers is becoming a real problem. Unless it is mandated, I'm not sure we will be able to create a national system. I think we need to model this concept to convince people of the value of playing together".

Individual rationality and efficiency: an impossibility theorem with a (discouraging) worst-case bound

- For every $k \geq 3$, there exists a compatibility graph such that no k-maximum allocation which is also individually rational matches more than $1/(k-1)$ of the number of nodes matched by a k-efficient allocation.

Proof (for k=3)
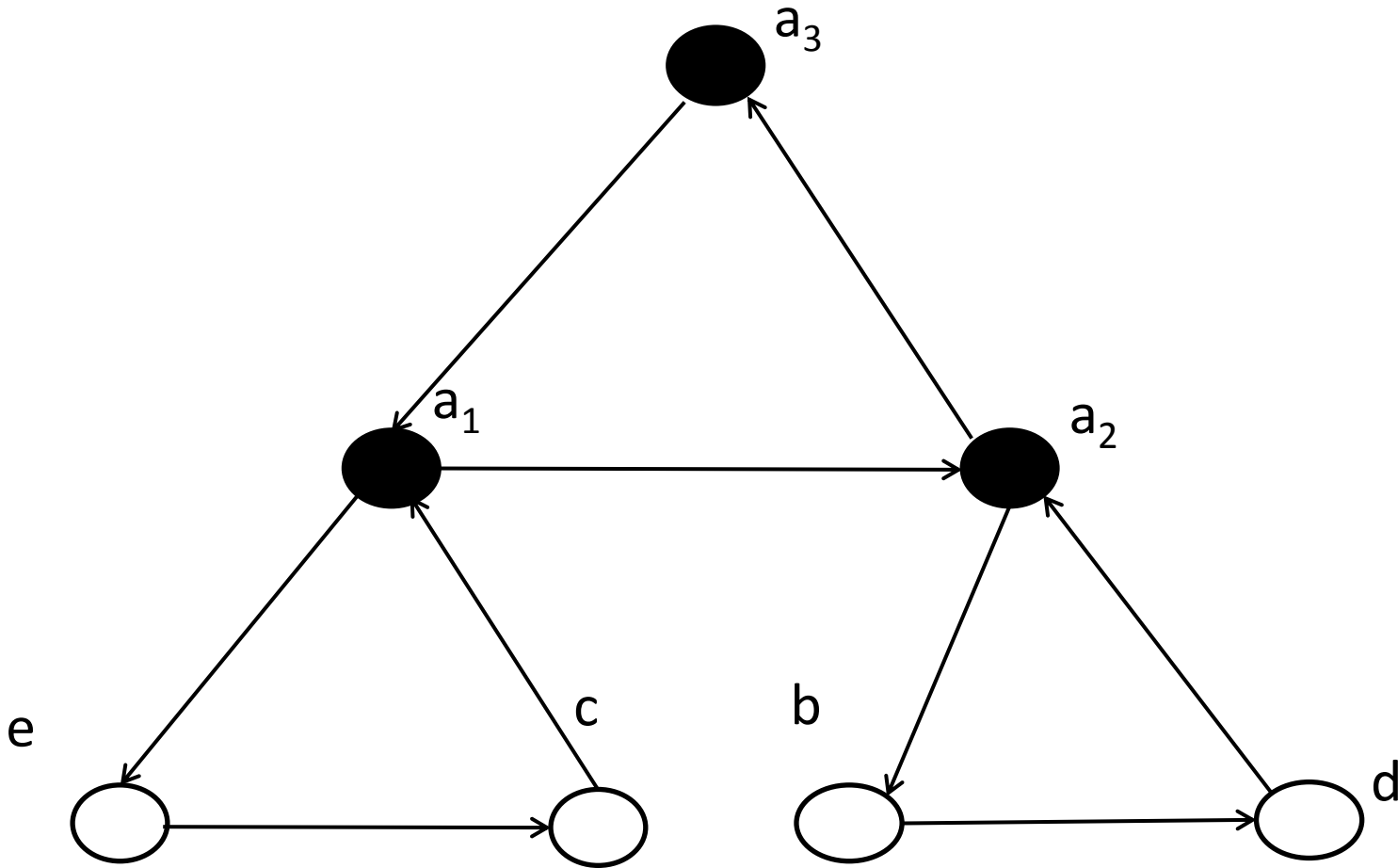
There are incentives for Transplant Centers not to fully participate even when there are only 2-way exchanges

The exchange A1-A2 results in **two** transplantations, but the exchanges A1-B and A2-C results in **four**.
(And you can see why, if Pairs A1 and A2 are at the same transplant center, it might be good for them to nevertheless be submitted to a regional match…)

(high priority)
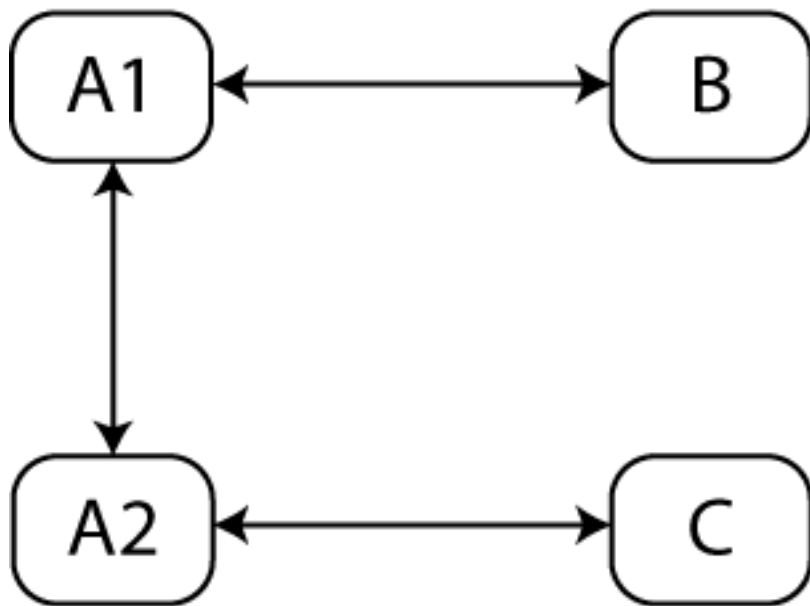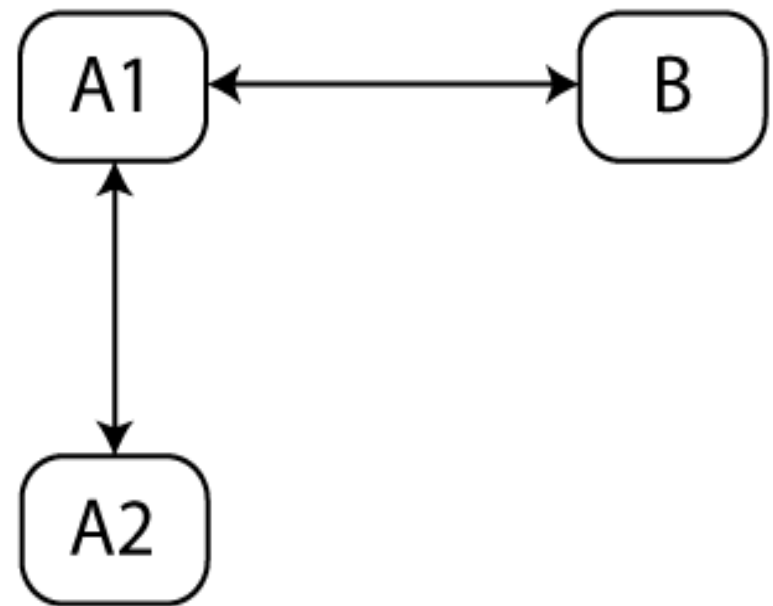
Fig. 1a   Fig. 1b

As kidney exchange has grown, we also have to worry about inefficient withholding of more complex exchanges

Why 4 way exchanges don't help:
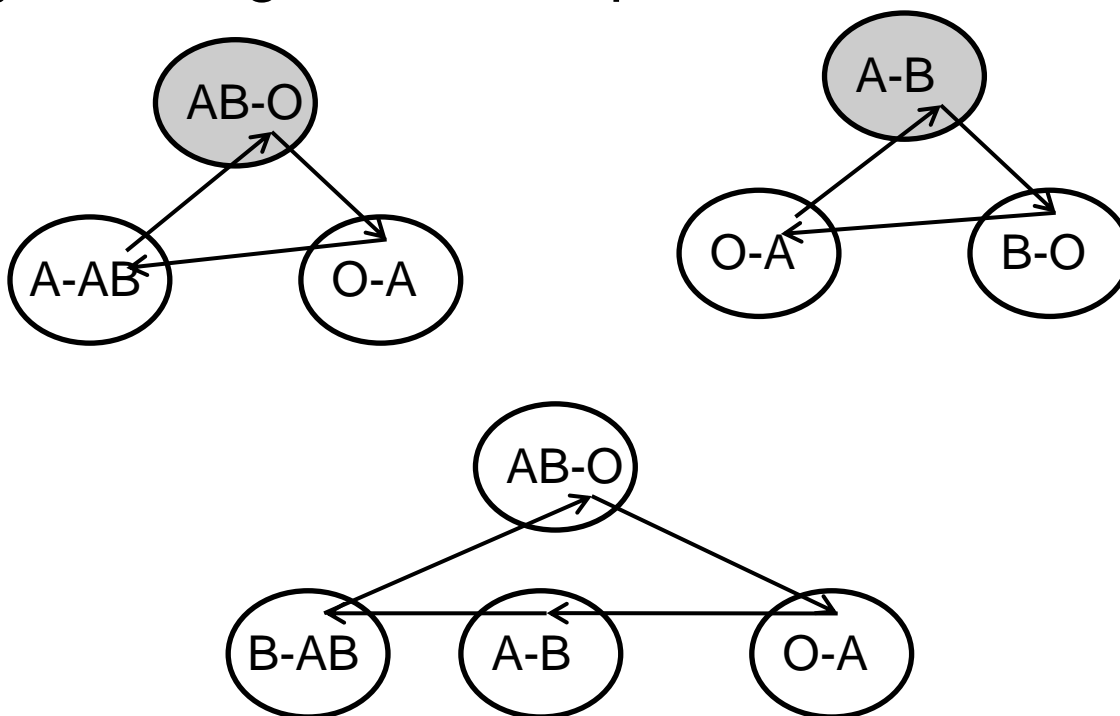
Individually Rational Allocations

Theorem: If every hospital size is regular and bounded than in almost every large graph the efficiency loss from a maximum individually rational allocation is at most $(1+\epsilon)\alpha_{AB-O}m + o(m)$ for any $\epsilon>0$ (less than 1.5%).

So the worst-case impossibility results don't look at all like what we could expect to achieve in large kidney exchange pools (if individually rational mechanisms are adopted).

"Cost" of IR is very small for clinically relevant sizes too - Simulations

No. of Hospitals	2	4	6	8	10	12	14	16	18	20	22
IR,k=3	6.8	18.37	35.42	49.3	63.68	81.43	97.82	109.01	121.81	144.09	160.74
Efficient, k=3	6.89	18.67	35.97	49.75	64.34	81.83	98.07	109.41	122.1	144.35	161.07

But the cost of not having IR could be very high if it causes centralized matching to break down

No. of Hospitals	Num Of Pairs	Decentralized k=2	Centralized k=2	Decentralized Exchange k=3	Centralized Exchange k=3
2	21	3.46	5.26	4.36	6.89
4	42	6.6	13.58	8.32	18.67
6	67	11.72	25.62	14.73	35.97
8	85	14.4	36.52	18.04	49.75
10	108	17.52	47.74	22.87	64.34
12	131	22.32	60.6	28.16	81.83
14	154	26.44	74.72	33.85	98.07
16	173	28.76	84.2	36.58	109.41
18	191	31.78	95.62	39.75	122.1
20	227	38.7	116.68	49.79	144.35
22	252	44.52	131.5	55.85	161.07

But current mechanisms aren't IR for hospitals

- **Current mechanisms:** Choose (~randomly) an efficient allocation.

Proposition: Withholding internal exchanges can (often) be strictly better off (non negligible) for a hospital regardless of the number of hospitals that participate.
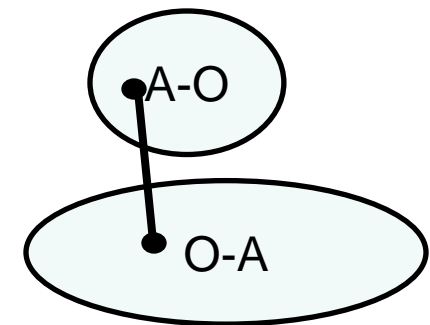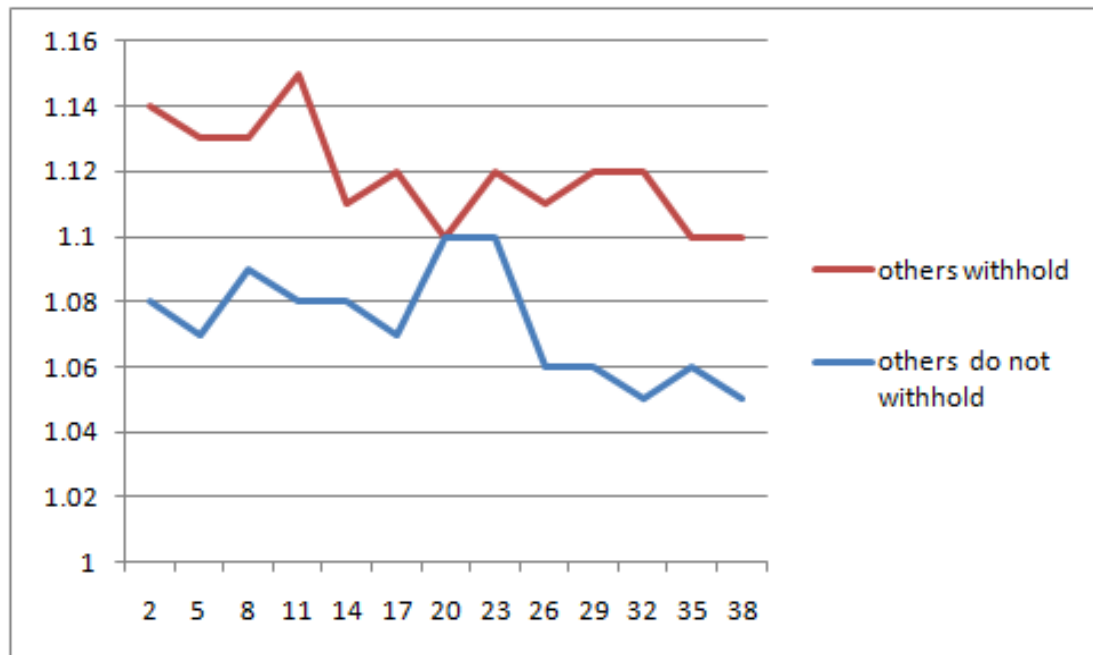
And hospitals can withhold individual overdemanded pairs

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Possible solution:

- "Frequent flier" program for transplant centers that enroll easy to match pairs.
- Their O patients can be included in exchanges with scarce O donors…
- Theorem: almost efficient mechanisms with truth-telling equilibria exist…

Other sources of efficiency gains

- Non-directed donors

P1 ← ND-D

P3 ← P2-D2 ← P1-D1 ← ND-D
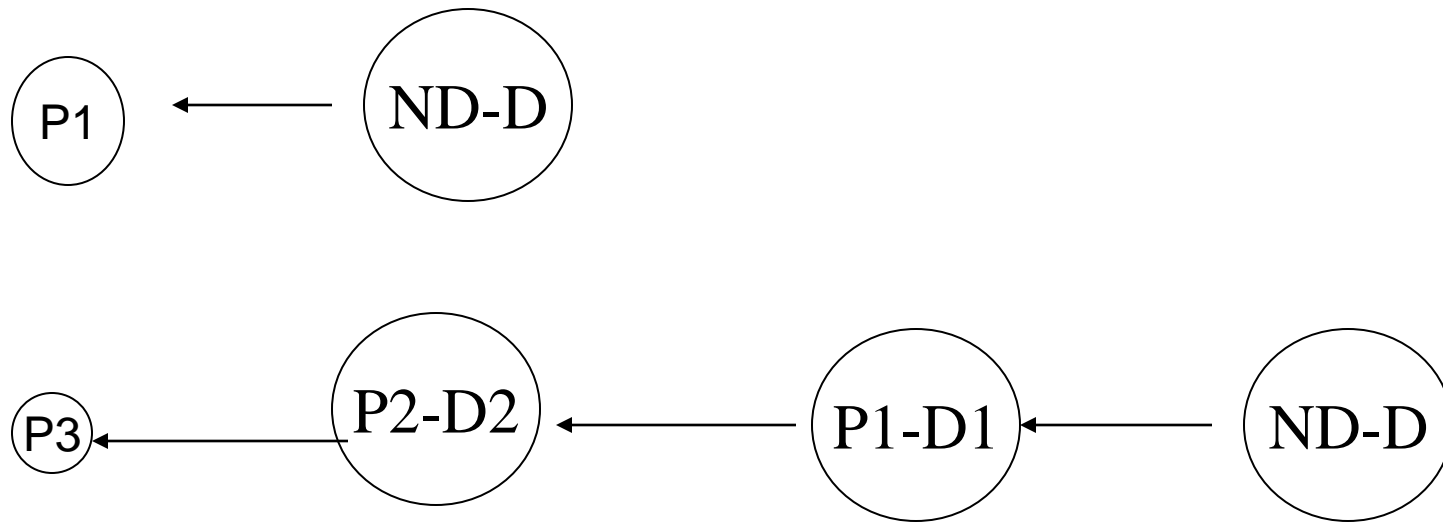
The graph theory representation doesn't capture the whole story

Rare 6-Way Transplant Performed

Donors Meet Recipients

March 22, 2007

BOSTON -- A rare six-way surgical transplant was a success in Boston.

NewsCenter 5's Heather Unruh reported Wednesday that three people donated their kidneys to three people they did not know. The transplants happened one month ago at Massachusetts General Hospital and Beth Israel Deaconess.

The donors and the recipients met Wednesday for the first time.

Why are there only 6 people in this picture?

Simultaneity congestion: 3 transplants + 3 nephrectomies = 6 operating rooms, 6 surgical teams…

Can simultaneity be relaxed in Non-directed donor chains?

- Cost-benefit analysis:
- "*If something goes wrong in subsequent transplants and the whole ND-chain cannot be completed, the worst outcome will be no donated kidney being sent to the waitlist and the ND donation would entirely benefit the KPD [kidney exchange] pool.*" (Roth, Sonmez, Unver, Delmonico, and Saidman) *AJT* 2006, p 2704).

Non-simultaneous extended altruistic donor chains (reduced risk from a broken link)
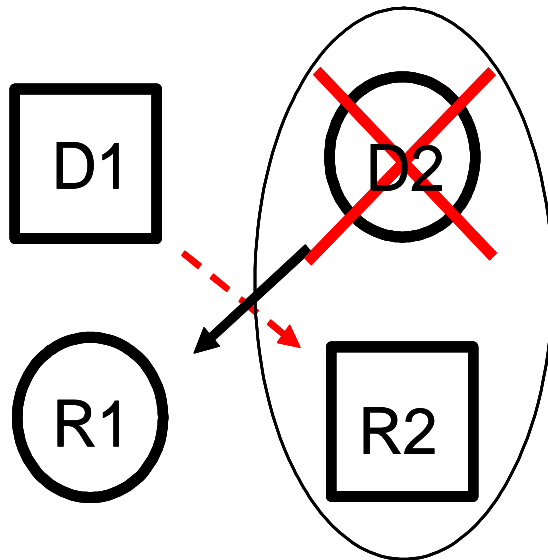
A. Conventional 2-way Matching
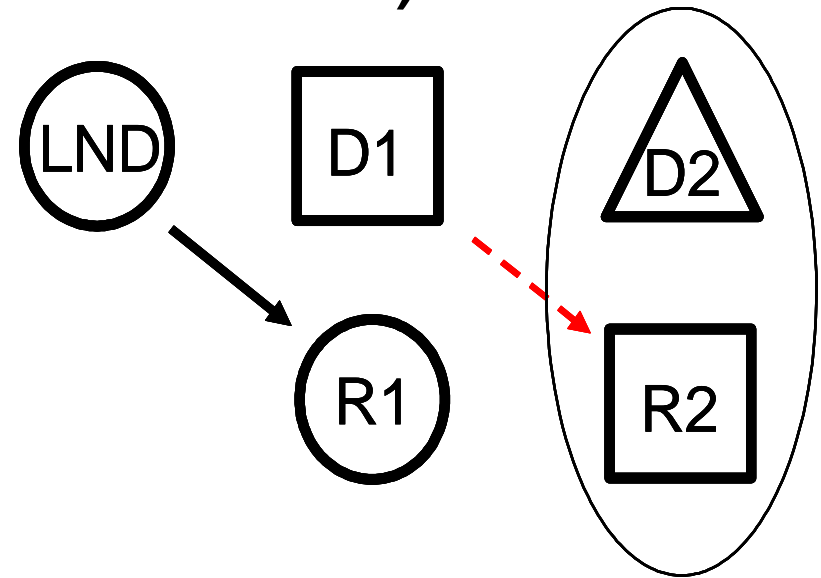
B. NEAD Chain Matching

Since **N**EAD chains don't require simultaneity, they can be longer…

A Nonsimultaneous, Extended, Altruistic-Donor Chain

Michael A. Rees, M.D., Ph.D., Jonathan E. Kopke, B.S., Ronald P. Pelletier, M.D.,
Dorry L. Segev, M.D., Matthew E. Rutter, M.D., Alfredo J. Fabrega, M.D.,
Jeffrey Rogers, M.D., Oleh G. Pankewycz, M.D., Janet Hiller, M.S.N.,
Alvin E. Roth, Ph.D., Tuomas Sandholm, Ph.D., M. Utku Ünver, Ph.D.,
and Robert A. Montgomery, M.D., D.Phil.

SUMMARY

We report a chain of 10 kidney transplantations, initiated in July 2007 by a single altruistic donor (i.e., a donor without a designated recipient) and coordinated over a period of 8 months by two large paired-donation registries. These transplantations involved six transplantation centers in five states. In the case of five of the transplantations, the donors and their coregistered recipients underwent surgery simultaneously. In the other five cases, "bridge donors" continued the chain as many as 5 months after the coregistered recipients in their own pairs had received transplants. This report of a chain of paired kidney donations, in which the transplantations were not necessarily performed simultaneously, illustrates the potential of this strategy.

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The First NEAD Chain (Rees, APD)

	July 2007 AZ 1	July 2007 OH 2	Sept 2007 OH 3	Sept 2007 OH 4	Feb 2008 MD 5	Feb 2008 MD 6	Feb 2008 MD 7	Feb 2008 NC 8	March 2008 MD 9	March 2008 OH 10
Recipient PRA	62	0	23	0	100	78	64	3	100	46
Recipient Ethnicity	Cauc	Cauc	Cauc	Cauc	Cauc	Hisp	Cauc	Cauc	Cauc	AA
Relationship	Husband Wife	Mother Daughter	Daughter Mother	Sister Brother	Wife Husband	Father Daughter	Husband Wife	Friend Friend	Brother Brother	Daughter Mother

* This recipient required desensitization to Blood Group (AHG Titer of 1/8).

\# This recipient required desensitization to HLA DSA by T and B cell flow cytometry.

THE KIDNEY CHAIN

How a single organ donation changed 20 lives and created the longest-running transplant chain

MATT JONES, 30
Petoskey, Mich.
First donor

— donated to →

BARBARA BUNNELL, 56
Phoenix

— married to →

RON BUNNELL, 56
Phoenix

— donated to →

ANGELA HECKMAN, 34
Toledo, Ohio

— daughter of →

LAURIE SARVO, 54
Toledo, Ohio

— donated to →

REYNALDO ESPINOZA, 59
Germantown, Md.

— donated to →

CLAUDIA ALAS, 32
Germantown, Md.

— daughter of →

JEAN STAYLOR, 53
Charleston, S.C.

— donated to →

RAYMOND STAYLOR, 53
Charleston, S.C.

— married to →

AVA ROBY, 54
Marysville, Ohio

— donated to →

GEORGE LEOHNER, 51
Chillicothe, Ohio

— brother of →

LINDA JANISIESKI, 42
Miamisburg, Ohio

— donated to →

CECILIA JANISIESKI, 71
Huber Heights, Ohio

— mother of →

ANONYMOUS RECIPIENT

— married to →

ANONYMOUS DONOR

— donated to →

BILL CORAM, 55
Lincolnton, N.C.

— friend of →

TIM SHAIN, 43
Lincolnton, N.C.

— donated to →

LINLEY BLENKENSOPP, 51
Patchogue, N.Y.

— brother of →

KURT BLENKENSOPP, 41
Patchogue, N.Y.

— donated to →

KATHERINE McKINNEY, 62
Toledo, Ohio

— mother of →

HELEENA McKINNEY, 29
Cincinnati
Donor-in-waiting

Dr. Mike Rees (center, left) and his team perform a kidney transplant.

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Why are NEAD chains so effective?

- In a really large market they wouldn't be…

Chains in an efficient large dense pool

O-O

A-A

B-B

AB-AB

B-A

AB-B | AB-A

AB-O | A-O | B-O

V_{A-B}

A-B

B-AB | A-AB

O-A | O-B | O-AB
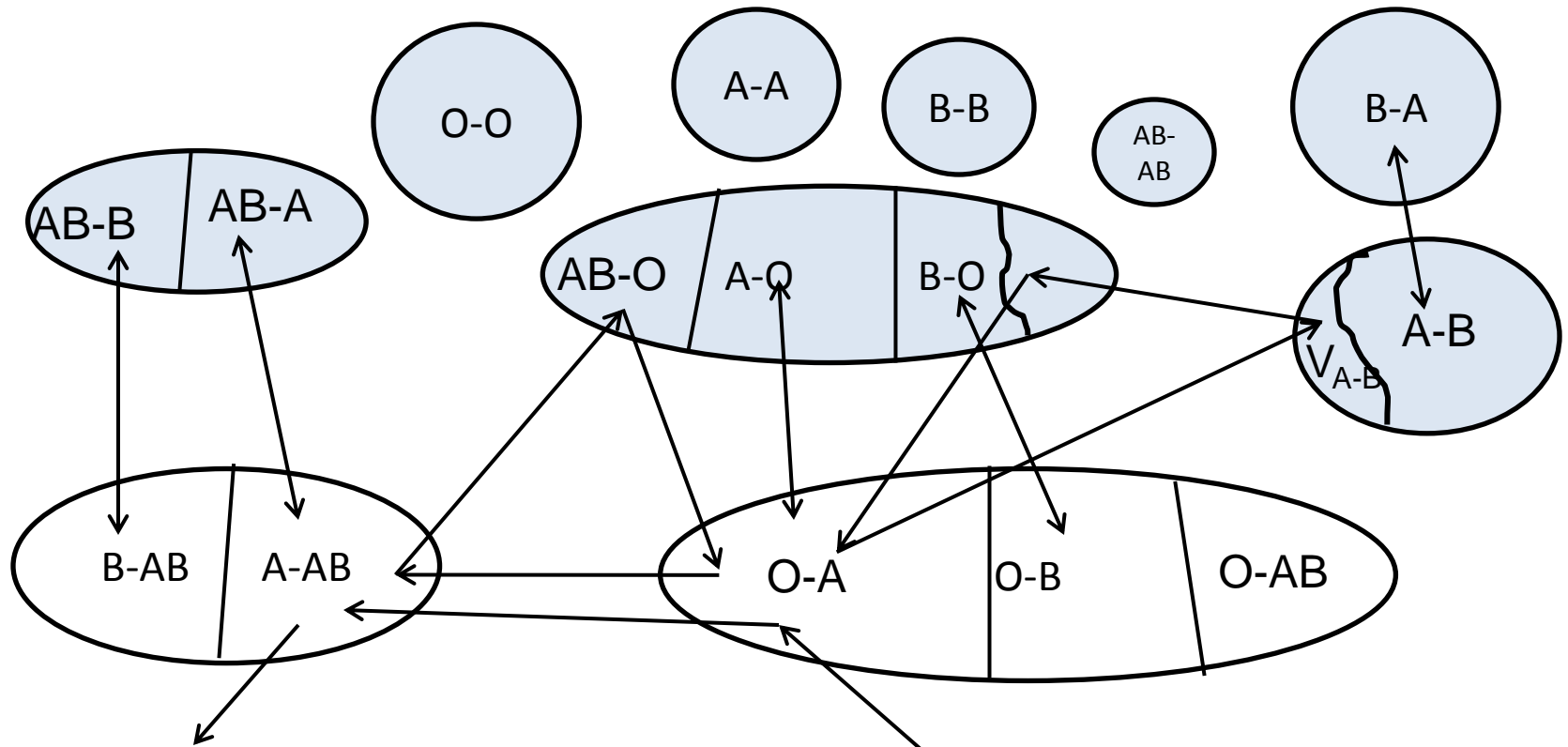
It looks like a non-directed donor can increase the match size by at most 3 ☹

Non-directed donor—blood type O

A disconnect between model and data:

- The large graph model with constant p (for each kind of patient-donor pair) predicts that only short chains are useful.

- But we now see long chains in practice.

- They could be inefficient—i.e. competing with short cycles for the same transplants.

- But this isn't the the case when we examine the data.

Why? Very many very highly sensitized patients

	UNOS	Previous simulations	Historical APD set	1st APD set	2nd APD set
Size			361	131	144
Low PRA	90.19	0.87	48.20	34.36	40.2
High PRA	9.81	0.13	51.80	65.64	59.8

Table 1: Percentages of Low and High PRA patients in various data sets and standard simulations

Previous simulations: sample a patient and donor from the general population, discard if compatible (simple live transplant), keep if incompatible. This yields 13% High PRA.

The much higher observed percentage of high PRA patients means compatibility graphs will be sparse

	O-O	B-B	A-A
Percentage of high PRA	73	69	81
Average PRA	76.1	82.6	78.3
Average PRA for high PRA	95.09	99.5	96.5
Average PRA for low PRA	41.3	60	44.3

Table 4: PRA statistics in subgraphs of pairs in which the donor and patient have the same blood type from the APD data (X-X pairs for X = A, B, or O). The first row is the percentage of High PRA patients in the pool, the second row is the average PRA. The third row provides the average PRA among high PRA patients and the fourth row gives the average PRA among low PRA patients.

Short cycles leave many highly sensitized patients unmatched

Size		$k = 2$	$k = 3$	$k = 4$	$k = 5$
	matches	36	52	58	58
83 (26, 57)	matched (L,H)	(21, 15)	(26, 26)	(26, 32)	(26, 32)

Table 3: Cycles in the graph induced by O donors and O patients in the historical data set (overall 83 O-O pairs). High PRA is considered to be 80. The number of patients in the pool is given in the first column, and in parentheseses the numbers of low and high PRA patients respectively. The rows in each column describe: (i) average number of matches obtained, and (ii) number of matched low and high PRA patients. 18 of the patients did not have any compatible donor in this graph.

Long chains in the clinical data: even a single non-directed donor can start a long chain

	$k=2$	$k=3$	$k=4$	$k=5$	$k=\infty$	$(3,\infty)$	(∞,∞)	$(3,5)$
Historical APD set	118	169	189	191	193	194.25	194.25	172
1st APD set	6	9	11	13	16	17.3	17.3	10.1
2nd APD set	8	12	16	16	16	17.9	17.9	14.1

Table 2: Allocations in Data. In the first 5 columns, we report the size of the k-efficient allocation where ∞ stands for unrestricted k. In the last 3 columns the average size of the (k,l)-efficient allocation is given using a single non-directed donor that is chosen at random, i.e. the average size of maximum allocation using cycles up to size k and a chain up to size l.

Long chains in the clinical data: even a single non-directed donor can start a long chain

MAX Cycle Cover

Subject to: Cycle length $\leq k$

	Set size	$k=2$	$k=3$	$k=4$	$k=5$	$k=\infty$
Historical 1st Set	361	118	169	189	191	193
1st Set (a)	131	6	9	11	13	16
1st Set (b)	251	8	12	16	16	16
2nd Set	74	22	42	45	47	51

One donor added

	Set size	$k=2$	$k=3$	$k=4$	$k=5$	$k=\infty$	$(3,\infty)$	(∞,∞)	$(3,5)$
Historical 1st Set	361	118	169	189	191	193	194.25	194.25	172
1st Set (a)	131	6	9	11	13	16	17.3	17.3	10.1
1st Set (b)	251	8	12	16	16	16	17.9	17.9	14.1
2nd Set	74	22	42	45	47	51	54.5	54.5	47.6

Graph induced by pairs with A patients and A donors. 38 pairs (30 high PRA).
Dashed edges are parts of cycles.
No cycle contains only high PRA patients.
Only one cycle *includes* a high PRA patient
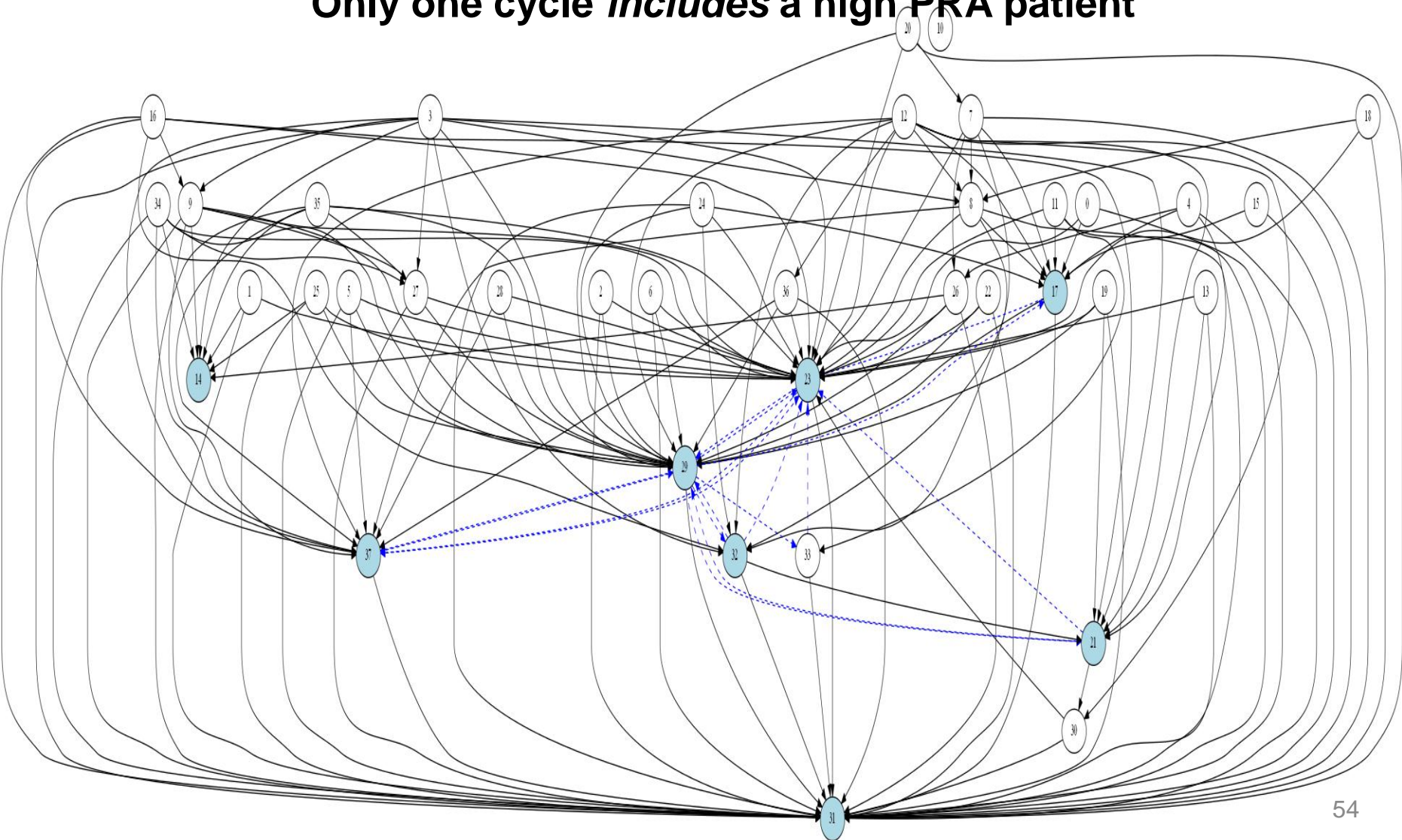
Jellyfish structure of the compatibility graph: highly connected low sensitized pairs, sparse hi-sensitized pairs

So we need to model sparse graphs…

- We'll consider random graphs with two kinds of nodes (patient-donor pairs): Low sensitized and high sensitized

- L nodes will have a constant probability of an incoming edge (compatible kidney)

- H nodes will have a probability that **decreases** with the size of the graph (e.g. in a simple case we'll keep the number of compatible kidneys constant, pH = c/n) more generally,

$$cn^{-1+\varepsilon} >> \left(\frac{\log n}{n}\right)$$

- In the H subgraph, we'll observe trees but almost no short cycles

- **A non-directed donor can be modeled as a donor with a patient to whom anyone can donate—this allows non-directed donor chains to be analyzed as cycles**

- (We also consider the effect of different assumptions about how the number of non-directed donors grows…)

Cycles and paths in random dense-sparse graphs

- n nodes. Each node is L w.p. $v \leq 1/2$ and H w.p. $1-v$

$$G\left(n, p_L, \frac{p_H}{n}, v\right)$$

- incoming edges to L are drawn w.p. p_L

- incoming edges to L are drawn w.p. $\dfrac{p_H}{n}$
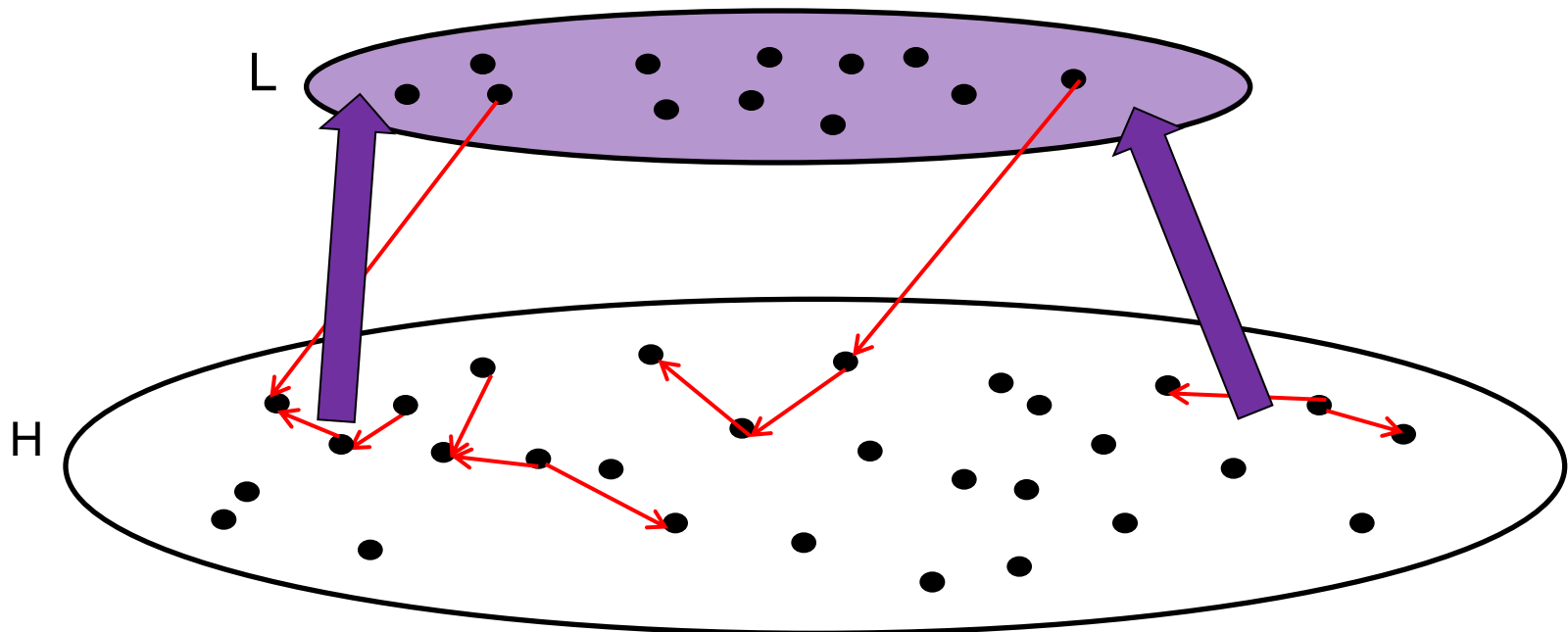
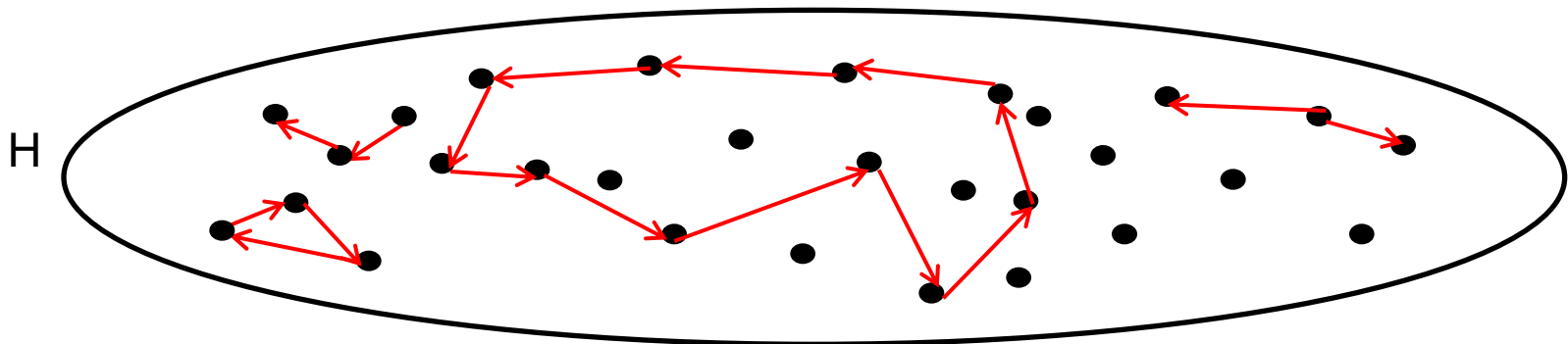# Cycles and paths in random sparse (sub)graphs
(v=0, only highly sensitized patients)

Theorem.
(a) The number of cycles of length **O(1)** is **O(1)**.
(b) But when p_H is a large constant there is cycle with length **O(n)**

"Proof" (a):

$$\mathbb{E}[\text{Cycles} \leq k] = \binom{n}{k} k! \left(\frac{p_H}{n}\right)^k = \frac{n(n-1)\cdots(n-k+1)}{k!} k! \left(\frac{p_H}{n}\right)^k$$

$$\approx p_H^k = O(1).$$

H

To be logistically feasible, a long cycle must be a chain, i.e. contain a NDD

Cycles and paths in random sparse graphs (v=0)

Theorem.
(a) The number of cycles of length **O(1)** is **O(1)**.
(b) But when p_H is a large constant there is path with length **O(n)**

*Since cycles need to be short (as they need to be conducted simultaneously) but chains can be long (as they can be initiated by an altruistic donor,) the value of a non-directed donor is **very large**!*
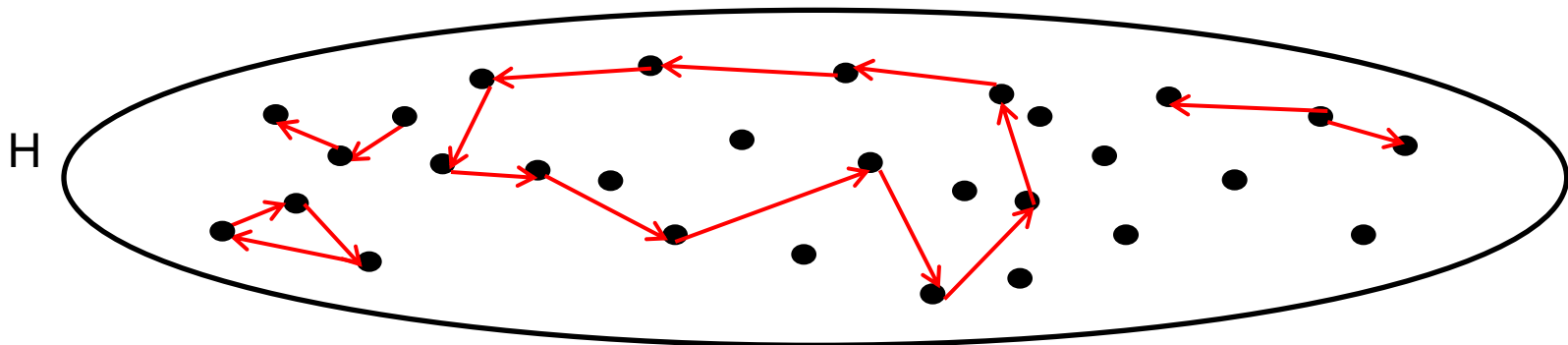
Case v>0 (some low sensitized, easy to match patients. Why increasing cycle size helps

Theorem. Let C_k be the largest number of transplants achievable with cycles $\leq k$. Let D_k be the largest number of transplants achievable with cycles $\leq k$ **plus one non-directed donor**. Then for every constant k there exists $\rho>0$

$$\mathbb{E}[C_{k+1}] \geq \mathbb{E}[C_k] + \rho n \qquad \text{and} \qquad \mathbb{E}[D_k] \geq \mathbb{E}[C_k] + \rho n$$

Furthermore, C_k and D_k cover almost all L nodes.
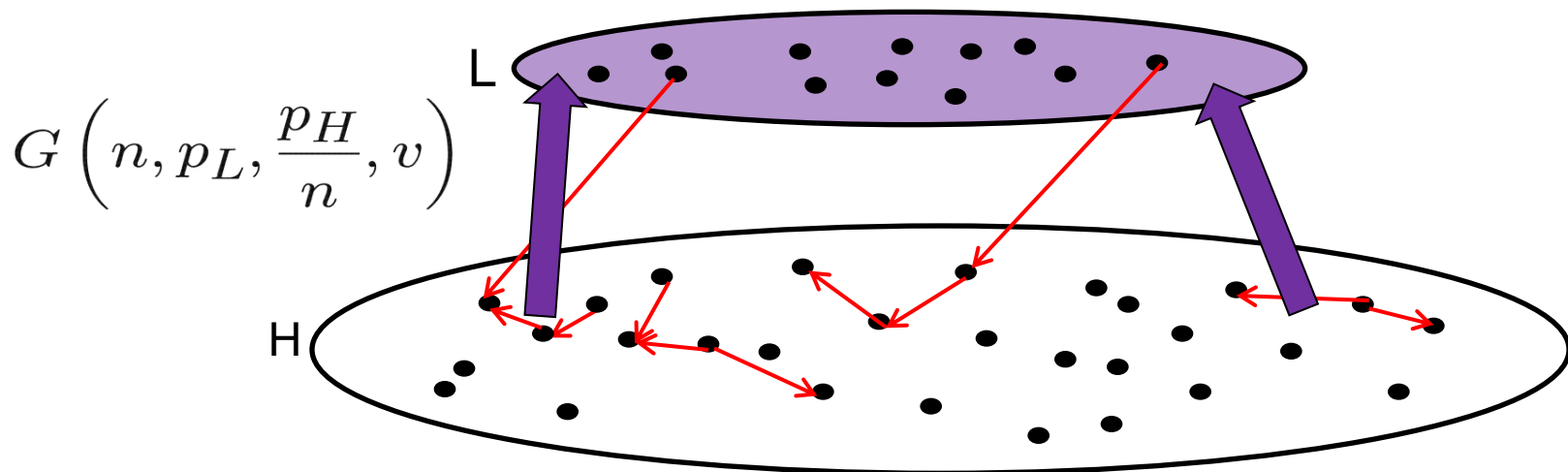
$$G\left(n, p_L, \frac{p_H}{n}, v\right)$$

Case $v>0$. Why increasing cycle size helps

Increasing cycle lengths significantly increases transplants. Highly sensitized patients are the principal beneficiaries.

Low sensitized pairs of all blood types are overdemanded: it's easy to start a cycle from L to H since there are many H, and easy to end it back in L since most blood type compatible donors will do…

Case $v>0$. Why increasing cycle size helps

Increasing cycle lengths significantly increases transplants. Highly sensitized patients are the principal beneficiaries.

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Simulations (re-sampling) with clinical data

Size	NDDs	$(2,3)$	$(3,3)$	$(3,4)$	$(3,\infty)$	(∞,∞)
40 (28.25)	0	10.63 (3.63)	15.71 (6.17)	15.71 (6.17)	15.71 (6.17)	18.63 (8.25)
	1	12.04 (4.04)	16.24 (6.29)	16.5 (6.48)	17.76 (7.41)	18.71 (8.1)
	5	16.5 (6.63)	18.86 (8.01)	19.54 (8.63)	20.54 (9.36)	20.61 (9.39)
60 (42.64)	0	18.18 (6.65)	26.3 (11.37)	26.3 (11.37)	26.3 (11.37)	30.64 (14.78)
	1	20 (7.42)	27.4 (11.74)	27.68 (11.97)	29.86 (13.83)	31.23 (14.91)
	5	24.82 (10.07)	29.87 (13.51)	30.72 (14.27)	32.73 (15.99)	32.81 (16.08)
80 (56.89)	0	26.89 (10.22)	38.8 (17.7)	38.8 (17.7)	38.8 (17.7)	43.82 (22.17)
	1	28.05 (10.84)	38.57 (17.46)	38.84 (17.69)	41.9 (20.45)	43.89 (22.22)
	5	34.81 (14.57)	43.05 (20.37)	44.25 (21.46)	46.68 (23.7)	46.73 (23.78)
100 (70.67)	0	35.26 (13.63)	50.3 (23.35)	50.3 (23.35)	50.3 (23.35)	56.56 (29.14)
	1	35.86 (13.82)	50.35 (23.32)	50.72 (23.65)	54.28 (27)	56.38 (29.01)
	5	43.39 (18.01)	54.31 (26.05)	55.51 (27.2)	58.91 (30.34)	58.98 (30.42)

Table 6: Average number of matched patients through (k,l)-efficient allocations (cycles up to length k and chains up to length l) with different numbers of non-directed donors (NDDs) and different size pools. Pairs are all blood type compatible, drawn from the O-O and A-O pairs. The number of matched high PRA patients is given in parenthesis.

Long chains benefit highly sensitized patients (without harming low-sensitized patients)
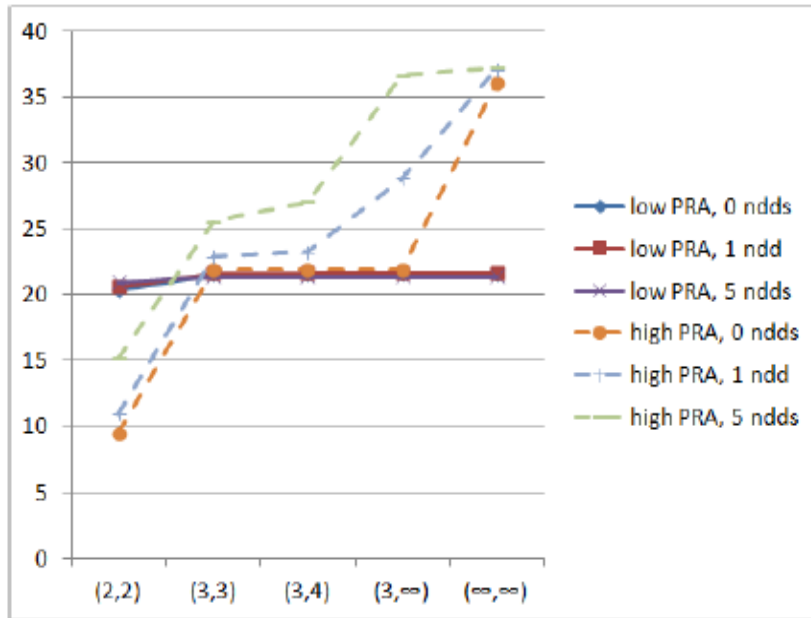
(a) 80 nodes.
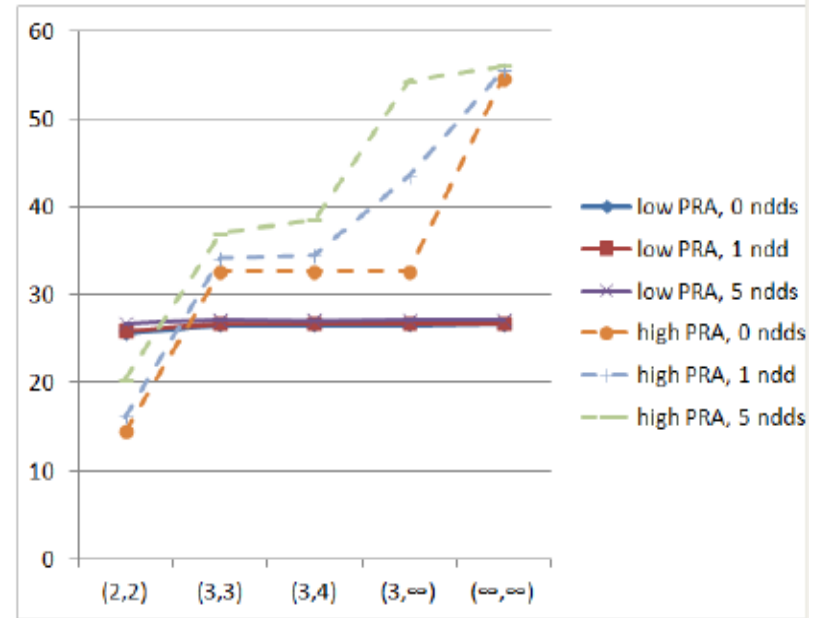
(b) 100 nodes.

Figure 4: Number of matches and high PRA matches (y-axis) obtained for different maximum cycle chain lengths (the x-axis indicates the type of efficient allocation). The percentage of L nodes is 0.27 the probability for an H node to have an incoming link from any given node is set to $p_H = 0.03$. Note low PRA patients are not harmed by the increase in cycle and chain length, while the benefits go al entirely to high PRA patients. (Note that all the low PRA lines almost completely overlap.)

NKR non-directed donor chain:2012. 60 lives, 30 kidneys: the practical implications are clear

What would it take to make long chains unnecessary?

- Many low-sensitized patients in the pool
- Many non-directed donors.

- With enough of those, small cycles (but not necessarily as small as k=3) and short chains would be sufficient.

But progress is still slow☹

- When we started, there were only 40,000 people on the US deceased-donor waiting list, and now there are over 90,000