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Endogenous Economic Fluctuations

Studies in the Theory
of Rational Beliefs

**With 4 Figures
and 19 Tables**

chapter 1: Endogenous Economic Fluctuations
and Rational Beliefs. A General Perspective

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1. Endogenous economic fluctuations and rational beliefs: A general perspective*

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1 On the diversity of probability beliefs

The formation of expectations by economic agents has been a controversial issue in economic theory over the last half a century. In the *General Theory*, Keynes [1936] states, on many occasions, the important role expectations play in his thinking. Yet, he never developed a formal and consistent theory of expectations. He states that agents do not have sufficient knowledge of the structure of the economy to form *correct* mathematical expectations (see Keynes [1936] pages 149–150) and he, therefore, concludes that "philosophically speaking" it is impossible for any formal theory, if developed, to lead to *unique* expectations that would be held by all agents (Keynes [1936] page 152). The failure of the Keynesian theory to incorporate a formal theory of expectations is the basis of most of its critiques (e.g. Lucas [1976]) and is the reason for the ultimate rejection of many of its key conclusions. However, Keynes' views about the importance of expectations were based on the observation that the distribution of expectations among agents affects aggregate investment and consumption and is therefore central to equilibrium allocation. This basic insight has been shared by most economists and has been bolstered by the overwhelming empirical evidence for the persistence of sharp differences of opinions and probability beliefs among agents. Empirical evidence on heterogeneity of beliefs is relevant since it is relatively easy to demonstrate that "expectations matter" when agents hold diverse beliefs but it is more subtle if all agents hold the same belief.

Both the Arrow-Debreu [1954] model as well as Savage's [1954] theory accommodate these empirical observations and permit agents to hold diverse probability belief about the exogenous states. However, within the Arrow-Debreu model this diversity has very limited implications; only the effects of the dispersion of such beliefs on equilibrium prices have been studied (see

Varian [1985], [1989]). The diversity of probability beliefs becomes a very significant question in any model in which trades occur sequentially and securities rather than contingent claims are the means of trading uncertainty. Arrow [1953] and Radner [1972], [1979] show that in order to permit agents to hold diverse probability beliefs about exogenous states and still attain the Arrow-Debreu [1954] allocations via securities, one needs to assume that the agents hold rational expectations which take the form of "conditional perfect foresight". It requires the agents to know the equilibrium map between future exogenous states and spot market prices at future dates and such knowledge implies that agents also know each other's probability beliefs (see Radner [1979]).

In most dynamic applications in economics and finance, the requirement that agents know the equilibrium map between states and prices (in fact, between states and all payoff relevant decisions of all agents) is replaced by the requirement that agents know the *true* equilibrium probability distribution of all variables. We refer to both forms of knowledge as "structural knowledge" to be distinguished from "empirical knowledge" or "information" about the state of the world. Most economists agree that both conditions impose unreasonable requirements on what an agent must know in order to act "rationally". Neither probability distributions nor equilibrium maps are observable and hence there are no guaranteed ways for agents to obtain such structural knowledge. But then, if we assume that agents do not possess structural knowledge, the Arrow-Radner security market equilibrium concepts do not apply. This fact leads to the natural question of how to extend the present theory of rational expectations to a theory that is applicable when agents do not possess structural knowledge and, yet, form expectations which are rational relative to what they observe and, therefore, can know.

This book consists of a collection of papers which propose the theory of Rational Beliefs (due to Kurz [1994a]) as the model of expectations which extends the theory of rational expectations to the postulated circumstances. The theory of rational beliefs implies that rational agents may hold diverse beliefs and such diversity leads to a paradigm with important implications to the behavior of agents in markets and in games. Consequently, the question of diversity of beliefs is central to this book. Because of the centrality of this issue, we first review the debate on the use of models with diverse beliefs. In Section 1.1 we briefly discuss the empirical evidence and the literature while in Section 1.2 we examine the validity of introducing assumptions such as asymmetric information and the presence of noise traders to explain the observed diversity of beliefs which is needed to explain, say, volatility and trade. Section 1.3 is very important in that it explains the crucial fact that a model with diverse beliefs does not aim to broaden the set of outcomes which are permitted under a corresponding, common beliefs, model. On the contrary, it aims to be *a substitute for some assumptions such as noise trading or asymmetric information when such assumptions are not warranted* and in that sense provide an alternative scientific paradigm for a wide range of market phenomena. Sections 2, 3 and 4 review the content of the papers in this book and Section 5 concludes with some comments on public policy issues.

1.1 Remarks on the empirical evidence and related literature

The observed diversity of beliefs in society has generated, over the last 50 years, an interest in the market impact of heterogeneity of beliefs. A large body of empirical work has studied alternative models which could approximate the way agents actually form expectations. Much of the post-war debate on the effect of expectations on market performance was conducted with different variants of the adaptive expectations model first introduced into macro-economics by Cagan [1956]. Heterogenous expectations were postulated in studies covering many areas such as consumer demand and saving behavior, investment behavior of firms etc. Brennscheidt [1993] provides a survey of empirical and experimental work in general areas of economics while Takagi [1991] reviews the results reported in surveys about exchange rate expectations. An influential line of research in international economics has focused on explaining the observed volatility of foreign exchange rates by the heterogeneity of beliefs and the reader may consult such papers as Frankel and Froot [1987],[1990], Froot and Frankel [1989] and the recent major surveys by Frankel and Rose [1995] and by Taylor [1995] for details and additional references. The papers in Thaler [1994] report on a rich structure of diverse beliefs among market participants while De Bondt and Thaler (discussed in Thaler [1994]) document strong disagreement among well informed security analysts. Kandel and Pearson [1995] conduct an empirical investigation of investors' heterogeneity of beliefs. They test and accept the hypothesis that agents in speculative markets interpret new public information in a differential way.

A constant stream of excellent work has analyzed the effects of heterogeneity of beliefs on the behavior of financial markets. In some papers beliefs are taken to be arbitrary and in others agents are assumed to be Bayesians with arbitrary priors. These papers also vary in their assumptions regarding the knowledge of the equilibrium map by the agents. This literature is large and cannot be reviewed here. Examples of such papers include Harrison and Kreps [1978]; De Long, Shleifer, Summers and Waldman [1990]; Harris and Raviv [1993]; Cabrales and Hoshi [1993]; Detemple and Murthy [1994] and Brock and Hommes [1996a], [1996b] (see additional references in these papers). In the specific area of general equilibrium Kurz [1974] introduces the concept of *Endogenous Uncertainty* defined as the uncertainty which is propagated internally by the beliefs and actions of agents. He then proposes an agenda for the reformulation of general equilibrium theory which treats endogenous uncertainty as the primary uncertainty and considers price contingent contracts as the main vehicle for trading such uncertainty. Svensson [1981] is the first to formulate a general equilibrium model with price uncertainty but without restrictions on beliefs. Henrotte [1992] and Kurz [1993] analyze more complex financial structures but with arbitrary beliefs. The aim of integrating the theory of endogenous uncertainty with principles of rationality of beliefs is one of the main motivations for this book. We, therefore, return to this question below.

On the opposite side, a very large literature on learning has studied the dynamics of heterogeneity of beliefs with the aim of showing that diversity of probability beliefs is a temporary phenomenon. This view proposes that heterogeneity vanishes as data becomes available since added data would enable agents to learn the true structure of the economy. Most of the work along this line adopted the Bayesian perspective which was inspired by martingale convergence theorems. A corresponding heated debate took place in the statistical literature under the heading of "Bayes Consistency" (see Diaconis and Freedman [1986]). The essential conclusion of the debate is that the convergence of the posterior to the true distribution is a rare occurrence. In two influential papers, Freedman [1963], [1965] shows that even when the statistician has a controlled experiment so that the data is generated i.i.d., the convergence of the posterior is a rare event. The problem is further compounded in typical learning situations in markets and games where the data is generated by an unknown process and the convergence of the posterior is even less likely (see Feldman [1991]). Blackwell and Dubins [1962] consider the very general case when the data is generated by a process with a true regular probability (over infinite sequences) which is denoted by Π. Let Q be the probability belief of an agent that may be thought of as a "prior" on the data. Then, given the history, the conditional probabilities of Q are shown to converge to those of Π for Π almost all histories, if the very strong condition of absolute continuity of Π with respect to Q is satisfied. But then how do you select a prior which satisfies this condition if your objective is to discover Π?

The extensive debate on learning in the statistics literature has had many important successes. For example, consider the analysis of a controlled experiment where the data is generated i.i.d. and when the statistician does not know the parameters but does know the distribution of the random variable under study (since the distribution is either known in advance or it can be deduced from the knowledge of the structure of the random mechanism which generates the data). If a known distribution is characterized with a small number of parameters and the data is known to be generated by an i.i.d. process then convergence of the posterior can be assured. On the other hand, the formation of expectations in markets or games do not fall into such a simple category of cases. We must, therefore, conclude that for the purpose of applications in economics the learning literature has accomplished exactly the opposite of what it set out to achieve: it exposed the weakness of the argument in favor of merging beliefs with increased data.

The attempt to use a learning approach as a foundation for rational expectations in markets has thus encountered a dual difficulty. The theoretical results of Bayes consistency show that rational expectations equilibrium is a possible but a most unlikely social outcome. The empirical evidence for the presence and persistence of diversity of beliefs appears, on the face of it, to contradict the predictions of a rational expectations equilibrium. Moreover, the theoretical results on Bayes consistency are compatible with the empirical evidence for the persistence of diverse market beliefs. In other words, the theoretical conclusion – holding that it is a rare event for the posterior

to converge to the truth – is compatible with the extensive empirical evidence for the persistence of heterogeneity of probability beliefs in the marketplace. But then, if diversity of beliefs persists, why try to show that learning will remove it? Instead, it is more important to establish a satisfactory explanation for the diversity. Moreover, since the explanation of the observed persistent diversity is central, it must also become an important component of the explanation of how expectations affect markets. We turn to this question now.

1.2 Diversity of information or diversity of beliefs?

Starting with financial markets, the most common explanation given in the finance literature for the observed persistent heterogeneity of beliefs *is the diversity of private information.* It is argued that agents do not possess different prior beliefs but, rather, that they have different private information resulting in different conditional beliefs. The theoretical and applied literature adopting this approach is very extensive (for recent papers see Kyle [1985], Wang [1993], [1994] and references there). This explanation is unsatisfactory from both theoretical as well as empirical perspectives. Purely theoretical considerations show that under the information revelation of rational expectations (e.g. Grossman [1981], Radner [1979]) prices make public all private information and therefore the introduction of asymmetric information, *by itself,* is not sufficient. It simply transforms the problem into other paradoxes. These include the problem of explaining why agents trade at all (e.g. Milgrom and Stokey [1982]), why asset prices fluctuate more than could be explained by "fundamentals" (indirectly generating an equity premium puzzle; see Mehra and Prescott [1985]) and why any resources are spent on the production of information (see Grossman and Stiglitz [1980]). To explain the observed heterogeneity and avoid such paradoxes researchers must introduce some additional assumptions of market structure that would remove the essential information revelation property of rational expectations. Consider, for example, the explicit introduction of uninformed noise traders or general "noise" which leads to a theory of "noisy rational expectations equilibrium". This is simply a negation of the theory of rational expectations since the assumption of noise in prices explicitly introduces *irrationality* of uninformed traders into the theory. Have you ever encountered an investor in securities who identified himself as a noise trader? Somehow, traders who generate noise are always *other* traders! The artificial, and unsatisfactory, assumption of irrationality then becomes the driving force of all the important conclusions.

Turning now to empirical considerations we note first the ample casual and scientific evidence for the opposite view that equally informed agents interpret differently the same information (see, for example, Frankel and Froot [1990], Frankel and Rose [1995] and Kandel and Pearson [1995]). This implies that the agents must have different probability beliefs which they use for conditioning on the same public information. However, focusing on asymmetric information, is there any empirical evidence to support the assumption of widespread use of private information in financial markets? We

think that the evidence is very questionable. Moreover, since it is illegal to employ private information when trading in financial markets, are we to conclude that the central characteristics of the behavior of financial markets arise as a result of *persistent and widespread* criminal behavior by market participants? The majority of firms whose securities are traded on public exchanges are being monitored carefully by a professional community of government regulators, brokers, analysts, financial managers etc. Based on that, there is ample evidence that, on the whole, the majority of firms take steps to avoid letting any one of the market participants either obtain or trade on private information. Furthermore, since modern financial markets are dominated by large institutions with vast resources which can be used to study and process all available information, elementary competitive behavior should lead us to conclude that all will possess the same information.

In some instances, like insurance markets, it is true that asymmetric information is an observed fact and is of central importance to the way these markets function. However, insurance markets are unique in that they can function reasonably well as pools of contingent contracts whereas we are concerned here with security markets which operate sequentially over time. Our theoretical discussion above has thus demonstrated that in most other models of microeconomic markets, asymmetric information – by itself – has a very little explanatory power due to the revelation mechanism of rational expectations equilibria. Hence, in order to generate results, asymmetric information must be supplemented by some additional assumptions about "friction", "noise" or other assumptions or "stories". These are used to avoid the standard paradoxes and thus become the main explanation for the observed diversity of beliefs. Models with noise traders and a mixture of informed and uninformed traders (who decide to trade anyway) may generate virtually any market behavior and hence it is easy to understand results which show that such traders could cause increased market volatility. It is much harder to see how asymmetric information arises in the first place and why a large number of noise traders continue to be present in the market. In sum, asymmetric information could be a very powerful tool for explaining market performance. However, such an assumption, together with the market structure within which it is made, must be rigorously justified and empirically supported.

Turning to macroeconomics, recall that the critique of the Keynesian theory by the rational expectations approach was associated with the rejection of the wage and price rigidities implicit in the Keynesian system. However, under the classical assumptions of price and wage flexibility and market clearing in equilibrium, rational expectations implies the usual conclusions of neo-classical analysis: in equilibrium the economy operates at full employment, GNP grows at the potential level, etc. This classical framework cannot explain the observed structure of aggregate fluctuations and, in particular, the observed cyclical correlation among economic variables such as the positive correlation between the price level and aggregate output (the "inflation – output tradeoff"). In order to explain the data, the New Classical Theory introduced asymmetric information and search processes which then became

the driving force of the theory. More specifically, although agents are assumed to have rational expectations, they either have asymmetric information or are unable to obtain information which is public in other parts of the economy. This rigidity in the transmission of public information leads to diverse models of Phelpsian or Lucasian islands (see Phelps [1970] and Lucas [1973]). To illustrate, consider the important example of The *Lucas supply curve* (Lucas [1973]). In a large market, P_t denotes the log of aggregate price level and P_t^i is the log of the price of firm i. Leaving trend aside, firm i is assumed to have a supply function of the form (in logs)

$$Y_t^i = Y^{i*} + \gamma(P_t^i - E_t(P_t|I_t(i))) \tag{1}$$

where Y^{i*} is the normal supply and E_t denotes conditional expectations by firm i given the information it has. The reason for this function is that firm i lives on a small island within the economy: it knows the price in its own competitive market but does not know the general price level and treats it as a random variable. Any deviation of firm i's price from P_t represents profit opportunities. The firm is then assumed to know that P_t is normally distributed and that the random variable z_t^i, which is the percentage deviation of P_t^i from P_t, is normally distributed with zero mean , independently of P_t. It follows that $P_t^i = P_t + z_t^i$ and it is routine to calculate the conditional probability of P_t given P_t^i and the mean of past price levels $\bar{P}_t$. Combining all these assumptions one shows that

$$Y_t^i = Y^{i*} + \theta\gamma(P_t^i - \bar{P}_t) \tag{2}$$

where θ is derived from the variances of the normal variables assumed. Holding rational expectations, the firm in (2) makes the "best" forecast of the unknown price given the history and its own price P_t^i. Nevertheless, the firm would often confuse aggregate price fluctuations for relative price fluctuations since it has imperfect information about P_t. Due to these persistent mistakes of all the firms, aggregation of (2) over all firms yields an aggregate supply function which predicts the desired positive correlation between Y_t and P_t.

Our aim here is not to review the state of macroeconomic theory. The important point to note in relation to the above example is that heterogeneity across agents is caused by an assumption (or a "story") about agents not being able to make rather simple observations and needing to form expectations about what they do not know. The logic is the same as above and we may question again the validity of the "islands" assumption and the rigid information structure which they impose. Using a term proposed by Lucas [1982], the model is "rigged" to generate the heterogeneity which induces the desired empirical implication. In addition, how do firms know that the aggregate price level has a normal distribution and how do they know the true distribution of z_t^i? Note that the above analysis of each firm would be entirely unchanged if firms had diverse prior beliefs and consequently derived different individual supply functions. The only minor difference would occur in the aggregation which requires common beliefs.

The argument presented here highlights the fact that although the rational expectations assumption insists on the common belief model, the direct

empirical implications of this common belief assumption – by itself – are rather absurd. The crucial empirical implications of all models which incorporate the common belief assumption are generated by an added set of assumptions which then drive the results. This may include asymmetry of information, lack of adequate knowledge, irrational behavior of some agents, etc. Most of these added assumptions introduce into the models "stories" with questionable theoretical and empirical foundations but *the questionable assumptions are those which drive the results*!

We finally turn to diversity in game theory. It is significant that the Bayesian developments in game theory encountered the same difficulties, or paradoxes, faced by the rational expectations theories as discussed above. For example, although one can derive "no-trade" theorems from signal extraction in markets one can also derive them from Aumann's [1976] theorem about common knowledge. Both approaches end up insisting on Harsanyi's "common prior" assumption which is in contrast with the spirit of the Axioms of Savage [1954]. Aumann [1987] takes the view that the common prior doctrine must be employed as a matter of scientific discipline since without it the "... equilibrium places very few restrictions on the possible outcomes" (page 15). We shall explain in the next section why we disagree with this assessment. Here we note that as in rational expectations equilibria, the discipline which Aumann seems to ascribe to equilibrium theory is entirely due to the very strong structural knowledge and informational assumptions made in constructing the equilibrium. Since in most applications agents do not have the kind of structural knowledge assumed in a Bayesian equilibrium and since the informational assumptions are often questionable in their realism, the discipline imposed by the equilibrium concept is illusory since it is driven by unrealistic assumptions to begin with. Aumann's and Harsanyi's insistence on the common prior assumption is also based on the wrong presumption that allowing for diversity of beliefs will come *in addition* to the unrealistic informational and structural knowledge assumptions rather than instead of these assumptions. We shall elaborate on this crucial issue below.

Aumann's comment is significant relative to one more issue. It reflects the fact that almost all objections to heterogeneity of beliefs are based on the false premise that if any heterogeneity is introduced then all beliefs should be permissible and then the theory places no empirical restrictions on the set of outcomes. In game theory the common belief assumption is known as the *Harsanyi Doctrine* and for some scholars in economics and game theory this assumption has turned into, what Kreps [1990] (page 111) calls, a "dogma". However, many serious scholars question its validity (see, for example, Kreps [1990] page 111 and Morris [1994], [1995]). We believe that one of the reasons for the positions taken by some on this issue is the mistaken view that the choice is between the two extremes represented by the Harsanyi-rational expectations view and the Savage approach permitting all priors. The papers in this volume demonstrate that reasonable criteria of rationality of beliefs can lead to a scientific middle ground placed between a theory which permits all possible beliefs and a theory which insists on a single common belief for all agents.

1.3 Diversity of beliefs: An alternative paradigm

In Section 1.2 we have shown that the observed diversity of beliefs among agents may be explained by a rational expectations (i.e. common belief) model which must then be supplemented by added assumptions which drive the results. This book argues that in many situations it is more plausible to accept the alternative paradigm based on the hypothesis that agents do not have structural knowledge and then, as a natural consequence, conclude that rational agents may have diverse beliefs about what they do not know. The empirical evidence for these two components of the new approach is substantial. Moreover, the scientific merit of this alternative paradigm is derived mostly from the fact that it offers new and useful economic insights with which we can answer difficult economic questions. In this Section we shall make the case in support of this new paradigm. We keep in mind, however, that the papers in this book enable the reader a direct evaluation of the usefulness of the new paradigm and for this reason our comments here are brief. For simplicity we use the terminology of "the diversity of beliefs theory" to refer to the combination of the hypothesis that agents do not have structural knowledge and the related theory which demonstrates that rational agents may have diverse beliefs.

The observation which needs to be made at the outset is that rational expectations is not a theory about expectations at all! It is a theory that assumes that expectations do not matter but, having done so, insists that other exogenous "fundamental" factors such as asymmetric information and irrational noise trading drive the real conclusions of the model. In contrast, under the diversity of belief theory "expectations matter" and the distribution of beliefs can have an important effect on the time series generated by the economy. On the more fundamental level, the diversity of beliefs theory rejects the validity of the formulation of uncertainty as only an *exogenous* phenomenon. It insists that both economic fluctuations as well as uncertainty have a large endogenous component which is propagated within the economy rather than caused by exogenous forces (i.e. "shocks"). Following Kurz [1974] we call it *endogenous uncertainty*. This uncertainty, which is probably the dominant form of uncertainty in our society, is indirectly the uncertainty about the beliefs and actions of other agents. It is the component of uncertainty about endogenous variables which is not resolved by the knowledge of the exogenous variables. Price uncertainty is the central form of endogenous uncertainty in a sequential economy. From the normative point of view the theory implies that endogenous uncertainty and interactions among the beliefs of agents opens up a new perspective for public policy since collective action can have an impact: whatever is endogenously propagated may be affected by collective action. In fact, the theory of endogenous uncertainty gives public policy new tools in affecting market performance but the use of these tools raises new and complex issues which are not addressed in this book.

We stress that the common belief assumption is a special case of a model with diverse beliefs. In addition, the assumption of asymmetric information is entirely compatible with diverse beliefs. It then follows that the diversity of

beliefs theory should be viewed as a more general paradigm relative to the model of common belief. Yet, the idea of diverse beliefs has been controversial. It would then be constructive to review some arguments against the critics of models with diverse beliefs and, by implication, review the case in favor of such a paradigm.

As explained above, those who object to the introduction of *any* diversity of beliefs insist that it reduces the predictive value of equilibrium analysis since it enlarges the set of individual actions which are viewed as optimal. We reject this criticism on the ground that it is based on a misunderstanding of the function of the diversity of beliefs theory. We stress at the outset that an important purpose of introducing models with diverse beliefs is to replace the artificial "rigging" of rational expectations based models when the added "stories" and assumptions, such as noise traders and asymmetric information in market, are of questionable validity. The diversity of beliefs does, indeed, enlarge the set of individual actions and market out-comes which are viewed as "rational" *as an alternative* to the way in which the assumptions of asymmetric information and noise trading add outcomes that would have otherwise been impossible in a rational expectations based equilibrium.

The last point is central to our viewpoint. Thus, at the risk of repetition we sharpen its statement. We have argued that without the "extra" assumptions, rational expectations based models cannot explain a large array of observed phenomena. Hence, if any theory under consideration is to have explanatory power, one must make a choice of which direction to proceed. One direction in which contemporary analysis has gone is to introduce such "fundamental" assumptions as private information, rigidity in the transfer of public informa-tion, noise trading, etc. We propose that the diversity of beliefs paradigm is a new direction that one may take. The papers in this book demonstrate that there is an extensive range of problems which can be studied with the tools of this paradigm, leading to new insights with implications both for positive analysis as well as for collective actions. Summing in simple terms, the diverse beliefs paradigm has been a useful scientific tool in two different ways. It has solved problems or explained observed facts when other theories could not accomplish the task and it has solved problems by replacing artificial and "rigged" assumptions which are currently added to rational expectations based models when those assumptions are unwarranted.

We return now to the criticism regarding the enlarged set of individual actions and market outcomes and show that the enlarged set is the main virtue of the new approach! The main issue under debate arises from the fact that the two paradigms offer profoundly different explanations for the observed facts and have different implications to collective action. Rational expectations based models hold the view that the sources of all risk and economic fluctuations are *exogenous to the economy.* This is true even of sunspot models where the sunspot process is exogenous to the economy and in no sense is endogenously selected by the agents. The diversity of beliefs paradigm points to endogenous uncertainty as an additional component of social risk. In

general equilibrium terms, it insists that the state space be endogenously expanded to include the "state of beliefs" so that variations in this component of the state space have a real impact on economic allocations over time. Since endogenous uncertainty entails added fluctuations on a microeconomic level, the presence of such uncertainty necessitates a larger set of individual actions. The implication is that the enlarged set of outcomes is exactly why endogenous uncertainty matters!

Next, focusing only on the enlarged set of individual actions misses a fundamental part of the approach. The theory which is proposed in this volume does indeed increase the set of individual actions which are rationalized, but does not necessarily lead to a theory of aggregate fluctuations. This is so since if beliefs are "independent" across agents then aggregation of demands and supplies in large markets act as a market "law of large numbers" rendering the belief of any one agent irrelevant to a theory of market performance. Hence the enlarged set of individual actions is, *by itself*, irrelevant to market risk and fluctuations. One of the important ideas advanced in this book is that heterogeneity of beliefs enables the emergence of a complex social interaction among agents. This interaction (or externality) takes the form of correlation of beliefs which then constitutes an essential cause of endogenous uncertainty and market fluctuations. The implication is that for an understanding of the behavior of the time paths of asset prices the distribution of beliefs in a market may be as important an explanatory factor as the conditions of resources and technology.

Finally, our rejection of the criticism is also methodological. Recall the comments regarding the availability of empirical evidence that equally informed agents disagree. Hence, it is a sound scientific procedure to explore the implications of any theory which is compatible with these facts even if it appears to alter the existing tools of analysis. Indeed, any such theory should have the model of common belief with full structural knowledge as a special case and comparisons with this special case should be important in determining which approach provides a deeper insight. Ultimately, it is scientific usefulness which should be the basis for a choice between the two approaches.

It then follows that accepting the theory of diverse beliefs immediately raises a large number of new questions of substance and technique. The most important one among them is: what theory of belief should we adopt to replace and perhaps extend the theory of rational expectations? The theory of Rational Beliefs developed by Kurz [1994a], [1994b] is the scientific "middle ground" proposed in this book. It specifies rationality principles which place restrictions on the beliefs of agents but do not lead to a single common belief. Indeed, the implied heterogeneity is exactly that heterogeneity which no amount of empirical evidence would remove. This theory is the basis for most of the papers in this issue and we shall briefly review the theory in the Section 2.1.

1.4 An outline of our review

Our review of the papers in this book does not follow exactly the sequence of papers in the book but rather, it concentrates on the development of the ideas.

That is, some papers contain material which covers multiple topics of interest. For example, the paper by Kurz and Beltratti [1997] contains both empirical work as well as a simulation of a Rational Belief Equilibrium. We thus review the econometric work in Section 4.1 while the simulation techniques and results are partly reviewed in Section 2.2 and partly in Section 4.2. We organize our review along the following topics:

Section 2.1 offers a self-contained and relatively simplified review of the theory of rational beliefs in Kurz [1994a]. Section 2.2 explains the basic technique of "private signals" which offers a simple method of describing stable but non-stationary systems which are compatible with the Main Theorem of Kurz [1994a]. Section 2.2 explains that the use of such signals is the central technique used in the papers of Nielsen [1996a], Kurz and Wu [1996], Kurz and Schneider [1996], Kurz and Beltratti [1997] and Kurz [1997b]. Section 2.3 reviews the results of Chuang [1996] and Nielsen [1996b] who study the topological properties of the set of rational beliefs and aim to gain a deeper understanding of the set of rational beliefs. We shall explain, however, that they address practical problems which are important in applications.

Section 3 discusses the concept of Rational Belief Equilibrium (RBE) introduced in Kurz [1994b]. These are equilibria in which the agents hold rational beliefs and this early paper will be briefly reviewed here. The papers by Nielsen [1996a], Henrotte [1996], Kurz and Wu [1996] and Kurz and Schneider [1996] also study the problems of general equilibrium with endogenous uncertainty and except for Henrotte [1996] they all study problems related to the formulation of RBE. In simple terms, the problem is that with endogenous uncertainty the price state space itself is endogenous and this is the central problem of general equilibrium analysis in this book. In an RBE the state space and the structure of beliefs of the agents must be consistent while the beliefs of the agents must satisfy the rationality conditions. The study of these consistency conditions is the central subject of the papers by Nielsen [1996a], Kurz and Wu [1996] and Kurz and Schneider [1996]. Henrotte's [1996] paper, which permits the agents to hold arbitrary beliefs, is focused on a financial structure of a collection of interrelated securities and studies the construction of the state space in a manner compatible only with the restrictions of no-arbitrage.

Section 4 is devoted to the review of applications of the theory of RBE. Section 4.1 reviews the papers by Kurz [1997a] and Kurz and Beltratti [1997] who spell out the econometric implications of the theory of rational beliefs. The first paper studies the implications of

the theory to the stock market and the second studies the asset allocation by managers of mutual funds. Section 4.2 presents a unified paradigm for the study of volatility in financial markets. The three papers reviewed are Kurz and Schneider [1996], Kurz and Beltratti [1997] and Kurz [1997b]: the first one studies the volatility of asset prices and shows how a GARCH type behavior emerges naturally in an RBE. The second paper studies the Equity Premium Puzzle and the third paper studies the volatility of foreign exchange rates and the forward discount bias in foreign exchange markets. The central conclusion of the three papers is that the GARCH behavior of stock prices, the Equity Premium Puzzle, the excess volatility of foreign exchange rates and the forward discount bias in foreign exchange markets are one and the same phenomenon. They all emerge naturally in an RBE and result from the way variations in the state of beliefs impact the dynamics of prices in an RBE.

Section 5 concludes with some comments about public policy.

2. Foundations: The theory of rational beliefs

The theory of rational beliefs was developed in Kurz [1994a] and is briefly summarized here as a background to the review in the sections which follow. Additional results are contained in the papers by Nielsen [1996a], [1996b] and Chuang [1996] and these are reviewed in this section as well.

2.1 A brief review of the theory of rational beliefs

We start with some notation. $x_t \in \mathbb{R}^N$ is a vector of N observables at date t and the sequence $\{x_t, t = 0, 1, \ldots\}$ is a stochastic process with true probability Π. Since every $x = (x_0, x_1, \ldots)$ is an infinite sequence in $(\mathbb{R}^N)^\infty$ we use the notation $\Omega = (\mathbb{R}^N)^\infty$ and denote by $\mathscr{B}$ the Borel σ-field of Ω. We thus think of the probability space $(\Omega, \mathscr{B}, \Pi)$ as the *true* probability space. A belief of an agent is a probability Q; such an agent is then adopting the theory that the probability space is $(\Omega, \mathscr{B}, Q)$. An agent who observes the data takes $(\Omega, \mathscr{B}, \Pi)$ as fixed but does not know Π. Using past data he will try to learn as much as possible about Π. The theory of Rational Beliefs aims to characterize the set of all beliefs which are compatible with the available data.

The basic assumption made is that date 1 has occurred a long time ago and at date t, when agents form their beliefs about the future beyond t, they have an *ample* supply of past data. We think of the vector $x = (x_0, x_1, x_2, x_3, \ldots)$ as the vector of observations generated by the economy. However, in studying complex joint distributions among the observables, econometricians consider *blocks* of data rather than individual, primitive observations. For example, if we study the distribution of $(x_{\text{today}}, x_{\text{today}+1})$ we would consider the infinite sequence of blocks (x_0, x_1),

$(x_1, x_2), (x_2, x_3), \ldots$ It is thus useful to think of the data from the perspective of date 0 as the infinite vector $x = (x_0, x_1, x_2, \ldots)$ and the data from the perspective of date n as $x^n = (x_n, x_{n+1, \ldots})$ where $x \equiv x^0$ and

$$x^n = Tx^{n-1}. \quad n = 1, 2, 3, \ldots$$

T is known as the *shift transformation*. The stochastic dynamical system at hand is denoted by $(\Omega, \mathscr{B}, \Pi, T)$ where Π is the unknown probability. Now for any $B \in \mathscr{B}$ consider the set $T^{-n}B$ which is the preimage of B under T^n defined by

$$T^{-n}B = \{x \in \Omega : T^n x \in B\}.$$

$T^{-n}B$ is the set in Ω such that if we shift it by n dates we enter B; $T^{-n}B$ is the event B occurring n dates later. A system $(\Omega, \mathscr{B}, \Pi, T)$ is said to be *stationary* if $\Pi(B) = \Pi(T^{-1}B)$ for all $B \in \mathscr{B}$. A set $S \in \mathscr{B}$ is said to be *invariant* if $S = T^{-1}S$; it is said to be *invariant* Π *a.e.* if $\Pi(S \Delta T^{-1}S) = 0$, (where $S \Delta T^{-1}S = (S \cup T^{-1}S) \backslash (S \cap T^{-1}S)$). The distinction between these two concepts is minimal and will be disregarded here. A dynamical system is said to be *ergodic* if $\Pi(S) = 1$ or $\Pi(S) = 0$ for any invariant S. In the discussion below we assume for simplicity of exposition that $(\Omega, \mathscr{B}, \Pi, T)$ is ergodic but this assumption is not needed (see Kurz [1994a] where this assumption is not made).

In order to learn probabilities agents adopt the natural way of studying the *frequencies* of all possible economic events. For example, consider the event B

$$B = \left\{ \begin{array}{l} \text{price of commodity 1 today} \le \$1, \text{ price of commodity 6 tomorrow} \ge \$3, \\ 2 \le \text{quantity of commodity 14 consumed two months later} \le 5 \end{array} \right\}.$$

Now using past data agents can compute for any finite dimensional set B the expression

$$m_n(B)(x) = \frac{1}{n} \sum_{k=0}^{n-1} 1_B(T^k x) = \left\{ \begin{array}{l} \text{The relative frequency that } B \text{ occurred among} \\ n \text{ observations since date 0} \end{array} \right\}$$

where

$$1_B(y) = \left\{ \begin{array}{ll} 1 & \text{if } y \in B \\ 0 & \text{if } y \notin B. \end{array} \right.$$

This leads to a definition of the basic property which the system $(\Omega, \mathscr{B}, \Pi, T)$ is assumed to have:

Definition 1: A dynamical system is called stable if for any finite dimensional set (i.e. cylinder) B

$$m = \lim_{n \to \infty} m_n(B)(x) \text{ exists } \Pi \text{ a.e.}$$

The assumption of ergodicity ensures that the limit in Definition 1 is independent of x. In Kurz [1994a] it is shown that the set function m can be uniquely extended to a probability m on $(\Omega, \mathscr{B})$. Moreover, relative to this probability the dynamical system $(\Omega, \mathscr{B}, m, T)$ is stationary. There are two crucial observations to be made at this point.

(a) Given the property of stability, in trying to learn Π all agents end up learning m which is a *stationary* probability. In general $m \neq \Pi$: the true dynamical system $(\Omega, \mathcal{B}, \Pi, T)$ may not be stationary. Π cannot be learned.

(b) Agents know that m may not be Π but with the data at hand m is the only thing that they can learn and agree upon.

Non-stationarity is a term which we employ to represent the process of structural change which cannot be explained by the statistical regularity of past data. Hence, a stable but non-stationary system is a model for an economy with structural change but in which econometric work can still be successfully carried out. If all agents knew that the true system is stationary they would adopt m as their belief. The problem is that they do not know if the environment is stationary and hence even if it is stationary, agents may still not adopt m as their belief.

It is important to see that m summarizes the entire collection of asymptotic restrictions imposed by the true system with probability Π on the empirical joint distributions of all the observed variables. It is shown in Kurz [1994a] that for each stable system with probability Π there is an entire set $B(\Pi)$ of stable systems with probabilities Q which generate the same stationary probability m and consequently impose the same asymptotic restrictions on the data as the true system does. The question is how one can determine analytically (i.e. without observing any data generated by a system) if a system $(\Omega, \mathcal{B}, Q, T)$ generates m as a stationary measure. To examine this question let us return to $(\Omega, \mathcal{B}, \Pi, T)$ and consider, for any cylinder B the set function

$$m_n^{\Pi}(B) = \frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}B).$$

It is important to note that $m_n^{\Pi}(B)$ has nothing to do with data: it is an analytical expression derived from $(\Omega, \mathcal{B}, \Pi, T)$.

Definition 2: A dynamical system $(\Omega, \mathcal{B}, \Pi, T)$ is said to be *weak asymptotically mean stationary* (WAMS) if for all cylinders $S \in \mathcal{B}$ the limit

$$\dot{m}^{\Pi}(S) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}S) \quad \text{exists.}$$

It is *strong asymptotically mean stationary* if the limit above holds for all $S \in \mathcal{B}$.

Kurz [1994a] shows that $\dot{m}^{\Pi}$ can be uniquely extended to a probability measure m^{Π} on $(\Omega, \mathcal{B})$. We then have the important theorem which is the main tool in Kurz [1994a]:

Theorem 1: $(\Omega, \mathcal{B}, \Pi, T)$ is stable if and only if it is WAMS. If m is the stationary measure calculated from the data, then $m(S) = m^{\Pi}(S)$ for all $S \in \mathcal{B}$.

The implication of Theorem 1 is that every stable system $(\Omega, \mathcal{B}, \Pi, T)$ generates a unique stationary probability m^{Π} which is calculated analytically from Π. This last fact is crucial since it is the foundation of the following:

Definition 3: A selection of belief Q cannot be contradicted by the data m if

(i) the system $(\Omega, \mathscr{B}, Q, T)$ is stable,
(ii) the system $(\Omega, \mathscr{B}, Q, T)$ generates m and hence $m^Q = m$.

We can finally state the two axioms which define the Rationality of beliefs:

Rationality Axioms

A selection Q by an agent is a *Rational Belief* if it satisfies

(I) *Compatibility with the Data:* Q cannot be contradicted by the data.
(II) *Non-Degeneracy:* if $m(S) > 0$, then $Q(S) > 0$.

One may interpret condition (II) to say that if a finite dimensional event S is observed infinitely often (generating a positive relative frequency) then one cannot be certain that S cannot occur from the perspective of today. Now, to express a belief in the non-stationarity of the environment, an agent may select a probability $Q^\perp$. This probability is said to be *orthogonal with* m if there are events S and S^c such that

(i) $S \cup S^c = \Omega, S \cap S^c = \varnothing$,
(ii) $m(S) = 1, m(S^c) = 0$,
(iii) $Q^\perp(S) = 0, Q^\perp(S^c) = 1$.

We are now ready to characterize the set $B(\Pi)$ of all Rational Beliefs when the data is generated by $(\Omega, \mathscr{B}, \Pi, T)$.

Theorem 2 (Kurz [1994a]): Let the data be generated by a stable dynamical system $(\Omega, \mathscr{B}, \Pi, T)$ with a stationary measure denoted by m. Every Rational Belief Q must satisfy $Q = \lambda Q_a + (1 - \lambda)Q^\perp$ where $0 < \lambda \le 1, Q_a$ and m are probabilities which are mutually absolutely continuous (i.e. they are equivalent) and $Q^\perp$ is orthogonal with m such that

(i) $(\Omega, \mathscr{B}, Q_a, T)$ and $(\Omega, \mathscr{B}, Q^\perp, T)$ are both stable,
(ii) $m^{Q_a} = m^{Q^\perp} = m$.

Moreover, any Q such that λ, Q_a and $Q^\perp$ satisfy the above is a Rational Belief.

The probability $Q^\perp$ is central since it represents the agent's theory on how the probability of an event at any date differs from the stationary probability at that date. This reveals a crucial characteristic of non-stationary systems: *the timing of events matters* in terms of the probabilities which are attached to them. Thus, the probability $Q^\perp$ permits an agent to assign to a given event different probabilities at different dates at which it may occur. Rationality of belief requires that averaging the probabilities assigned to this event over all dates must yield the stationary probability assigned to it by m. However, non-stationary systems can give rise to an unbounded number of such events which are different from each other. Consequently, a Rational Belief Q may induce forecasts which are different from the forecasts of m at all dates and the

difference between the forecasts of Q and m need not converge to zero. $Q^{\perp}$ places positive probabilities on events on which m places zero probability.

Theorem 2 then says that two economic agents who are equally intelligent and who have *identically the same information* may make two different rational forecasts because they hold two competing theories which are compatible with the data (i.e. they are both in $B(\Pi)$). The agents may disagree on how much weight should be placed on the possibility that the environment is stationary (represented by the subjective parameter λ). They may also disagree about the probabilities of an event occurring at any particular date and about the likelihood of important but rare events. Disagreement among rational agents must, therefore, arise from their having different theories about the nature of the fluctuations of the system rather than about the behavior of its long term averages.

The fact that agents may have different probability beliefs about a dynamical system with an unknown probability law has a very simple implication: when such disagreement between two agents occurs then at least one of them is wrong. Hence, the theory outlined here insists that agents who are declared "rational" may be wrong and conversely, that agents who hold wrong rational beliefs are not irrational. Since this observation is important, Kurz [1997a] defines $(\Pi\text{-}Q)$ – the difference between the true probability and the probability belief of an agent – to be the "mistake" function of the agent. Mistake functions play a central role in the empirical implications of the theory in that we can, ex-post, estimate these functions and thus study with some statistical confidence the pattern of market mistakes which agents make. Such analysis can teach us a great deal about the structure of market volatility and its causes. Also, we noted above that the forecasts made under a rational belief Q may be different at an infinite number of dates from the forecasts made under the stationary measure m. In that case there would typically exist a subsequence of dates along which one can compare the forecasts under Q with the realizations at those dates and, ex-post, demonstrate with some degree of statistical confidence that this rational belief was wrong. But then, if a belief can be statistically falsified ex-post, should we not take this to be a demonstration that the belief should not have been adopted to begin with and thus not be a member of $B(\Pi)$? The answer is "no" and is explained in the various papers in this book. Without a more concrete structure, we briefly sketch four of the reasons but elaborate further in the next section. First, assuming that at date t a finitely lived agent adopts a belief about events beyond t, then any demonstration that his theory is false can be made only with finite data and only with some statistical confidence. In a typical application it is almost certain that the agent never acquires sufficient data (after date t) to provide him a reasonable level of confidence that his belief is wrong. Second, suppose that the amount of data available over n dates after t is sufficient (relative to the level of confidence required by the agent) to demonstrate that the distribution of the data between date t and $t+n$ was not what he thought. If his belief is non-stationary, represented by a sequence of different distributions over different time intervals, then the demonstration that he was wrong between

t and $t + n$ does not necessarily provide a convincing case against the distributions implied by his beliefs for dates after $t + n$. Third, even if at date $t + n$ the agent has convinced himself that his belief was wrong then, in a typical application he would have already made all the important economic decisions which cannot be reversed; the falsification of his belief comes too late to be useful. Finally, even if the agent becomes confident with a small amount of data that his model is wrong, there does not exist a procedure to ensure the selection of a better model so that the best he can do is select an alternate member of the set of rational beliefs $B(\Pi)$ which is likely to be wrong as well. The reader may further examine these points in the context of the specific models discussed below.

2.2 On generating processes and private signals: Nielsen [1996a], Kurz and Wu [1996], Kurz and Schneider [1996], Kurz and Beltratti [1997] and Kurz [1997b]

In order to use the theory of rational beliefs we need to work with processes which are non-stationary but stable. The problem is that in infinite horizon economies a "belief" is a very complex probability measure and the condition of stability can be difficult to check. This, in turn, can make it difficult for the agent to select a rational belief. It would, therefore, be very useful to develop general analytical tools which allow stable probability beliefs to be constructed *sequentially* by selecting at each date from a finite or a countable collection of one-period probabilities (or transition matrices) together with a rule of how to select future one-period probabilities for each realization of the system up to the date of choice. It turns out that this way of describing the belief of an agent has a natural economic interpretation. From the technical point of view, the problem is to describe the *changes* over time in the probability of the process while ensuring that it is a stable process in the sense of Kurz [1994a]. A simple example will help clarify the nature of this procedure.

Pick a process $\{y_j, j = 0, 1, \ldots\}$ of i.i.d. random variables taking values in $\{0, 1\}$ with probability of 1 being, say, 1/4 and generate an infinite sequence of observations $y^* = (y_0^*, y_1^*, \ldots)$ of the process. The realizations $y_j^* = 1$ or $y_j^* = 0$ are now treated as *parameters* of the non-stationary process $\{x_t, t = 0, 1, \ldots\}$ to be defined. That is, there is a set of parameters $\{\alpha, \beta\}$ and a map associating the value $y_j^* = 1$ with the value α and the value $y_j^* = 0$ with β such that the process $\{x_t, t = 0, 1, \ldots\}$ where $x_t \in X = \{0, 1\}$ is a sequence of *independent* random variables satisfying

$$P\{x_t = 1\} = \begin{cases} \alpha & \text{if } y_t^* = 1 \\ \beta & \text{if } y_t^* = 0. \end{cases} \tag{3}$$

Suppose that α and β satisfy $(1/4)\alpha + (3/4)\beta = .35$ then the long term average of the x_t's is .35. Given the specification of y^* and the assumption of independence over time, (3) will determine Π_{y^*}, the probability of infinite sequences $x \in \{0, 1\}^\infty \equiv X^\infty$ given y^* and hence the dynamical system of the x_t's is

$(X^\infty, \mathscr{B}(X^\infty), \Pi_{y^*}, T)$. This system is stable and has a stationary measure m represented by the i.i.d. process $\{v_t, t = 0, 1, \ldots\}$, $v_t \in X$ with $P\{v_t = 1\} = .35$. The stationary measure is independent of the specific realization of y^*. The i.i.d. process $\{y_j, j = 0, 1, \ldots\}$ is called a *generating process* and $y^* = (y_0^*, y_1^*, \ldots)$ a *generating sequence*. A particular realization y^* is thought of as a sequence of "structural" parameters which generate the probability Π_{y^*}.

Theorem 2 above (Kurz [1994a]) implies that the dynamical system $(X^\infty, \mathscr{B}(X^\infty), m, T)$ is a rational belief and, for brevity, we say that "the probability m is a rational belief". At each t_0 there are, however, other rational beliefs Q about $\{x_t, t = t_0, t_0 + 1, t_0 + 2, \ldots\}$ such that $m^Q = m$. For example, an agent may have a process $\{z_t, t = t_0, t_0 + 1, \ldots\}$ of *private signals* (or generating variables) which are i.i.d. coin tossings with probability of 1 being ξ and a sequence of signals (or parameters) z^* in the same way as above with the frequency of $\{z_t^* = 1\}$ being ξ. Define now the *perceived process* $\{x_t', t = t_0, t_0 + 1, \ldots\}$ by

$$P\{x_t' = 1\} = \begin{cases} \alpha' & \text{if } z_t^* = 1 \\ \beta' & \text{if } z_t^* = 0 \end{cases}$$

independently over time. This independent but non-stationary sequence of random variables defines a stable measure P with m as its stationary measure if $\xi\alpha' + (1 - \xi)\beta' = .35$. By Theorem 2 all probabilities $Q = \lambda m + (1 - \lambda)P$ with $\lambda \in (0, 1]$ are Rational Beliefs. We often do not assume the technical Axiom II in Kurz [1994a] in which case P is also a rational belief since $\lambda = 0$ is permitted.

The above example is fully generalized by Nielsen [1996a] as follows. The space of observable economic variables is the measurable space $(X, \mathscr{B}(X))$ where $X \subseteq \mathbb{R}^N$ and $\mathscr{B}$ denotes the Borel σ-field of the specified set. Let $Y = \mathbb{N}$ (i.e. the integers) and let $\mathscr{P} = \{P_1, P_2, \ldots\}$ be a countable collection of probability measures on $(X, \mathscr{B}(X))$. Now pick any ergodic and stationary dynamical system $(Y^\infty, \mathscr{B}(Y^\infty), Q, T)$ and a realization y^* of this system. Let $(P_{y_0^*}, P_{y_1^*}, \ldots)$ be the corresponding sequence of one-period probabilities on $(X, \mathscr{B}(X))$. The product measure $\mu = \overset{\infty}{\underset{t=0}{\times}} P_{y_t^*}$ on $(X^\infty, \mathscr{B}(X^\infty))$ is called a Simple Independently Distributed Stable (SIDS) measure. It is useful to think of the generating sequence y^* as the set of parameters of the non-stationary SIDS dynamical system $(X^\infty, \mathscr{B}(X^\infty), \mu, T)$ in the sense that if at date t $(Y^\infty, \mathscr{B}(Y^\infty), Q, T)$ selects y_t^*, then at date t the probability $P_{y_t^*}$ is the operative one. Nielsen [1996a] studies the structure of such systems and shows that for Q almost all realizations they are stable and the stationary measure is independent of the particular realization.

Kurz and Schneider [1996] generalize the use of generating processes to any joint system $((X \times Y)^\infty, \mathscr{B}((X \times Y)^\infty), \Pi, T)$ where the data x and the parameters y are interrelated. Their "Conditional Stability Theorem" then states that if the joint system is ergodic and stable (and in most applications one may take the joint system to be stationary) then the conditional system on the data $(X^\infty, \mathscr{B}(X^\infty), \Pi_{y^*}, T)$ given any realized sequence y^* is stable for

Π almost all realizations y^*. The stationary measure of this stable system is the stationary measure of Π_X which is the marginal measure on $(X^\infty, \mathscr{B}(X^\infty))$. When the joint system is stationary, then Π_X is stationary and hence the stationary measure of Π_X is Π_X itself.

The method of generating variables is a central technical device used in several of the papers of this book and at various points in this paper we have stressed their dual interpretation. From the mathematical point of view, generating variables are simply the parameters of a non-stationary process and have been introduced as a *technical device* used for the tractable description of non-stationary but stable systems. Thus, when used to describe the true (and objective) probability of economic variables this is exactly the interpretation that should be given to them. On the other hand, for the formulation of a rational belief as a joint dynamical system $((X \times Y)^\infty, \mathscr{B}((X \times Y)^\infty), Q, T)$ we have proposed to think of the values taken by the process $\{y_t, t = 1, 2, \ldots\}$ either as a description of the subjective probability perception of an agent or as *private signals* that an agent sends himself (e.g. an internal report prepared by the agent). Given a signal y_t^* at date t, the date t belief of the agent is represented by the conditional probability of Q given y_t^* as well as the history of the process up to t. In this context the distribution of private signals at t may depend upon all past values of the observed variables and all past private signals of the agent. Given the subjective nature of the belief, we stress that private signals *should not be confused with objective data*: they are neither observable by other agents nor communicable to other agents. Being subjective perception variables used by the agents to specify their beliefs and not objective data, they are not subject to any rationality conditions. Kurz and Schneider [1996] apply these tools to the study of systems in which economic variables follow finite state Markov processes and in that case each value taken by a private signal leads to a selection, by an agent, of a different transition matrix. In such a Markov economy private signals provide a tractable mathematical representation of *the state of belief in the economy*. Hence, they play a crucial role in the construction of the price state space which is one of the central problems attacked by the papers in this book. We shall discuss this issue in Section 3 below.

The simplicity and tractability of private signals may appear to question the argument presented in the last paragraph and suggest that the agent who knows these parameters should use future observations to test the validity of his belief parameters. To illustrate, return to the example above and note that along the subsequence of dates where $z_t^* = 1$ the relative frequency of $x_t = 1$ in the observed data is .35 while the agent believes it to be α'. This enables him *an ex-post* estimation of his mistake and shows that a rational belief may be falsified ex-post with some statistical confidence depending upon the data availability after t_0. At the end of Section 2.1 we gave four general reasons why this should not be viewed as a violation of rationality and now that we have the more concrete model of private signal we make here two additional comments. First, the model of two regimes is a simple approximation for more complex Markov models. Such models may have either an infinite number of regimes or

such a large number of possible regimes that the frequency of any one regime is very small (or zero in the case of an infinite number of regimes) enlarging the data requirement. Second, the arguments given here and at the end of Section 2.1 insist on *more* than the conclusion that the possibility of ex-post falsification of a rational belief should not negate its rationality; it also insists that there may be some situations where agents may be allowed to "change" their beliefs but, in general, there is little to be gained from allowing such changes after date t_0. This is so since allowing such a procedure implies that *a belief* over the finite life of an agent must be defined as a *sequence of probabilities* $(Q^0, Q^1, \ldots, Q^N)$ where $Q^j \in B(\Pi)$ and where the dates when an agent switches probabilities is random, depending upon the realization of the true process. From the point of view of endogenous uncertainty the random variability of the probabilities used by the agent constitutes an additional descriptive component of endogenous variability and this strengthens our basic theory. However, from the formal point of view, if an agent specifies the statistical rule by which he decides to reject a measure and the rule by which he selects a new member of $B(\Pi)$, then this information can be used to generate a composite probability measure which incorporates the rules for random selection of members of $B(\Pi)$ where the probabilities for such additional randomization are defined by the realizations of the process. Given that $B(\Pi)$ is convex, this procedure amounts to a selection of a fixed member of $B(\Pi)$. The consequence of such a composition is that this single probability may not be describable with the simple structure of a finite number of Markov signals. Hence, from the perspective of general equilibrium theory, allowing agents to change probabilities over time has the analytical result of inducing an infinite dimensional state space.

2.3 Further examination of the set of rational beliefs: Chuang [1996] and Nielsen [1996b]

These papers study the structure of the set $B(\Pi)$ where $(\Omega, \mathscr{B}, \Pi, T)$ generates the data. Kurz [1994a] shows that $B(\Pi)$ is a convex set and asserts (page 886), in error, that $B(\Pi)$ is also compact in the topology of weak convergence. We retract this claim; Nielsen [1996b] provides a counter-example. In the present study Nielsen [1996b] shows that both the set of stable measures as well as the set of rational beliefs are closed in the topology of strong (sup norm) convergence but this is too restrictive for economic applications. The interest in the closedness properties of $B(\Pi)$ is motivated by a desire to clarify the technical foundations of the theory. We, however, prefer to stress here the economic importance of the problem under consideration.

The theory of rational beliefs does not aim to either explain or predict the specific beliefs which economic agents hold. It does aim to identify that boundary of the class of probability beliefs which is determined by criteria of rationality. Agents who adopt a rational belief must, therefore, either explicitly apply or implicitly employ additional criteria which lead to the determination of the beliefs which they hold. The requirement of compatibility between the

belief and the empirical experience of an agent implies that with his one very long (but of finite length) time series he must calculate the relative frequencies of a generating collection of cylinders. This collection may be the class of all cylinders or, for an agent who maximizes over a future of length N, the class of cylinders of dimension less or equal to N. It is then plausible for the agent to subjectively restrict his considerations to probability measures in $B(\Pi)$ for which the speed of convergence of his data is uniformly high on the specific class of cylinders with which he is concerned. These considerations lead Nielsen [1996b] to examine various collections of probability measures which are *uniformly* stable. If we then view such sets as containing the distribution of beliefs in a given market then the interest in the topological properties of such sets is clear. It is a basis for comparing how "close" beliefs of agents are in terms of their forecast functions, how well can a probability measure be approximated by others in the class and finally, for proofs of the existence of RBE. In essence, Nielsen's [1996b] result is that sets of probability measures which are uniformly stable are closed in the topology of weak convergence.

Chuang [1996] also studies the properties of $B(\Pi)$ but in terms of the long term average of the *conditional forecast* function. Thus suppose that Q is a rational belief with respect to Π and a bounded function $f(x_t, x_{t+1}, \ldots, x_{t+L})$ is given. The problem is to characterize the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{t=0}^{N-1} E_Q(f(x_t, x_{t+1}, \ldots, x_{t+L}) | I_t) \tag{4}$$

where $I_t = (x_0, x_1, \ldots, x_{t-1})$ and to show that the limit (4) exists. Chuang's [1996] main theorem says that if the process $\{x_t, t = 1, 2, \ldots\}$ is stable and if f is bounded, then the limit (4) exists for Q almost all histories and is equal to the unconditional expectation of f under the stationary measure m.

To see why the above problem has important empirical implications consider the typical portfolio problem of an agent (used in Kurz [1997a]) who maximizes an expected utility over consumption streams (c_t) given a rational belief Q and information I_t. His first order conditions are of the well known form

$$E_Q \left[\frac{u'(c_{t+1})}{u'(c_t)} \frac{P_{t+1} + r_{t+1}}{P_t} \,\Big|\, I_t \right] \left(\frac{1}{1+\delta} \right) = 1 \tag{5}$$

where P_t is the price of the asset, r_t is the profit or dividend and δ is the discount rate. In the next section we shall review Kurz [1997a] who shows that when there are diverse beliefs in asset markets the standard orthogonality conditions do not hold. Hence (5) implies that there exist functions $\eta_t \neq 0$ which are not orthogonal to the subspace spanned by I_t such that

$$\frac{u'(c_{t+1})}{u'(c_t)} \left(\frac{P_{t+1} + r_{t+1}}{P_t} \right) \frac{1}{1+\delta} - 1 = \eta_t(I_t, \varepsilon_{t+1}). \tag{6}$$

The functions η_t which depend upon the pure noise ε_{t+1} and on I_t, measure the mistakes of rational agents who hold beliefs which are rational but are not equal to the true equilibrium probability Π. We then want to know the limit of

the long term average of the η_t. Chuang's [1996] main theorem implies that if consumption is a time invariant function which depends on information variables of bounded length so that the expression in (6) can be written as the function $f(\cdot)$ in (4), then the long term average of the η_t tends to 0 for Q almost all histories $\{I_t, t = 1, 2, \ldots\}$. This result is not strong enough in that it does not assert the conclusion for Π almost all histories. It can be shown that if the process is ergodic and if the belief Q satisfies Rationality Axiom II of Kurz [1994a] then Chuang's [1996] conclusion holds Π a.e. as well. When agents hold non-stationary beliefs then consumption is not time invariant and what we have on the left side of (6) is a sequence of functions $f_t(\cdot)$. The problem of convergence of such a sequence of functions is of great importance but is open.

3 Rational belief equilibrium and the endogenous formation of the price state space

The concept of Rational Belief Equilibrium (RBE) was introduced by Kurz [1994b]. The model used is the same model used in many of the papers on rational expectations (e.g. Muth [1961], Bray and Savin [1983]). In that model a demand function is exogenously specified and a continuum of risk neutral producers with quadratic cost functions must select their output at any date before observing the price. Once produced, the output is sold at the prevailing price. The demand function is non-stationary in the sense that it shifts by a deterministic sequence of unpredictable constants. Kurz [1994b] shows that any RBE has a higher price volatility than the unique rational expectations equilibrium. Even in an RBE where agents select, as their price forecast, the forecast implied by the stationary measure (which is also the best stationary linear forecast) the volatility of equilibrium prices is higher than their volatility when the agents *know* the exact demand and supply functions. Although the model used is very simple, it provides an early example for the general mechanism which causes endogenous uncertainty. In the case at hand, this mechanism increases price volatility through the interaction among the mistaken but rational forecasts of the diverse agents. These mistaken forecasts cause the on-going increased imbalance between the shifting demand and the outputs of the producers which are based on these mistaken forecasts.

As we pointed out earlier, in an RBE the state space itself is endogenously determined and this fact renders the analysis of RBE relatively more complex than the study of a standard Arrow-Debreu model where the state space is exogenously given. However, we also pointed out that the technique of generating variables enables a great simplification of the construction of the price state space and such a construction is used repeatedly in this book. This method of constructing the price state space is an innovative result of Nielsen [1994][1] (it is also contained in Nielsen [1996a]) and we shall outline here an intuitive explanation of the procedure employed.

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[1] Nielsen [1966a] is based, in part, on material from his dissertation Nielsen [1994]. He uses the terminology of "Rational Belief Structure" for an object which is essentially the same as the "Price State Space" which we describe here.

Start by noting that at date t the private signal of each agent is a parameter in his demand function and hence the market clearing conditions of a market with K agents depend upon the vector $y_t^* = (y_t^{1*} \ldots, y_t^{K*})$ of these K variables representing the "state of belief". If s_t is a vector of exogenous variables and p_t are market clearing prices at t, then consider the class of market clearing time-invariant functions of the form

$$p_t = \Phi(s_t, y_t^*). \tag{7}$$

If all exogenous variables and private signals take only a finite number of values then only a finite number of prices will ever be observed in this economy. Therefore, the rationality conditions require that a rational belief places positive stationary probability only on the finite number of M prices which are ever observed. Now, since placing probabilities on $(p_1, p_2, \ldots, p_M)$ is equivalent to placing these probabilities on the price state space $\{1, 2, \ldots, M\}$ one can then state all the rationality conditions on the price state space together with the state space for the process of exogenous variables without specifying the *values* of the exogenous variables and equilibrium prices. Hence, the deeper part of the problem of constructing a consistent price state space is the statement of the rationality of beliefs conditions of the agents in such a way that they are stochastically compatible with the market clearing conditions. This is where Nielsen's [1996a] SIDS construction and the conditional stability theorem of Kurz and Schneider [1996] are used. Given the specified equilibrium functions (7) one shows that under some technical conditions if the exogenous variables and the states of belief (i.e. the generating variables) are either jointly SIDS or Markov and if the private signals which are used by the agents to form beliefs are SIDS or Markov, then the equilibrium prices and the exogenous variables indeed become jointly SIDS or Markov. Thus the SIDS and Markov classes of processes are "closed" in the sense that if the beliefs and the exogenous variables are jointly in the class, then the equilibrium realization remains within the same class of stochastic processes. This argument can easily be extended to the case where all variables take countable rather than finite values but needs to be substantially modified to handle the open case where the number of values taken by the endogenous variables is of the order of the continuum.

We remark that in the case of a "small" economy, an RBE which is constructed as above suffers from a natural problem which arises from the lack of "anonymity". To see the problem note that it follows from (7) that when the state of belief of an agent changes, he changes his demand and a different market price is set. However, if the agent acts competitively he must disregard his effect on market prices. But then, if he disregards his effect he may discover empirical evidence to falsify this assumption since over time he *does* have some effect on prices. This effect becomes negligible as the market grows in size but for small markets the equilibrium may exhibit such distortions. The standard solution to this problem is to introduce homogenous "types" and thus we can interpret the equilibria with small number of agents (as is the case for several papers in this monograph) as containing a large number of *identical* agents of

the same type. If we examine economies with richer structure where agents have imperfectly correlated private signals, an RBE will depend upon the *distribution* of beliefs within each type and across types. Such equilibria will have the desired "anonymity" property if the number of agents is large.

We turn now to commenting on the specific treatment of equilibrium and the price state space by Nielsen [1996a], Kurz and Wu [1996], Kurz and Schneider [1996] and Henrotte [1996]. Nielsen [1996a] works out the properties of SIDS processes and shows that for this class of processes, if the equilibrium price function is as specified in (3) then the price state space can be consistently constructed. Keep in mind, however, that the existence of the price state space must be established *jointly* with the structure of the beliefs of the agents that will be defined on that space. Nielsen [1996a] also provides a characterization of all economies for which a consistent price state space can be constructed under the restriction that the exogenous variables form an SIDS process and the beliefs of the agents are also SIDS. The paper demonstrates the existence of an RBE of an OLG economy with K agents, a homogenous consumption good and money. However, the model allows agents only very limited opportunities to trade endogenous uncertainty. This brings up the important general question of how we should reformulate general equilibrium theory to allow agents to trade endogenous uncertainty in general and price uncertainty in particular. An outline of such a program is presented in Kurz [1974] where the proposal is made to put aside the Arrow-Debreu system of *exogenous state* contingent contracts and replace it by a system of *price* contingent contracts (in short, PCC). Utilizing the Bewley [1972] fixed point theorem, Svensson [1981] formulates a temporary equilibrium model with endogenous uncertainty in which agents can trade PCC. Temporary equilibria with PCC but with more complex financial structures are analyzed by Kurz [1993] and Henrotte [1992], [1996].

The paper in this volume which is closest in structure to the model proposed by Kurz [1974] is the paper by Kurz and Wu [1996]. Although it is cast in the context of a relatively simple OLG economy, it studies the full problem of general equilibrium with endogenous uncertainty in which the ownership shares of a firm are traded in a stock market, agents hold rational beliefs and can trade uncertainty using a full set of PCC. Such PCC are contracts traded at date t and permit an agent to receive or deliver units of the common stock in period $t + 1$. The quantity received or delivered is specified in the contract to be contingent upon the price of the common stock in period $t + 1$. Since these PCC are "derivative" securities Kurz and Wu [1996] introduce non-arbitrage pricing restrictions which, in their case, can be directly imposed on the set of allowable prices. They then assume that the probability of the exogenous endowments and the beliefs of the agents are jointly SIDS and this assumption enables them to construct a consistent price state space. The paper considers in detail a simple special case aiming to explain the construction of the state space and to demonstrate the exact restrictions which the rationality conditions impose on the beliefs of the agents. Having constructed the state space, Kurz and Wu [1996] give a precise

definition of "endogenous uncertainty" in terms of the price states. First time readers may find Section (3.a) of this paper helpful for understanding some of the basic ideas of this book.

The general existence result proved in all four papers is that an equilibrium exists whenever a consistent price state space is constructed. However, the trading of PCC presents new technical difficulties and an important aspect of the Kurz and Wu [1996] paper is the existence proof. It demonstrates that contrary to Svensson [1981], the utilization of the rationality conditions for the OLG economy enables a reduction in the dimension of the price state space and this simplifies the structure of the PCC so that finite dimensional methods of proof become applicable. Kurz and Wu [1996] also show that endogenous uncertainty is generic in an RBE and an RBE is constrained Pareto Optimal in the restricted sense that the endowment risk of the unborn young cannot be reallocated.

Kurz and Schneider [1996] utilize the technique of generating variables to study the RBE of an economy with a single, infinitely lived, agent and of an OLG economy with a single consumption good but multiple firms whose shares are traded on a stock market. The paper studies economies where the joint process of exogenous variables and private signals follow a finite state Markov process and demonstrates for such economies two sets of conclusions. First, using the conditional stability theorem, the paper works out the consistency conditions between the exogenous variables, the private signals and prices which must be satisfied if a consistent price state space can be constructed. Second, they show that for the two types of economies an RBE exists whenever a price state space can be consistently constructed. The second part of Kurz and Schneider [1996] studies the effect of correlation among the beliefs of agents on the structure of price volatility. This part of the paper will be reviewed in Section 4.2 below.

Henrotte's [1996] paper demonstrates some of the difficulties, in the construction of the price state space, which we must confront if we have a continuum of distinct equilibrium prices. Rejecting the assumption of structural knowledge by the agents, the paper assumes that they have hetero-genous probability beliefs on the space of possible prices. It sets up a complex structure of interrelated securities: common stocks and derivative securities where the return on an investment in a derivative security is a known function of future *prices* of other securities. Henrotte [1996] studies the construction of a consistent price state space by defining it to be the set of no-arbitràge prices. To define this term let a positive dynamic portfolio be an initial portfolio together with a reinvestment strategy which ensures non-negative payoff in all states of the world. The condition of no-arbitrage requires that the price of any positive dynamic portfolio be non-negative. However, since the prices of all securities are part of the state space on which derivative securities are defined, we have a self referential definition: the price states determine the positive dynamic portfolios which, in turn, define the no-arbitrage prices. The main result of Henrotte [1996] establishes under some technical conditions the existence of a non-arbitrage price state space.

Henrotte's [1996] paper is different from the other three in this group since he does not incorporate any rationality restrictions. However, the interest in non-arbitrage prices arises from the fact that it is a "minimal" rationality condition which should be imposed on the beliefs of the agents. The supports of their probability beliefs are then required to be subsets of non-arbitrage prices. This leaves a continuum of configurations which constitute the price state space.

4 Applications

4.1 Econometric implications of the theory of RBE: Kurz [1997a] and Kurz and Beltratti [1997]

The common methodological approach of the two papers in this group is derived from the econometric implications of the "mistake function" of an agent which is the difference between his forecast function and the forecast function under the equilibrium probability. The econometric implications focus on the orthogonality conditions of the theory. To explain these, return to equation (6) above which we reproduce in the explicit form

$$\frac{u'(c_{t+1})}{u'(c_t)} \left(\frac{P_{t+1} + r_{t+1}}{P_t} \right) \frac{1}{1+\delta} - 1 = \eta_t(I_t, \varepsilon_{t+1}) \tag{6a}$$

$$\eta_t(I_t, \varepsilon_{t+1}) = \xi_t(I_t) + \varepsilon_{t+1}. \tag{6b}$$

The functions η_t which depend upon the noise ε_{t+1} and on I_t, measure the mistakes of an agent who holds a rational belief which may not be equal to the true equilibrium probability Π. When $Q = \Pi$ the usual orthogonality theorem of conditional expectations imply that $\xi_t = 0$ a.e. Under rational beliefs typically $\xi_t \neq 0$ and this means that excess utility returns are present in the market and are predictable in the sense that there are information variables in I_t which are correlated with stock returns.

We note that the technique of analyzing the innovation functions is not new and has been used extensively in the rational expectations literature to measure "unanticipated" shocks such as shocks to the money supply (see, for example, Barro [1977]). However, in the rational expectations literature the "unanticipated" component must (by construction) satisfy the orthogonality conditions since no agent is allowed to make a mistake that can, even in retrospect, be explained by information available at the time. Under rational beliefs that restriction does not hold and consequently $\xi_t \neq 0$.

Kurz [1997a] extends the theory of asset pricing from its standard formulation under rational expectations to a natural reformulation under rational beliefs. The objective, however, is to study the theoretical as well as the empirical implications of the theory. This is done by formulating a simple model of asset prices, followed by a discussion of the theoretical implications of the model and leading to an econometric examination of the theory. The paper stresses the fact that the theory of asset prices which is proposed here

utilizes information efficiently as required by the efficient market theory. The central and crucial difference between the two theories is that the new theory allows two agents to *interpret the same information differently*: agents have their own probability measures and consequently derive different conditional probabilities given the same information. Thus, the hypothesis of diversity of rational beliefs is a tool to model differences in judgements of agents who face new and unfamiliar circumstances-typical to the process of investments.

To describe the non-stationarity of the environment, Kurz [1997a] proposes a model of a sequence of regime shifts where each regime is represented by a fixed joint distribution of the stochastic process of stock prices and dividends. Due to the stability of the process agents know the stationary measure which provides the long run average behavior of returns. However, each regime is a new and unknown distribution that remains in effect for a limited duration; all that agents can do is take actions based on their best assessment of the available opportunities. The theory insists that it is the non-stationarity of the equilibrium process which is both the challenge as well as the opportunity of investors. However, the lack of structural knowledge by the agents means that they adopt theories about stock prices and dividends which are different from the true distribution of the process. Kurz [1997a] then defines the "mistake functions" of the agent as in (6a)–(6b) and explains that the mistakes of the agents are the mechanism which generate the central phenomena predicted by the theory.

The central implications which Kurz [1997a] focuses on are the orthogonality conditions of the theory as explained earlier. In the context of the asset pricing model the condition $\xi_t \neq 0$ is interpreted to mean that within each regime there are excess returns and such returns are predictable in the sense that there are information variables in I_t which are correlated, within each regime, with stock returns. If $\xi_t = \xi$ for all t, then it follows from Chuang's [1996] theorem that $\xi_t = 0$ a.e. It then follows that if $\xi_t \neq 0$ then the theory predicts that ξ_t *will be different across regimes*. Hence the essential predictions of the theory, which Kurz [1997a] tests for the U.S. stock market for the period 1947–1992, are that for each regime $\xi_t \neq 0$ and ξ_t are statistically significantly different across regimes. Kurz [1997a] then shows that the bulk of the fluctuations in stock prices represent endogenous uncertainty propagated by the mistakes of the agents.

The paper by Kurz and Beltratti [1997] addresses the extensive literature on the equity premium puzzle initiated by the insightful work of Mehra and Prescott [1985]. However, here we shall review only the econometric work reported in Section 4 of that paper; the analysis of the equity premium in an RBE framework will be discussed in Section 4.2 below.

In Section 4 of the paper, Kurz and Beltratti [1997] study the asset allocation of mutual funds. The motivation for the study is the observation that two mutual funds with the same objective may have different asset allocations due to different risk aversion coefficients. However, if they have diverse rational beliefs then they will also have diverse mistake functions. Hence, utilizing the observed asset allocation of the funds we can estimate the

mistake functions which depend upon information variables which are known at the time of decision as in (6b) above. The empirical work of Kurz and Beltratti [1997] uses a data file covering the period 1982–1995 on 63 U.S. mutual funds all of which are in the same category of "balanced" funds whose objective is to seek an optimal asset allocation among stocks, bonds and short term debt instruments (cash). They formulate a model of optimal asset allocation by funds and estimate the mistake functions $\xi_t(I_t)$ which are implied by the optimality conditions. The estimates show that most of the risk aversion coefficients of the fund managers are between 2 and 4 and the mistake functions are significantly different from zero. These functions also exhibit significant heterogeneity across funds, a result which is predicted by the theory.

4.2 Endogenous uncertainty: A unified paradigm for the study of volatility in financial markets

We now return to the simulation work reported in Kurz and Schneider [1996] Kurz and Beltratti [1997] and Kurz [1997b]. These three papers investigate different problems. The paper by Kurz and Schneider [1996] studies the effect of correlation among beliefs on asset price volatility and discovers that a typical RBE will exhibit a GARCH type of behavior in the time series of stock prices. This theoretical result is consistent with the ample empirical evidence for time varying variance of stock prices. The paper by Kurz and Beltratti [1997] studies the equity premium puzzle in an RBE context and by simulating the time series behavior of an RBE example, it shows that there is a large set of model parameters which can replicate the historical moments of asset returns which have been the center of the equity premium puzzle. The paper by Kurz [1997b] studies the volatility of foreign exchange rates in an OLG monetary economy with a stock and a bond market. The model thus enables him to study the volatility of foreign exchange rates in conjunction with the equity premium on risky assets and the observed "forward discount bias". Yet, together these three papers present a unified view of the basic mechanism which generates asset price volatility. Indeed, together the three papers reveal that the GARCH behavior of asset prices, the equity premium puzzle, the volatility of foreign exchange rates and the forward discount bias *are one and the same phenomenon*: all result from the mechanism which generates endogenous uncertainty in asset markets. In order to explain this general outlook, it would be helpful if the reader is a bit more familiar with the details of the three papers. We, therefore, first review the content of the papers and only then return to discuss the unified view proposed here.

We start with Kurz and Schneider [1996]. The paper studies the effects of correlation among agents on the volatility of stock prices in an RBE. The motivation for this important problem arises from the observation that if agents have diverse beliefs which are "independent," then aggregation over agents acts as a "law of large numbers" which reduces the potential impact of heterogeneity on endogenous price volatility. Thus if heterogeneity matters,

there must be some form of correlation across the beliefs and actions of the agents. To study this problem, Kurz and Schneider [1996] construct a simulation model for which equilibrium prices and probability distribution of prices can be calculated. The model assumes an economy with one firm, two agents, a dividend process taking two values, two processes of private signals each taking two values and a Markov joint structure with Markov marginals.

To be able to discuss "correlation" between the agents, private signals in the Kurz and Schneider [1996] simulation model are given a specific meaning: given each of the two values that a private signal can take, the agent selects a different Markov transition matrix over the observables (i.e. prices and dividends). In each matrix an agent can become either optimistic or pessimistic about the states of "high" dividends in the subsequent period. The *intensity* of such optimism or pessimism is measured by the displacement of the probabilities of the high or low dividend states relative to the corresponding values of the stationary measure. A second measure of correlation are parameters which directly influence the statistical correlation between the private signals. Kurz and Schneider [1996] view both forms of correlation as consequences of the process of communication in society. The results of Kurz and Schneider [1996] show that the presence of correlation has a dramatic effect on asset price volatility. Moreover, correlation among the generating variables of the agents tend to endogenously create a non-stationarity of the GARCH type in the time series of prices which takes the form of multiple "regimes" of different price volatility. The high volatility regimes consist of those states in which the agents "agree" and their agreement (in optimism or pessimism) induces increased price volatility. The low volatility regimes are the "disagreement" states in which the disagreements of the agents tend to cancel each other out and consequently reduce price volatility.

The paper by Kurz and Beltratti [1997] evaluates, in the context of RBE theory, the equity premium puzzle introduced by Mehra and Prescott [1985]. These authors observe that the large mean spread of about 6% between the return on common stocks (about 7%) and short term debt instruments (about 0–1%) over the last century is incompatible with the results generated under rational expectations by a standard aggregate model of an optimizing single agent. This same results were observed in other studies which found it difficult to explain the size of the premium.

Kurz and Beltratti [1997] argue that the mistaken component in the equity premium puzzle debate is the assumption that agents hold rational expectations and that consequently, the only risk in the stock market is the risk represented by the exogenous variability of the dividends. By replacing rational expectations with rational beliefs they argue that the amount of uncertainty in equity holdings would be allowed to include endogenous price uncertainty. In that case the demanded premium may be higher than could be accounted for by an aggregate model in which the only allowed and perceived uncertainty is the uncertainty of future dividends.

To make their case Kurz and Beltratti [1997] construct a dynamic model of the economy with stock and "bill" (one period riskless debt instrument)

markets and with two agents. The parameters of the real part of this economy are chosen to correspond to the Mehra and Prescott [1985] specifications of the model and in accord with the empirical evidence. Moreover, under rational expectations the two models become identical and hence a comparison of the results is possible. Kurz and Beltratti [1997] compute the asymptotic moments implied by the model first for the rational expectations equilibrium specified by Mehra and Prescott [1985] and then for an RBE under a particular specification of rational beliefs by the two agents. They show that the calculations for the rational expectations equilibrium lead to the usual unsatisfactory results of the equity premium puzzle whereas for the specified RBE the moments closely correspond to the historical record.

Kurz and Beltratti [1997] provide a simple intuitive explanation for their results. They point out that the riskless rate is determined by the average price level of the "bill" while the premium is determined by the structure of capital gains and losses in the model. However, capital gains and losses can occur only if the model exhibits price volatility and some asymmetry in the transition across price states. They show that the parametrization of the beliefs in the RBE example given in the paper results in two very revealing facts: First, the variance of stock prices in the Kurz and Beltratti [1997] RBE *is 630 times larger* than the variance of asset prices in the Mehra and Prescott [1985] model. Second, the transition probabilities between price states (as endogenously determined in the RBE) are drastically different from the transition probabilities implied by the dividend process which drives the Mehra and Prescott [1985] dynamics. Since the premium is entirely determined by capital gains and losses generated by the transition between price states, an RBE is capable of explaining the premium while a Mehra and Prescott [1985] type model which is driven by fundamentals, cannot.

Kurz [1997b] extends the model used in Kurz and Beltratti [1997] into a two country OLG model with a stock, a real bond and two currencies. He introduces two exogenous shocks: a random net output growth in the home economy and a non-capital endowment growth shock in the foreign economy. Monetary policies in both economies are selected to be as neutral as possible: domestic monetary policy adjusts the growth of the money supply in response to shocks in domestic output growth while foreign monetary policy adjusts the growth of the foreign money supply to respond to the foreign non-capital endowment shocks. The stock and the bond are used as stores of value while money is used only for transactions within each period. Market equilibria are computed for different parameter values and compared. These show that endogenous uncertainty is a major explanation of the volatility of the foreign exchange rate. This component of exchange rate volatility cannot be explained either by "fundamental" causes or by shocks to the money supplies of the two economies: it operates in addition to the volatility induced by monetary policy. The simulations also reveal that increasing the endowment shocks in the foreign economy has a spillover effect on the financial markets of the home economy. In REE this effect increases the volatility of commodity and asset prices, it lowers the riskless rate and increases the equity premium. In RBE the

effect of increased endowment shocks depends upon the beliefs of the agents. An example shows that such spillover effects may reduce price volatility in RBE. Also, in RBE these effects have a non-monotonic impact on the riskless rate, on the equity premium and on the volatility of the foreign exchange rate. Kurz [1997b] exhibits a family of RBE's in which the foreign exchange rate is volatile, the equity premium is high and a forward discount bias is present.

We now return to the unified view proposed in the introductory remarks. In all three papers the mechanism which generates volatility is the variability of the state of beliefs of the agents. Such states of beliefs are defined by the vector of private signals or "generating variables" of the agents. Although each agent perceives his own signal to be generated by an i.i.d. process, these signals are correlated and may also jointly depend on other observables in the economy. Pure price volatility may be generated by uncorrelated private signals if the agents interpret their signals so as to vary their conditional probability beliefs about future prices with these private signals. However, pure price volatility is not sufficient to generate a low riskless rate, a high equity premium and a forward discount bias of the foreign exchange rate. There are two important factors which come into play. First, when a joint variability is present among the private signals, agents cannot know the correct correlation since each one of them knows only his own signal; this correlation is a form of social externality. Secondly, given a private signal the interpretation of the signal by each agent may be price dependent in the sense that the transition matrix which is selected given that signal is different from the stationary matrix and the difference is price dependent. The three papers under discussion show that the correlation among signals and the price dependency of the signal interpretation are the mechanism which induce price fluctuations with transitions among price states that result in a high equity premium, a low riskless rate, a strong forward discount bias in foreign exchange markets and a GARCH like behavior of asset prices

In general, the above mechanism of endogenous generation of volatility is very complicated. However, Kurz and Schneider [1996] show that under certain conditions one can define a simple formal process which provides an insight into the structure of volatility. They call it "the regime process" and we have already mentioned it above. Since all three papers use models where a private signal determines if an agent is "optimistic" about some future states or "pessimistic" about them, the regime process is defined to be a sequence of random variables which take the value 1 if the agents "agree" and 0 if they "disagree". Naturally, when they agree they are *both* either pessimistic or optimistic. As we have already explained, their private signals may be correlated and agents do not know the joint distribution of these signals (which is influenced by the social communication among the agents). Due to the nature of this correlation, the regime process may have a high degree of *persistence* which each agent takes as given. In the "agreement" regime (regime "1") the agents have similar demands and thus either both want to sell stocks or both want to buy stocks and as a result regime "1" is a high volatility regime. On the other hand in regime "0" their excess demands diverge and by canceling each

other they cause aggregate demand to be relatively smooth resulting in regime "0" being the low volatility regime.

Kurz and Schneider [1996] show that the regime process is a simple way to describe the mechanism which causes the GARCH phenomenon in stock prices and the duration of each volatility regime is determined by the correlation between the signals of the agents. Also, Kurz and Beltratti [1997] calculate the transition matrix of the regime process to exhibit the structure of capital gains and losses that are induced by endogenous uncertainty. This is a useful way of thinking about the premium since the pattern of capital gains and losses is exactly the mechanism which determines the premium of stock returns.

We have made the important theoretical observation that the market performance of an RBE depends upon the *distribution* of beliefs of agents. We observe, in addition, that we have little empirical evidence regarding the characteristics of this distribution in real markets. Since in all three papers reviewed here simulation models aim to study the behavior of RBE's and the models are generally calibrated to the real economy, the natural methodological question arises as to the criteria used for selecting the distributions of beliefs in the simulated models. In response to this question we note that none of the papers suggest that the postulated distributions are based on empirical evidence. All three papers start by specifying some empirically observed set of phenomena for which no satisfactory explanation is available with current models. Next they construct commonly used real models which are generally calibrated to the real fundamental structure of the economy. The papers then ask whether one can find a distribution of rational beliefs (i.e. allowable model parameters) that would generate the observed phenomena as an equilibrium outcome. This is exactly the sense in which the theory of rational beliefs is proposed as an alternative scientific paradigm to the approach of seeking exogenous "fundamental" explanations to these phenomena.

5 Some remarks on rational beliefs and public policy

The question whether variations in equilibrium prices and quantities are due only to variations in exogenous "fundamental" factors has been a long standing controversial issue in the theory of value. The theory advanced in this book derives an internally propagated component of economic fluctuations from principles of individual rationality. The usefulness of these principles must be judged either by the positive restrictions which the theory places on the data generated by the economy and/or by the policy implications which it has. The papers in this book explore both the predictions and positive implications of the theory as well as its econometric implications. The implications of the theory to public policy have not been addressed in this book but due to the great importance of the issue, we shall make a few concluding remarks on this subject.

It can be seen that in a market with endogenous uncertainty, public policy and collective action can have an important effect on the characteristics of

economic fluctuations. Consider, for example, a collective action which aims to restrict price fluctuations in a specified market to a narrow target range. Under any theory where the state of belief has no effect on prices, such a policy implies that with probability 1 the public sector will end up needing to directly intervene (for example, by either buying or selling) in order to ensure the success of the policy. Under rational beliefs the situation changes. First, consider the range of price fluctuations when no public policy is in effect. Typically, there exists a narrower range of price fluctuations that would have been realized if only fundamental exogenous factors caused prices to vary. All price fluctuations outside this narrow ("fundamental") range are propagated endogenously. Second, there exists a credible public stabilizing policy such that the policy will be effective – causing prices to fluctuate in this narrower target range – but where the policy *will never require any actual market intervention (i.e. buying or selling)*. Furthermore, the larger is the endogenous component of economic fluctuations the larger is the range of effectiveness of public policy. We remark that, in comparison to the points made here, if a market is dominated by irrational traders, one cannot make any prediction as to the effect of public policy.

Without structural knowledge by agents, individuals and firms who hold rational beliefs make decisions based on their differing expectations. In such an economy prices and quantities change not only in response to changes in fundamental states but also in response to changes in the state of belief. A credible public policy will then alter the beliefs of agents and will therefore guide the economy onto an alternative RBE in which the level of fluctuations of prices and quantities is *permanently* altered. This fundamental fact expresses perhaps the most important difference between rational expectations and rational beliefs. Under rational expectations public policy has no effect since agents can neutralize the effect of such policy by using their full structural knowledge to predict correctly the effect of the policy. In some cases this is an accurate description of the policy environment. However, under rational beliefs agents have different beliefs about the future of the economy and will usually have different predictions as to the effects of any particular policy. In such an RBE the pattern of economic fluctuations is partly determined by the beliefs of the agents and consequently *one of the objectives of public policy is to change the beliefs of the agents.* That is, inflationary processes and deep recessions can be substantially aggravated by the structure of expectations in an RBE and under some criteria such expectations may not be socially desirable. In such situations, an important objective of public policy would be to alter the beliefs of market participants by drastic and credible policies rather than to take the beliefs of the agents as fixed. The implications of this perspective is one of the important topics for future research.

We hasten to add that the observation made above does not mean that public policy under rational beliefs is a simple matter since in the postulated environment there are important difficulties in identifying criteria for determining what public policies could be declared as *desirable* policies. It is entirely clear that the Pareto principle is much too weak a guide for such choices since

for any proposed countercylical policy in an RBE, there exists a distribution of beliefs under which some agents would be worse off in an RBE with lower aggregate volatility. As a result, public policy under rational beliefs is not likely to be defined by the Pareto criterion.

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