Matching with Short Preference Lists

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A stylized fact: in matching markets participants usually submit short preference lists.

- The market is too large: difficult to express preferences over all possible choices.
- Choice is contrained by the mechanism:
  - New York City School Match (12 choices max)
  - College admission in Spain (8 choices max)
  - Academic job market in France (5 choices max for departments, until 2009)
- Participants cannot include someone/institution without a prior interview.
- Participants find many choices as unacceptable.

Constrained choice is the most disturbing case, we loose strategyproofness (revealing one's true (complete) preferences is no longer an option).

Advantages?

- Gives a "target" of the number of choices one would expect
- Easier to think about one's preferences over a small set of alternatives than a large one.
- By limiting choice participants only put alternatives they really care about
  $\Rightarrow$ less "no-show" when enrolling.

This lecture

- Theoretical & Experimental investigation of constrained choice in school choice problems.
- See how short preference lists bring additional information and how we can use it.

First part: Constrained choice

Study the effects of a **quota** k on the length of submittable ordered lists for:

- Boston
- Gale-Shapley
- Top Trading Cycle

base on

- Haeringer & Klijn, **Journal of Econ. Theory**, 2009
- Calsamiglia & Haeringer & Klijn, **American Econ. Rev.**, 2010.

(First account: Romero-Medina, *Rev. Econ. Design*, 1998).

The model

A **school choice problem** (Abdulkadiroğlu & Sönmez, *AER*, 2005) consists of

- a set of **students** $I = \{i_1, \ldots, i_n\}$
- a set of **schools** $S = \{s_1, \ldots, s_m\}$
- a **capacity vector** $q = (q_{s_1}, \ldots, q_{s_m})$
- a profile of **students preferences** $P = (P_{i_1}, \ldots, P_{i_n})$
- a **priority structure** $f = (f_{s_1}, \ldots, f_{s_m})$.

Results (Informally)

Constrained School Choice – Main questions for the three prominent mechanisms β (Boston), γ (Gale-Shapley), τ (TTC):

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  NO. $\rightarrow$ Study of Nash equilibria of preference revelation games.
- Do Nash equilibria (in pure strategies) exist?
  YES.
- Are NE outcomes always stable?
  Boston: YES, but DA and TTC: NO.
- Can stability be recovered?
  DA and TTC: via well-known but restrictive conditions on priorities.

Matching: definition

An outcome of a school choice problem is called a **matching** and is a mapping $\mu : I \cup S \rightarrow 2^I \cup S$ such that for any $i \in I$ and any $s \in S$,

- $\mu(i) \in S \cup \{i\}$;
- $\mu(s) \in 2^I$;
- $\mu(i) = s \quad \text{if and only if} \quad i \in \mu(s)$;
- $|\mu(s)| \leq q_s$.

The quota game

Fix the priority structure f and the capacity vector q. Let $\varphi \in \{\beta, \gamma, \tau, \ldots\}$ be a mechanism.

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We obtain a strategic form game

$$G^{\varphi}(P, k) = \langle I, \mathcal{Q}(k)^n, P \rangle.$$

Notation:
$\mathcal{E}^{\varphi}(P, k)$ = set of k-Nash equilibria
$\mathcal{O}^{\varphi}(P, k)$ = set of k-Nash equilibrium outcomes

Incentives

Proposition

φ a strategyproof mechanism, φ^k its "constrained version". Ordering the declared acceptable school in the true order dominates any other re-ordering of those schools.

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- etc.

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- etc.

Boston and Gale-Shapley differ on the notion of "rejection".

Boston

- A school s chooses the students who applied to it that have the highest priority, up to the capacity q_s.

  If quota attained, reject all other students who applied to s.

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- New applications (from students rejected by other schools):

  Repeat the first step with considering **only the remaining available slots and the new applications.**

Deferred Acceptance

- A school s chooses the students who applied to it that have the highest priority, up to the capacity q_s.

  If quota attained, reject all other students who applied to s.

- New applications (from students rejected by other schools):

  Repeat the first step with considering **all the q_s slots and the students previously accepted.**

The Boston Mechanism

Theorem

Let P be a school choice problem.
For any quota k,

$$\emptyset \neq S(P) = \mathcal{O}^{\beta}(P, k).$$

Proof straightforward adaptation of Ergin and Sönmez's (*J. Pub. Econ.*, 2006)

Gale-Shapley Mechanism

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For any quotas $k < k'$,

$$\mathcal{E}^{\gamma}(P, k) \subseteq \mathcal{E}^{\gamma}(P, k').$$

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- Let $\hat{Q}_i = \gamma(Q_i', Q_{-i})(i)$.

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- Student i has a profitable deviation Q'_i.
- Let $\hat{Q}_i = \gamma(Q'_i, Q_{-i})(i)$.
- $\gamma(\hat{Q}_i, Q_{-i})(i) = \gamma(Q'_i, Q_{-i})(i)$ —Roth (1982), Roth and Sotomayor (1990).

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- $\gamma(\hat{Q}_i, Q_{-i})(i) = \gamma(Q'_i, Q_{-i})(i)$ —Roth (1982), Roth and Sotomayor (1990).
- $\hat{Q}_i \in \mathcal{Q}(k)$. So Q is not a k-Nash equilibrium, contradiction.

Gale-Shapley Mechanism: stability?

Proposition

$S(P) = \mathcal{O}^{\gamma}(P, 1)$.

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Proof

If $k = 1$ then Gale-Shapley = Boston. Since Boston implements stable matchings so does Gale Shapley for $k = 1$.

Gale-Shapley Mechanism: stability?

P_{i_1}	P_{i_2}	P_{i_3}	f_{s_1}	f_{s_2}
s_2	s_1	s_1	i_1	i_3
s_1	s_2	s_2	i_2	i_1
			i_3	i_2

For any profile $Q = (P_{i_1}, Q_{i_2}, P_{i_3})$ with $Q_{i_2} \in \mathcal{Q}(2)$, $\gamma(Q)(i_2) = i_2$.

Take $Q_{i_2} = s_2$, then $\gamma(Q) = \{\{i_1, s_2\}, \{i_3, s_1\}, \{i_2\}\}$.

So, $Q \in \mathcal{E}^{\gamma}(P, 2)$, but... $\gamma(Q^*)$ is not stable w.r.t. P.

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- $\gamma(P)$ may not be efficient.
- Ergin (*Econometrica*, 2002) introduces the concept of weak acyclicity (of school priorities).

$$\text{Priorities weakly acyclic} \Rightarrow \gamma(P) \text{ efficient.}$$

Gale-Shapley Mechanism: stability?

Given f, an **Ergin-cycle** is constituted of distinct $s, s' \in S$ and $i, j, l \in I$ such that:

- **cycle condition** $f_s(i) < f_s(j) < f_s(l)$ and $f_{s'}(l) < f_{s'}(i)$;

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A priority structure is **Ergin-acyclic** if no cycles exist.

Gale-Shapley Mechanism: stability?

Theorem

Let $k \neq 1$*.*

$$f \text{ Ergin-acyclic}$$

$$\Updownarrow$$

$$\Gamma^{\gamma}(P,k) \text{ implements } S(P) \text{ in Nash equilibria for any } P.$$

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- Since $\gamma(P_i, Q_{-i})R_i\gamma(Q)$ and $\gamma(Q_i', Q_{-i})(i) = \gamma(P_i, Q_{-i})(i)$, we have $\gamma(Q_i', Q_{-i})(i) = \gamma(Q_i, Q_{-i})(i)$.
  γ strategy-proof $+$ Q equilibrium.

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- Rewriting we get $\gamma(P_i, Q_{-i}) \notin S(P_i, Q_{-i})$, contradiction.

TTC

Theorem

For any quotas $k < k'$,

$$\mathcal{E}^{\tau}(P, k) \subseteq \mathcal{E}^{\tau}(P, k').$$

TTC

proof

- We first show that if a mechanism φ is *individually idempotent* then the equilibria are nested:

$$\varphi(\varphi(Q)(i), Q_{-i}) = \varphi(Q) \Rightarrow \mathcal{E}^{\varphi}(P, k) \subseteq \mathcal{E}^{\varphi}(P, k')$$

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- TTC is individually idempotent: show that under Q and $(\tau(Q)(i), Q_{-i})$ the same cycles form.

TTC

Theorem

For any quota $k \geq 2$*,*

$$\emptyset \neq \mathcal{O}^{\tau}(P, 1) = \mathcal{O}^{\tau}(P, k)\,.$$

Note: we can have $S(P) \cap \mathcal{O}^{\tau}(P, 1) = \emptyset$.

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- $\tau(P)$ may not be stable.
- Priorities Kesten-acyclic $\Rightarrow \tau(P)$ stable.
  (Kesten, *JET*, 2006)

Given f, a **Kesten-cycle** (Kesten, JET, 2006) is constituted of distinct $s, s' \in S$ and $i, j, l \in I$ such that:

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- f is Ergin-acyclic (Lemma 1 of Kesten, 2006).

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- f is Ergin-acyclic (Lemma 1 of Kesten, 2006).
- Since $O^\gamma(P, k) \in S(P)$ (our result about γ), we have $O^\tau(P, k) \in S(P)$.

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Proof of $\Leftarrow$

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- τ strategy-proof, so P is an m-equilibrium.
- For each $k \leq m$, there exists a k-equilibrium Q such that $\tau(Q) = \tau(P) \notin S(P)$ (our result about τ).

Eq. of undominated "truncations"

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	f_{s_1}	f_{s_2}	f_{s_3}
s_1	s_2	s_3	s_1	i_3	i_1	i_2
s_2	s_3	s_1	s_2	i_1	i_2	i_4
s_3	s_1	s_2	s_3	i_2	i_3	i_3
				i_4	i_4	i_1

Let $k = 2$. Let Q be such that each student submits his 2 best schools. Then,
$\gamma(Q) = \tau(Q) = \{\{i_1, s_1\}, \{i_2, s_2\}, \{i_3, s_3\}, \{i_4\}\}$.

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So, even (strong) Nash equilibria in (undominated) "truncations" may yield unstable matchings!

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Nash Implementation of the stable correspondence through mechanism

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- TTC: for $k \geq 1$: if and only if priority structure is acyclic à la Kesten (JET, 2006).

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  γ and τ outperform β in terms of efficiency.
- Ergin and Sönmez (J. Pub. Ec., 2006):
  β implements set of stable matchings in NE
  $\rightarrow$ transition from β to γ would lead to unambiguous efficiency gains.
- As the acyclicity conditions are restrictive, current transitions from β to γ or τ with quota are unlikely to be as successful as they could be.

Conclusion

Equilibrium analysis of matching games

	Players		
	Students	Students-Schools	Schools
β	Ergin-Sönmez (*J. Pub. Econ., 2005*)	$\varnothing$	$\varnothing$
γ	This paper	Alcalde (*JET, 1996*) $IR(P)$	Roth (*JET, 1984*) $S(P)$
τ	This paper	$\varnothing$	$\varnothing$

The experiment

Reconduct the Chen-Sönmez experiment with two treatments:

- First treatment: like Chen-Sönmez, no constraint.
- Second treatment: a **quota** k on the length of submittable ordered lists is imposed.

Note: No after market for unassigned students.

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- Constrained case: can put only 3 schools.
- 2 sets of payoffs: one designed, one random.
- For each mechanism (BOS, SOSM, TTC) and each payoff matrix, 2 sessions.
- A total of $2 \times 3 \times 2 \times 2 \times 36 = 872$ subjects

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- For each school, the students of the district were placed on the top of the school priority list, in the order given by the draw.

The district school and priorities

Each student was assigned a “district school”

- For each school, the number of students whose district school is this school = capacity of the school.
- Once subjects’ choices were collected, a random order of the student was drawn from an urn.
- For each school, the students of the district were placed on the top of the school priority list, in the order given by the draw.
- Other students were ranked in the school priority list below the district students in the order given by the draw.

District Schools

- For SOSM and TTC, the district school is a "safety" school.
- For Boston, the district school is a "safety" school only if put first in choices.

Sub-samples

We split the set of subjects into two sub-samples:

- **High district**: the district school is ranked 1st, 2nd or 3rd in the subject's preferences.
- **Low district**: the district school is ranked 4th or less in the subject's preferences.

The experiment

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The experiment

- During a session, each subject was given his payoff vector (her gain depending on the school she would be matched to)
- Subjects were given a mini-course on about the mechanism at hand.
- Subjects had to make a choice list (7 schools in one treatment and 3 schools in another treatment).
- Choices were collected and a matching was computed. Subjects were paid just at the end of the experiment. Average duration: 45 minutes.

Hypothesis 1

For SOSM and TTC:

Constraint implies more **rational** behavior.

(relative order of schools in choices same as in preference)

Hypothesis 2

For SOSM and TTC:

Constraint implies less **truncated truthtelling**.
(choices are the 3 most preferred.)

For BOS:

Constraint implies less (but not significant) **truncated truthtelling**.

Hypothesis 3

For SOSM and TTC:

Constraint implies more **District School Bias** and more **Small School Bias**.

For BOS:

Constraint implies more (but not significant) **District School Bias** and more **Small School Bias**.

Hypothesis 4

For BOS, SOSM and TTC:

Constraint implies more **Safety School Effect**.

Effect smaller for BOS than for SOSM and TTC.

Hypothesis 5

Under all three mechanisms, the constraint produces an efficiency loss.

The inefficiency of the three mechanisms in the constrained case is similar.

Hypothesis 6

SOSM is "more stable" than TTC or Boston in the unconstrained case.

SOSM more stable in the unconstrained case.

Hypothesis 7

Individuals will be assigned to their district school more often in the constrained than in the unconstrained case.

Rational Behavior

More rationality under constrained SOSM and TTC

	Constrained	Unconstrained	*p*-value
BOS_d	34.7	37.5	.37
BOS_r	37.5	44.4	.2
$SOSM_d$	95.8	73.6	.0001
$SOSM_r$	91.7	81.9	.043
TTC_d	93.1	84.7	.057
TTC_r	90.3	88.9	.4

Rational Behavior

Low-district subjects more sensitive to the constraint.

	Low-district sample		High-district sample	
	Cons.	Uncons.	Cons.	Uncons.
$SOSM_d$	95.2	57.1	96.7	96.7
$SOSM_r$	88.6	81.8	96.4	82.1
TTC_d	90.5	78.6	96.7	93.3
TTC_r	90.9	86.4	89.3	92.9

Rational behavior

Without low capacity schools

Treat.	$SOSM_d$	$SOSM_r$	TTC_d	TTC_r
Cons. (%)	100	100	100	100
Unons. (%)	100	100	100	100

Truncated truthtelling

Less truncated truthtelling under constrained choice

	Constrained	Unconstrained	*p*-value
BOS_d	18.1	18.1	.5
BOS_r	8.3	22.2	.0102
$SOSM_d$	25.0	58.3	.000
$SOSM_r$	18.1	56.9	.000
TTC_d	22.2	62.5	.000
TTC_r	19.4	73.6	.000

In the constrained setting, the level of truncated truthtelling does not significantly vary among SOSM, TTC and BOS-d.

Truncated truthtelling

Low-district optimize more.

	Low-district sample		High-district sample	
	Cons.	Uncons.	Cons.	Uncons.
BOS_d	16.7	19.0	20.0	16.7
BOS_r	9.1	25.0	7.1	17.9
$SOSM_d$	2.4	45.2	56.7	76.7
$SOSM_r$	6.8	26.8	35.7	57.1
TTC_d	0	64.3	53.3	60.0
TTC_r	6.8	79.5	39.3	64.3

Two types of misrepresentation

- **District School Bias** (DSB)
  A participant puts his district school into a higher position than that in the true preference order.

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- **District School Bias** (DSB)
  A participant puts his district school into a higher position than that in the true preference order.

- **Small School Bias** (SSB)
  A participant puts school A or B (or both) into lower positions than those in the true preference ordering.

Misrepresentations

District School Bias:

SOSM and TTC: 15 (d) – 20 (r) % $\rightarrow$ 70 (d) – 75 (r)%

BOS: 60 (r) – 70 (d) % $\rightarrow$ 75 (r) – 80 (d) %

Small School Bias:

SOSM and TTC: 20 (d) –35 (r) % $\rightarrow$ 60 (d) – 40 (r)%

BOS: 37 (r) – 70 (d) % $\rightarrow$ 52 (r) – 77 (d) %

Low-district more biased than high-district.

Misrepresentations

Low-district and high-district subjects exhibit different patterns of manipulation:

- Low-district subjects: DSB dominates in the constrained case, SSB dominates in the unconstrained case.
- High-district subjects: DSB dominates in both cases (const./unconst.), and SSB $\Rightarrow$ DSB.

Safety school effect

Proportion of subjects having the district school ranked 4th or more in preferences (low-district subjects) and ranked 3rd or less in choices.

Mechanism	Constrained	Unconstrained	p-value
$SOSM_d$	91	12	**0.009**
$SOSM_r$	89	18	**0.0076**
TTC_d	86	14	**0.00**
TTC_r	89	9	**0.00**
BOS_d	81	57	**0.000**
BOS_r	75	50	**0.000**

Safety School effect

- Constrained case: DSB $\equiv$ Safety School Effect (by definition).
- Unconstrained case: DSB and Safety School Effect do **not** measure the same thing.

  However, we observe DSB $\approx$ Safety School Effect.

  $\Rightarrow$ First three choices are "focal".

Safety School Efffect even if the district school is the worst school (constrained case).

Recombinant technique

- Each treatment = one shot game
- Each treatment was run twice, so we have two strategy profiles.

⇒ to compute the outcomes for a treatment, we can use any combination of the two strategy profiles, i.e., 2^{36} different combinations (Mullin-Reiley, *Games Econ. Behav.*, 2006).

We use 14,400,000 recombinations.

Efficiency

	Observed	1-2	2-3	1-3
Uncons.-d	TTC $>$ SOSM $>$ Bos	R	R	A
Uncons.-r	TTC $\gg$ SOSM $>$ Bos	A	R	A
Cons.-d	TTC $>$ SOSM $\gg$ Bos	R	A	A
Cons.-r	TTC $>$ SOSM $\gg$ Bos	R	A	A

The efficiency loss between the unconstrained an unconstrained cases is significant for the three mechanisms.

Stability

Average number of blocking pairs.

	Constrained	Unconstrained	p-value
BOS_d	10.6	11.4	.2
BOS_r	14.9	12.6	.05
$SOSM_d$	7.6	4.7	.001
$SOSM_r$	9.6	7.8	.07
TTC_d	10.4	15.5	.04
TTC_r	13.4	9.8	.01

Segregation

Proportion of students assigned to their district school.

Mechanism	Constrained	Unconstrained	p-value
SOSM_d	65	54	**0.008**
SOSM_r	44	28	**0.0002**
TTC_d	59	46	**0.007**
TTC_r	31	23	**0.039**
BOS_d	68	31	**0.026**
BOS_r	45	50	**0.008**

Increase milder than for District School Bias.

Conclusion

- Experimental study of a situation in which agents are constrained: some of their strategies are "deleted".
- Agents tend to choose "safe" strategies:
  - Secure their prospects (district school),
  - Flee competition (small school bias).
- Subjects without easily (easily identifiable) dominant strategy tend to show greater signs of optimizing behavior.
- Trade-off when restricting agents' strategies:
  - Increase agents' rationality,
  - Efficiency loss.

Two-Sided Matching with One-Sided Preferences

(or how take advantage short preference lists)

with Vincent Iehlé (Université Paris-Dauphine)

The student-optimal stable matching μ_I

- students' most preferred stable matching;
- Strategyproof (for the students)
- **Not necessarily Pareto optimal**

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- students' most preferred stable matching;
- Strategyproof (for the students)
- **Not necessarily Pareto optimal**

Proposition (Kesten, 2010, QJE)

There is no Pareto-efficient and strategy-proof mechanism that selects the Pareto-efficient and stable matching whenever it exists.

The origin of inefficiency

i_1	i_2	i_3
s_2	s_1	s_1
$\underline{s_1}$	$\underline{s_2}$	$\underline{s_3}$

s_1	s_2	s_3
i_1	i_2	i_3
i_3	i_1	
i_2		

The origin of inefficiency

i_1	i_2	i_3
$\underline{s_2}$	$\underline{s_1}$	
s_1	s_2	$\underline{s_3}$

s_1	s_2	s_3
i_1	i_2	i_3
i_3	i_1	
i_2		

Not asking a school I won't get can make other students better off.

The origin of inefficiency

i_1	i_2	i_3
$\underline{s_2}$	$\underline{s_1}$	
s_1	s_2	$\underline{s_3}$

s_1	s_2	s_3
i_1	i_2	i_3
i_3	i_1	
i_2		

Not asking a school I won't get can make other students better off.

Kesten's mechanism finds those "critical" students, eliminates them, but looses strategy-proofness.

A matching μ is not stable if there exists a pair of agents (i, j) such that

$$i \; P_j \; \mu(j) \qquad \text{and} \qquad j \; P_i \; \mu(i),$$

or there is an agent i such that $iP_i\mu(i)$.

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or there is an agent i such that $iP_i\mu(i)$.

$\Rightarrow$ Checking stability involves preferences from **both** sides of the market.

Objective of the paper

Propose a mechanism that:

- Pareto dominates the Student-Optimal Stable Matching (SOSM)
- Selects SOSM whenever it is efficient
- that is "pseudo strategyproof."

How we do it

Given a matching problem:

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- We go to a more general problem where we ignore students' preferences

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Given a matching problem:

- We go to a more general problem where we ignore students' preferences
- Extract information about stable matchings
- Feed back that information to the original problem.

- Take the preferences from both sides of a matching market (schools and students).

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- Consider only school's preferences and for each student the list of acceptable schools (**but not their preferences**)

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- for each pair student-school, (i, s), say whether there exists **a** student preference profile such that i can be matched to s for **some** stable matching. If not, i is a **dummy** for s.

- Take the preferences from both sides of a matching market (schools and students).
- Consider only school's preferences and for each student the list of acceptable schools (**but not their preferences**)
- for each pair student-school, (i, s), say whether there exists **a** student preference profile such that i can be matched to s for **some** stable matching. If not, i is a **dummy** for s.
- A new mechanism: If a student is a dummy for a school, delete that student from that school's preferences. Then run Gale-Shapley.

This paper adds to a series of paper that extract information from partial matching data:

- Stable matchings $\longrightarrow$ preferences:
  Roth and Sotomayor (1985), Echenique, Lee, Shum and Yenmez (2012).

- Preferences $\longrightarrow$ stable matchings:
  Martínez, Massó, Neme and Oviedo (2012), Rastegari, Condon, Immorlica, and Leyton-Brown (2012).

Example

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}
i_1	$\cdot$	$\cdot$	$\cdot$
$\cdot$	i_2	i_3	i_4
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	i_1	i_2	i_3
$\cdot$	$\cdot$	$\cdot$	
$\cdot$	i_4	$\cdot$	

There is no preference profile and a stable matching (for that profile) such that i_1 is matched to s_2.

A **matching problem**, $(I, S, \succ_I, \succ_S, q_S)$, is defined by:

- A set S of schools
- A set I of students.
- A vector q_S of schools' capacities.
- Each school s has a preference relation $\succ_s$ over the set of students. (responsive prefs. over sets of students)
- Each student i has a preference relation $\succ_i$ over the set of schools and himself.

A **pre-matching problem**, (I, S, P_S, q_S), is defined by:

- A set S of schools
- A set I of students.
- A vector q_S of schools' capacities.
- Each school s has a preference relation P_s over a set $A_s \subseteq I$ of students. (responsive prefs. over sets of students)

$$\begin{array}{ll} & A_s = \text{set of students acceptable for } s \\ \Rightarrow & A_i = \text{set of acceptable schools for } i. \end{array}$$

Example

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$	$\succ_{s_6}$
i_1	i_3	$\cdot$	i_2	i_2	i_2
i_2	i_2	i_3	i_4	i_3	$\cdot$
i_3	$\cdot$	$\cdot$	$\cdot$	$\cdot$	i_3
$\cdot$	i_1	i_2	i_3	i_4	$\cdot$
i_4	$\cdot$	i_1	$\cdot$	i_1	i_1
$\cdot$	i_4	i_4	i_1	$\cdot$	i_4

$\succ_{i_1}$	$\succ_{i_2}$	$\succ_{i_3}$	$\succ_{i_4}$
s_1	s_2	s_3	s_2
s_2	s_3	s_4	s_4

Example

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	P_{s_5}	P_{s_6}
i_1	·	·	·	·	·
·	i_2	i_3	i_4	·	·
·	·	·	·	·	·
·	i_1	i_2	i_3	·	·
·	·	·	·	·	
·	i_4	·	·	·	

Given a pre-matching problem P, a matching problem $\succ$ is **P-compatible** if

- for each student i and each school s,

$$s \succ_i i \;\Leftrightarrow\; i \in A_s$$

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$\Theta(P) =$ the set of matching problems that are P-compatible.

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- it is **individually rational**: I prefer my match than being unmatched.

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- it is **non wasteful**: If I prefer a school to my match, that school is full.
- there is no **justified envy**: If I prefer a school to my match, that school has no student less preferred than me.

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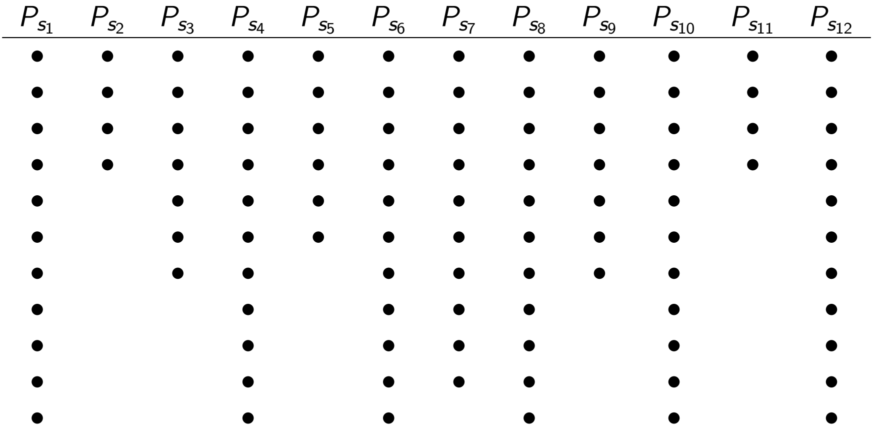
- it is **non-wasteful**: If a school does not fill its capacity, all the students acceptable for that school are matched to some school.
- there is no **justified envy**: If a student is matched to a school, all the students preferred to him by that school are matched to a school.

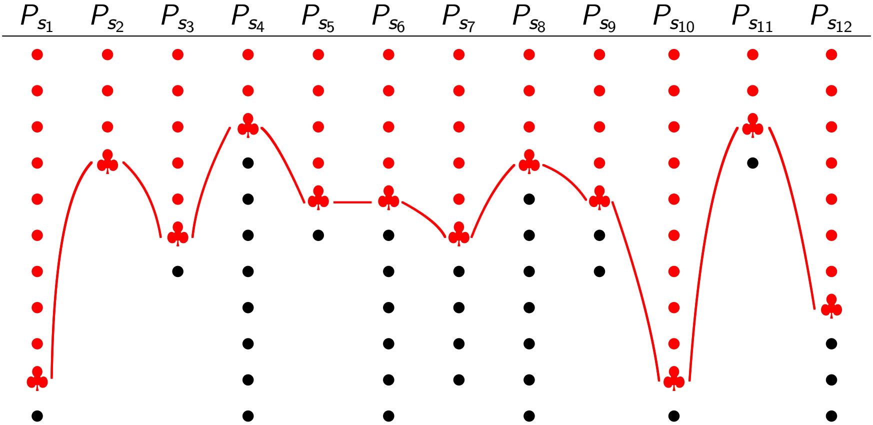
Example



Example

P_{s_1}	P_{s_2}	P_{s_3}	P_{s_4}	P_{s_5}	P_{s_6}	P_{s_7}	P_{s_8}	P_{s_9}	$P_{s_{10}}$	$P_{s_{11}}$	$P_{s_{12}}$
●	●	●	●	●	●	●	●	●	●	●	●
●	●	●	●	●	●	●	●	●	●	●	●
●	●	●	♣	●	●	●	●	●	●	♣	●
●	♣	●	●	●	●	●	♣	●	●	●	●
●		●	●	♣	♣	●	●	♣	●		●
●		♣	●	●	●	♣	●	●	●		●
●		●	●		●	●	●	●	●		●
●			●		●	●	●		●		♣
●			●		●	●	●		●		●
♣			●		●	●	●		♣		●
●			●		●		●		●		●

Example



Dummy students

A student i is a **dummy** for school s at the pre-profile P if for any matching problem $\succ \in \Theta(P)$, there is no matching μ stable for $\succ$ such that $\mu(i) = s$.

- If μ is stable for $\succ$ then μ is stable for P, with $\succ \in \Theta(P)$.
- If μ is stable for P, then there exists $\succ$ in $\Theta(P)$ such that μ is stable for $\succ$.

$\longrightarrow$

$$i \text{ is dummy for } s$$
$$\Updownarrow$$
$$\text{there is no pre-matching stable for } P \text{ such that } \mu(i) = s.$$

Identifying dummy students

Given P, let $P^{i,s}$ be P obtained by deleting i to each $P_{s'}$ with $s' \neq s$.

Proposition

Student i is a dummy for s if, and only if, there is no maximum and stable matching μ for $P^{i,s}$ such that $\mu(i) = s$.

Proof

- Take μ, stable for P^i but not maximum.

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$\Rightarrow$ There is an augmenting path π

- If the resulting matching is not stable, then we can select a subpath of π that will avoid the violating the stability condition:

$$\pi = (i_1, s_1, i_2, \dots, i_h, s_h, \dots, i_k, s_k),$$

$$\text{but } iP_{s_h}i_h,\ j \notin \pi, \text{ and } \mu(j) = j$$

$$\pi' = (j, s_h, \dots, i_k, s_k).$$

Proof

- Take μ, stable for P^i but not maximum.

$\Rightarrow$ There is an augmenting path π

- If the resulting matching is not stable, then we can select a subpath of π that will avoid the violating the stability condition:

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$$\text{but } iP_{s_h}i_h,\ j \notin \pi, \text{ and } \mu(j) = j$$

$$\pi' = (j, s_h, \ldots, i_k, s_k).$$

Keep doing with π'', π''', etc. until we have a problem-free augmenting path.

A sufficient condition

Intuition we want to capture:

i is a dummy for s and $\mu(i) = s \Longrightarrow$ however we match the other students (filling schools' capacities) there is always a student i' and a schools' such that

$$\mu(i') = i' \quad \text{and} \quad i' P_{s'} i'' \text{ for some } i'' \in \mu(s').$$

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The **truncation of P at i** is the pre-profile $\bar{P}^i$ such that

- If $i \notin A_s$ then $\bar{P}^i_s = P_s$,
- If $i \in A_s$ then $\bar{P}^i_s$ is a truncation of P_s at i (including i).

A **block** at (i_0, s_0) is a set $\mathbf{J} \subseteq I \backslash \{i_0\}$ of students such that:

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(b) $\mathbf{J}_0 := \mathbf{J} \cap \{i \ : \ iP_{s_0} i_0\} \neq \emptyset$ with $|\mathbf{J}_0| \geq q_{s_0}$

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(b) $\mathbf{J}_0 := \mathbf{J} \cap \{i \ : \ iP_{s_0} i_0\} \neq \emptyset$ with $|\mathbf{J}_0| \geq q_{s_0}$

(c) If we match i_0 to s_0, we need to "get rid" of some $j \in \mathbf{J}$.

$\Rightarrow$ For any $j \in \mathbf{J}$, it is not possible to match **all** students in $\mathbf{J} \backslash \{j\}$ such that j does not block the matching.

$\Leftrightarrow$ For any $j \in \mathbf{J}$, it is not possible to match **all** students in $\mathbf{J} \backslash \{j\}$ in the pre-profile $\bar{P}^j$.

A **block** at (i_0, s_0) is a set $\mathbf{J} \subseteq I \backslash \{i_0\}$ of students such that:

(a) $|\mathbf{J}| = \sum_{s \in A_{\mathbf{J}}} q_s$ and there exists a perfect match between $\mathbf{J}$ and $A_{\mathbf{J}}$

(b) $\mathbf{J}_0 := \mathbf{J} \cap \{i \ : \ iP_{s_0} i_0\} \neq \emptyset$ with $|\mathbf{J}_0| \geq q_{s_0}$

(c) for each $i \in \mathbf{J} \backslash \mathbf{J}_0$, for the pre-matching problem P^i,

$$\exists \ T \subseteq \mathbf{J} \backslash \{i\} \text{ such that } |T| > \sum_{s \,\in\, A^i_T} \bar{q}_s \qquad (\star)$$

where $\bar{q}_s = q_s$ if $s \neq s_0$ and $\bar{q}_{s_0} = q_{s_0} - 1$.

Illustration of condition $(\star)$

P_{s_0}	P_{s_1}	P_{s_2}	P_{s_3}
a	b	b	b
i_0	c	c	d
	a	d	a
	$\cdot$	a	$\cdot$
		$\cdot$	

Illustration of condition $(\star)$

P_{s_0}	P_{s_1}	P_{s_2}	P_{s_3}
a	b	b	**b**
i_0	c	**c**	d
	a	d	a
	$\cdot$	a	$\cdot$
		$\cdot$	

There is no block at (i, s_1): b and c can "eliminate" d and let a be matched to s_2 so that i can be matched to s_1.

Here condition *(c)* is not satisfied for d.

Proposition

Let P_S be a profile. If there is a block at (i_0, s_0) then student i_0 is dummy for school s_0.

Proof

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  - Now we have j_1 matched but not j_2. Using $(\star)$, $\exists\, j_3 \in \mathbf{J} \backslash \{j_2\}$ such that $\mu(j_3) = j_3$.
- Repeat for $j_4, j_5, \ldots$ until we hit $j_k \in \mathbf{J}_0$, contradicting μ being stable.

Corrolary for school choice problems

School choice usually endow each student with a "district school": a school for which the student has the highest priority.

Assumption

Each student always puts his district school in his submitted preference list.

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There exists an order partition of schools, $\{S_1, S_2, \ldots, S_k\}$ *such that students whose school district is in* S_h *only put in their submitted preferences schools that are in* $S_1, S_2, \ldots, S_h$.

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Proposition

For any stable matching, students whose district school is in S_h *are matched to a school in* S_h*.*

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3. Run students' DA with with the "cleaned" preferences. Output = $\bar{\mu}_I$.

Proposition

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Once a student has chosen which schools to put in his submitted preferences, it is a dominant strategy to put them in the correct order.

$\rightarrow$ Students can manipulate but only by declaring some schools as unaccepable.

Wrap up: Look at the data before doing anything

- Under not so severe circumstances, knowing preferences of both sides of the market is not necessary to identify unstable matchings;
- Stable mechanisms are not necessarily the best way to promote district mobility in school choice;
- Scrutinizing the data before running the algorithm can help to enhance one side's welfare.