Matching: The Theory

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NRMP match variations:

- Couples: Submit rank oders over pairs of programs, and must be matched to two positions
- Applicants who match to 2nd year positions and have supplemental lists which then must be consulted to match them to 1st year positions
- Some residency propgrams require odd or even number of matches
- Some residency programs require reversions of unfilled positions from one program to another.

Theorems about simple markets whose conclusions are wrong with NRMP

1. The set of stable matchings is always nonempty
2. The set of stable matchings always contains a "program optimal" stable matching, and an "applicant optimal" stable matching.
3. The same applicants are matched and the same positions are filled at every stable matching.

Some descriptive statistics

	1987	1993	1994	1995	1996
Applicants					
ROLS	20071	20916	22353	22937	24749
Appl that are coupled	694	854	892	998	1008
Programs					
Active programs	3170	3622	3662	3745	3758
Total quota	19973	22737	22801	22806	22578
Programs: reversion	69	247	276	285	282
positions to be reverted	225	1329	1467	1291	1272
programs: even matching	4	2	6	7	8

Difference between program and applicant proposing algorithm

	1987	1993	1994	1995	1996
Applicants					
Affected Appl	20	16	20	14	21
Prefer Appl. Prop	12	16	11	14	12
Prefer Program prop	8	0	9	0	9
new matched	0	0	0	0	1
new unmatched	1	0	0	0	0
Programs					
affected	20	15	23	15	19
Prefer Appl. prop.	8	0	12	1	10
prefer program prop	12	15	11	14	9
prog. w. new matched	0	0	2	1	1
prog w. new unmatched	1	0	2	0	0

Upper limit of the number of applicants who could benefit by truncating their lists at one above their original match point: (for students, truncation is exhaustive)

	1987	1993	1994	1995	1996
Program-proposing	12	22	13	16	11
Applicant proposing	0	0	2	2	9

Upper limit of the number of hospital programs that could benefit by truncating their lists at one above their original match point (for hospitals, truncation is not exhaustive: dropping strategies is)

	1987	1993	1994	1995	1996
Program-proposing	15	12	15	23	14
Applicant proposing	27	28	27	36	18

Estimate of the Upper Bound of the Number of Programs That Could Improve Their Remaining Matches By Reducing Quotas

	1987	1993	1994	1995	1996
Program-proposing	28	16	32	8	44
Applicant proposing	8	24	16	16	32

This will be worth thinking about again—a small cloud on the horizon—when we consider what temptations may exist for residency programs to hire some of their people early, before the match. If there are such temptations, they may not be counterbalanced by a tendency to do worse in the match, on the contrary, reducing demand may have small spillover benefits in the match for the remaining candidates...

- Set of stable outcomes appears to be small: few applicants or programs are affected by a switch from program proposing to applicant proposing
- The opportunities to misrepresent preferences or capacities seem small

But we don't really understand the structure of the set of stable matchings when there
are couples, supplementary lists, and reversion of positions from one program to another.

- We know that program and applicant optimal stable matchings no longer exist.
- Maybe the set of stable matchings isn't all located between these two matchings when all the match variations are present; maybe the set of stable matchings just appears to be small because we don't know where to look.

Two approaches to study these issues further:

- Empirical: examine some simple markets
- Theoretical/computational: explore some artificial simple markets

The Thoracic surgery match is a simple match, with no match variations. It exactly fits the college admissions model; those theorems all apply

Descriptive statistics and original Thoracic Surgery match results

	1991	1992	1993	1994	1996
Applicants	127	183	200	197	176
Program	62	86	90	93	92
Quota	93	132	136	146	143
positions filled	79	123	136	140	132
Difference	0	2 app better 2 prog worse	2 app better 2 prog worse	0	0

Alternative approach: Use simulations to compute the size of the core for well-behaved problems: Roth and Peranson (1999)

Simple model: n firms, n workers, (no couples) uncorrelated preferences, each worker applies to k firms.

Let $C(n)$ = number of workers matched differently at firm and worker optimal stable match.

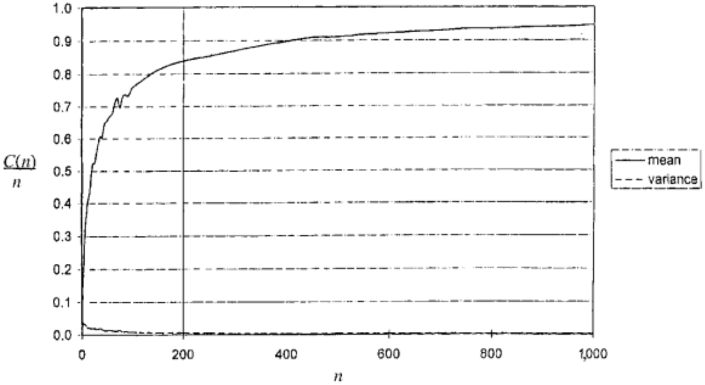

FIGURE 1. SIZE OF THE SET OF STABLE MATCHINGS AS A FRACTION OF n, WHEN $k = n$ (UNCORRELATED PREFERENCES

Note: $C(n)$ is the number of applicants who get different stable matches, when the market size is n.

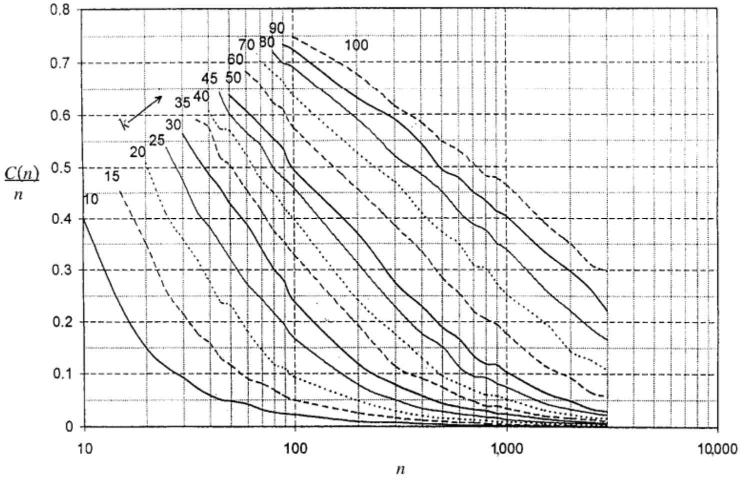

FIGURE 2. SIZE OF THE SET OF STABLE MATCHINGS AS A FRACTION OF n FOR DIFFERENT VALUES OF k (UNCORRELATED PREFERENCES)

Notes: $C(n)$ is the number of applicants who get different stable matches, when the market size is n; k is the number of programs on an applicant's ROL.

The numerical results show us that $C(n)/n$ gets small as n gets large when k is fixed, (even) for uncorrelated preferences.

In these simulated markets, the core gets small not because of strategic behavior—these are the true preferences.

This also implies that in large markets it is almost a dominant strategy for every agent to reveal his true preferences—only one in a thousand could profit by strategically mis-stating preferences (if they had full information about all preferences).

From simulations to theory...

The new (and current) NRMP algorithm, called the Roth-Peranson algorithm, is based on student-proposing DA, but try to accommodate couples.

The algorithm allows couples to express preferences on pairs of hospital programs.

First run DA without couples, and then add couples one at a time.

If someone is displaced, then such an agent is allowed to apply later in the algorithm.

The basic idea is based on Roth and Vande Vate (1989) on one-to-one matching.

Nicole Immorlica and Mohammad Mahdian, “Marriage, Honesty and Stability,” SODA 2005, 53-62.

Theorem (Immorlica and Mahdian, 2005)
Consider a marriage model with n men and n women, in which each woman has an arbitrary complete preference list, and each man has a random list of at most k women as his preference list (chosen uniformly and independently).

Then, the expected number of women who have more than one stable husband is bounded by a constant that only depends on k (and not on n). So, as n gets large, the proportion of such women goes to zero. . .

What about the manipulations by firms?

Theorem (Kojima and Pathak, AER 2009): In the limit, as n goes to infinity in a regular sequence of random (many-to-one) markets, the proportion of employers who might profit from (any combination of) preference or capacity manipulation goes to zero in the worker proposing deferred acceptance algorithm.

To provide an intuition about the proof: Recall: How does truncation affect the outcome in a DA algorithm?

Consider a simple one-to-one market: Suppose we consider a man-proposing DA. So, women have an incentive to truncate.

How does that work? By declaring the man-optimal stable match partner unacceptable, the woman rejectes that man, who in turn, applies to another woman, who rejects a man...who then makes an offer to the initial woman so, she receives more offers, eventually from a man she prefers to the one she rejected. The question is, how likley are such rejection chains to come back to the women who launch them?

- There is a finite sets S of students and C of colleges.
- Each student can be matched to at most one college, and college c can be matched with at most q_c students (many-to-one matching).
- Assume there are no match variations (no couple, etc).

There are constants $q; \widetilde{q}; k$ (independent of n). G^n is a game of incomplete information such that:

- there are n colleges, with quota at most q.
- there are at most $\widetilde{q}n$ students.
- Preferences of colleges are common knowledge (the result holds under incomplete information as well).
- Utility of a college is additive in all students, and every student is acceptable (that is, the utility for any student is positive). Furthermore, the utility of the most desired student, for all n is finite (value of the most desired student grow to infinity as n increases).

- Preferences of students are private information. A student's preference list is drawn from a uniform distribution over preference lists of length k, independently across students (more general cases are analyzed in the paper.)
- Timing of the game: Students and colleges submit their preference lists and quotas simultaneously. DA is applied under the reported preferences.

Given $\varepsilon > 0$, a strategy profile is an ε-Nash equilibrium if no player gains more than ε by unilateral deviation.

Theorem

For any $\varepsilon > 0$, there exists n such that truth-telling by every agent is an ε-Nash equilibrium for any game with more than n colleges.

Theorem

The expected proportion of colleges that can manipulate DA when others are truthful goes to zero as the number of colleges goes to infinity.

The expected proportion of colleges that are matched to the same set of students in all stable matchings goes to one as the number of colleges goes to infinity.

DA is strategy-proof for students, so truthtelling is an optimal strategy for students.

Strategic rejection by a college causes a chain of application and rejections. Some of the rejected students may apply to the manipulating college, and the college may be made better off if these new applicants are desirable.

In a large market, there is a high probability that there will be many colleges with vacant positions. So the students who are strategically rejected (or those who are rejected by them and so on) are likely to apply to those vacant positions and be accepted. So the manipulating college is unlikely to be made better off.

Sketch of Proof (Step 1): Dropping Strategy

$(\succ'_c, q'_c)$ is a dropping strategy of $(\succ_c, q_c)$ if

1. $q'_c = q_c$ and
2. $\succ'_c$drops some acceptable students from $\succ_c$, but does not change orders between remaining students.

Lemma
If c cannot manipulate the student-proposing DA successfully by a dropping strategy, then c cannot manipulate it successfully by any strategy.

This lemma simplifies analysis by narrowing down the class of strategies to consider.

Sketch of Proof (Step 2): Rejection Chains

Given c and dropping strategy $\succ'_c$, consider the following *rejection chains algorithm*: an algorithm similar to student-proposing DA:

1. First, run DA under true preferences.
2. Then let c reject students matched to c who are unacceptable under $\succ'_c$.
3. Each rejected student applies to next choice, just as in DA. The rest proceeds as in DA.

The rejection chain *returns* to c if some student applies to c at Step (3).

Lemma
If no rejection chains return to c, then no dropping strategies are successful manipulations for c.

Sketch of Proof (Step 3): Vanishing Market Power

Lemma (Vanishing market power)
For any $\varepsilon > 0$, if the number of colleges n in the market is sufficiently large,

$$\Pr(\text{at least one rejection chain returns to } c) < \varepsilon$$

for any college c in the market.

Intuition: In a large market, with high probability there are many colleges with vacant positions. So the rejected students (or those who are rejected by them and so on) usually apply to those vacant positions and are accepted, ending a rejection chain. That is, the chance that the rejection chain returns to that college c becomes very small.

Lemmas 1-3 show the theorem.

Does this provide an answer why we see so few differences between hospital and student proposing DA in the NRMP data?

Possible issues:

- Couples (and possibly other match variations) make it possible that no stable match exists.
- Furthermore, constant employment may not be guaranteed.
- Conclusions such as non-manipulability of DA in large markets are not directly applicable. Even worse, DA may not be strategy-proof even for students.
- How large do markets have to be for the theorems to apply?

An additional odd result: Looking at NRMP: despite the presence of couples, stable matchings mostly exist.
Kojima, Pathak and Roth (2011) consider a model similar to Kojima and Pathak but assume there are a small number of couples.

Theorem

The probability that there exists a stable matching converges to one, as the size of the market (number of colleges) goes to infinity with the number of couples being fixed.

Theorem

For any $\varepsilon > 0$, there exists n such that truth-telling by every agent is an ε-Nash equilibrium under the Roth-Peranson algorithm for any game with more than n colleges.

Alternate approach to understand whether agents manipulate preferences in a DA algorithm.

Run experiments to assess the extent to which agents manipulate preferences.

Featherstone and Mayefsky (2011): Stability and Deferred Acceptance: Strategic Behavior in Two-Sided Matching

Compare the behavior in two mechanisms:

- Deferred acceptance (DA): Based on the Gale-Shapley algorithm
- Priority: Order all possible matches, and then go down this order and implement each match if it is still feasible

Use the following representative:
Lexicographic order: first man then woman preference.

- Step 1: All men make an offer to their first choice woman. Women are permanently matched to their favorite acceptable offer, and reject all else.
- Step k: Rejected men make an offer to their favorite acceptable woman that has not rejected them. Matched women reject all offers. Unmatched women are permanently matched to their favorite acceptable offer.
- The algorithm stops the first time no new offers are made.

Comparing DA to priority:

Criterion	M-proposing DA	M-Proposing Priority
Stable: submitted pref	Yes	No
Truth-telling: Eq	M: Yes, W: No	No
Stable: incomplete info	No	No

What if W's ruthfully reveal preferences (out of equilibrium)?

In both mechanisms: W should truncate:

Intiuition in DA;

- DA matches a W to its least-preferred stable match partner that it declares acceptable.
- With complete information, a W should declare all but one stable match partner unacceptable.
- With incomplete information, a W must also worry about truncating all stable match partners

Intuition for Priority:

- Say a W knows that a favored M will propose in Round 2, but a less favored M will propose in Round 1
- The W should truncate the M it likes less to avoid matching too early.
- With incomplete information, the W must also worry about whether the favored M will actually be proposing in Round 2

Will these two different intuitions lead to different responses in the lab?

Experimental Design:

- 5 M's, 5 W's, one-to-one matching: Truthful revelation by M's will be rationalizable as a best response
- Participants play the role of W's
- Common knowledge that truth-telling computers play the role of the Ms
- Between subjects, 40 rounds, feedback on the match in each round
- Participants know their own preferences: ordinal preferences for all agents are i.i.d. uniform, re-drawn every round
- Payoffs are decided by the true rank of a participant's match

True Rank	1st	2nd	3rd	4th	5th	Unm.
Payoff	32	16	8	4	2	0

- Payoffs: Designed to equalize expected gains from truncation across mechanisms
- Expect truncation under both mechanisms.

Observation:
In the uncorrelated market, under M-Proposing DA, any Bayes-Nash equilibrium in label-independent, weakly undominated strategies has any $m \in M$ truthfully revealing and any $w \in W$ playing a truncation.

Observation:
In the uncorrelated market, under M-Proposing Priority, any Bayes-Nash equilibrium in label-independent, weakly undominated strategies has any $m \in M$ reporting all members of W as acceptable and any $w \in W$ playing a truncation.

Suppose we observe a higher rate of truth-telling under DA than under Priority: How should we observe these results?
Second preference environment where both mechanisms give equilibrium incentives for truth-telling, so they should yield identical equilibrium outcomes.

Give Ms a common preference (drawn from the uniform distribution like before)

- DA: Common preferences imply a unique stable match
- Priority: With common preferences, a W receives all proposals in only one of the steps of the implementing algorithm

Correlated preference environment: All M share common preferences. Payoffs for W like before.

Truthtelling rates:all 40 rounds

	DA		Priority
Truth-tell. treat.	66 %	(0.38)	58.4 %
	(0.2)		(0.001)
Truncation treat.	56.7%	(0.00)	25.3%

- Truth-telling is significantly higher under DA in the truncation treatment
- No difference in truth-telling rate in truth-telling treatment

Moving to the truncation treatment:

- Priority: truth-telling drops significantly
- DA, truth-telling do not drop significantly

Truthtelling rates: last 20 rounds

	DA		Priority
Truth-tell. treat.	70.3 %	(0.33)	60.8 %
	(0.05)		(0.001)
Truncation treat.	54.7%	(0.00)	19.3%

- Seems that manipulation of preferences can be learned under DA, but, very slowly, even in an environment with lots of feedback.Truth-telling is significantly higher under DA in the truncation treatment

Non-truncation rates

	DA		Priority
Truth-tell. treat.	16.3 %	(0.23)	11.1 %
	(0.67)		(0.2)
Truncation treat.	14.3%	(0.49)	17.9%

Blocking pairs

	DA		Priority
Truth-tell. treat.	0.47	(0.57)	0.59
	(0.05)		(0.00)
Truncation treat.	0.49	(0.00)	1.87

Furthermore, under priority, more participants are unmatched.

What have we learned from theory, experiments, computation and data?

- Stable mechanisms are more likley to be used sucessfully in the field
- While DA is stable with repect to submitted preferences, no stable mechanisms exists that makes it a dominant strategy for all participants to report preferences truthfully.
- In more complicated markets (many-to-one matching), no stable mechanism makes it a dominant strategy for firms to submit preferences truthfully.

So why does DA survive and why do other unstable mechanisms not?

- In large markets, the gains from manipulations vanish.
- Empirically, markets often seem to have only few stable matches
- Manipulation under DA seems much harder to learn than under priority mechanisms