

# **Endogenous Uncertainty and Rational Belief Equilibrium: A Unified Theory of Market Volatility**

by

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**Abstract.** The theory of Rational Belief Equilibria (RBE) offers a unified paradigm for explaining market volatility by the effect of "Endogenous Uncertainty" on financial markets. This uncertainty is propagated within the economy (hence "endogenous") by the beliefs of asset traders. The theory of RBE was developed in a sequence of papers assembled in a recently published book (Kurz [1997]) and the present paper provides an exposition of both the main ideas of the theory of RBE as well as a summary of the main results of the book regarding market volatility.

Section I starts by reviewing the standard assumptions underlying models of Rational Expectations Equilibria (REE) and their implications to market volatility. The paper then reviews four basic problems which have constituted puzzles or anomalies in relation to the assumptions of REE : (i) Why are asset prices much more volatile than their underlying fundamentals? (ii) The equity premium puzzle: why under REE is the predicted riskless rate so high and the equity risk premium so low? (iii) Why do asset prices and returns exhibit the "GARCH" behavior without exogenous fundamental variables to explain it? (iv) The "Forward Discount Bias" in foreign exchange: why are interest rate differentials poor predictors of future changes in the exchange rates? Section II outlines the basic assumptions of the theory of RBE and the main propositions which it implies for market volatility. Section III develops the simulation models which are used to study the four problems above and explains that the domestic economy is calibrated, as in Mehra and Prescott [1985], to the U.S. economy. Then for each of the four problems the relevant simulation results of the RBE are presented and compared to the results predicted by a corresponding REE and to the actual empirical observations in the U.S.

The paper concludes that the main cause of market volatility is the dynamics of beliefs of agents. The theory of RBE shows that if agents disagree then the state of belief of each agent, represented by his conditional probability, *must fluctuate over time*. Hence the distribution of the individual states of belief in the market is the root cause of all phenomena of market volatility. The GARCH phenomenon of time varying variance of asset prices is explained in the simulation model by the presence of both *persistence* in the states of beliefs of agents as well as *correlation* among these states. Correlation makes beliefs either narrowly distributed (i.e. "consensus") or widely distributed (i.e. "non-consensus"). In a belief regime of consensus (due to persistence it remains in place for a while) agents seek to buy or sell the same portfolio leading to high volatility. In a belief regime of non-consensus there is a widespread disagreement which causes a balance between sellers and buyers leading to low market volatility. In short, the GARCH phenomenon is the result of shifts in the distribution of beliefs in the market induced by the dynamics of the individual states of belief.

Turning to the equity risk premium, the key question is what are the distributions of beliefs which ensure that the average riskless rate is low and the average equity risk premium is high. It turns out that *the only circumstances* when the mean riskless rate falls to around 1% and the mean equity premium rises to around 7% arise when, on the average, the majority of agents are relatively optimistic about the prospects of capital gains in the subsequent period. In such a circumstance the rationality of belief conditions imply that the pessimists (who are in the minority) must have *a higher intensity of pessimism than the intensity of the optimists*. In a large economy with this property the state of belief of any one agent may fluctuate but on the average there will be a *minority of intensely pessimistic agents*. This asymmetry between optimists and pessimists flows directly from the rationality conditions of beliefs and implies that at most dates the pessimists have a stronger impact on the bill market. At those dates the pessimists protect their wealth by increasing their purchases of the riskless bill. This bids up the price of the bill, lowers the riskless rate and results in a higher equity risk premium. In sum, the theory of RBE offers a very simple explanation to the observed riskless rate and equity premium. It says that the riskless rate is, on average, low and the premium high because at most dates there is a minority of pessimist who, *by the rationality of belief conditions*, have the higher intensity level of belief about high stock prices in the future. These agents drive the riskless rate lower and the equity premium higher.

The "Forward Discount Bias" in foreign exchange markets is the result of the fact that in an RBE agents often make the wrong forecasts although they are right on the average. Hence, in an RBE the exchange rate fluctuates excessively due to the errors of the agents and hence at almost no date is the interest differential between two countries an unbiased estimate of the rate of depreciation of the exchange rate one period later. The bias is positive since agents who invest in foreign currency demand a risk premium on endogenous uncertainty which is above and beyond the risk which exists in an REE. The size of the bias is equal to the added risk premium due to endogenous uncertainty.

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**Key Words:** rational expectation equilibrium (REE), rational beliefs, rational belief equilibrium (RBE), endogenous uncertainty, state of belief, market volatility, equity risk premium, riskless rate, GARCH, forward discount bias, foreign exchange rates, OLG economy, correlation among beliefs, simulations.

# **Endogenous Uncertainty and Rational Belief Equilibrium: A Unified View of Market Volatility**

by

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## **I. The Basic Issue**

This paper surveys the unified view of market volatility that flows from the insight that volatility has two components. One is generated by "fundamental" forces which are outside the economy and I refer to them as *exogenous*. The second is propagated within the economic system and I refer to it as the *endogenous* component. Since the nature of exogenous shocks is well known, explaining the endogenous component is the main task of this paper. My exposition style in this survey is mostly non-technical but several sections are mathematically formal in order to ensure that a precise statement of the main concepts is provided.

Before explaining my theory, I briefly outline the perspective of the Market Efficiency Theory, or Rational Expectations Equilibrium (REE) on market volatility. My aim is to use REE as a *reference point for the evaluation of the problems which market volatility generates*. My account is brief since the theory of REE is well known.

The standard formulation of an equilibrium of an economy starts with the dynamic portfolio and consumption choices of households and the production, investment and dividend decisions of firms. The theory is closed with market clearing conditions but given the random nature of the underlying economy it follows that equilibrium quantities are all stochastic processes with an underlying probability law. I call this probability the "true" probability law. Most of what is done in modern research depends upon the utilization of this probability for computing objects like forecasts, theoretical covariances or security prices. Thus, the idea that equilibrium is represented by a *true* stochastic process is fundamental to modern thinking in economics.

The REE theory is based on several assumptions, but three of them are fundamental to my discussion here. These are:

- (A.1) The true probability law of the economy is stationary. In a *stationary* economy the joint probabilities of economic variables remain the same over time.
- (A.2) Economic agents know the true probability law underlying the equilibrium variables of the economy. This is the first component of "structural knowledge" which the agents

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are *assumed* to possess.

(A.3) Agents know the demand and supply functions of all other agents. They can compute equilibrium prices of commodities and assets in the present and in the future given any possible exogenous fundamental information in the future. This is the second component of structural knowledge which they possess.

When formulating uncertainty, the standard theory specifies an *exogenous* "state space" which describes all that the agents are uncertain about with respect to the external environment. Examples of exogenous variables include: weather conditions, earthquakes, technological changes, fire destruction etc. All equilibrium magnitudes depend upon the realization of the exogenous state but according to (A.3) all agents know the map between equilibrium magnitudes (e.g. production decisions of firms, prices, etc.) and the state. Consequently, all economic magnitudes vary only with the variability of the exogenous state over time. Moreover, it is then *an* assumption that given any observed information, all agents agree on the *interpretation* of such information.

The implication of these assumptions is that *all* financial risks and observed volatility arise from causes which are *external to the economy*. I call such uncertainty "Exogenous Uncertainty". Under the above theory, no risk can be propagated from within the economic system via human beliefs or actions and the volatility of equilibrium variables is exactly equal to the level justified by the variability of exogenous conditions.

The above discussion enables me to offer a simple summary of the conclusions of the theory of REE with respect to the nature of market volatility:

1. For each state of the exogenous fundamentals there is a *correct* equilibrium price of all securities in the market.
2. If you possess all exogenous fundamental information you are able to compute the correct prices of securities and hence all uncertainty about prices is resolved. By implication, hedging against the risks of all exogenous fundamentals is possible, in principle, and can control all risk associated with market volatility.
3. All market volatility is caused by exogenous forces.

These conclusions of REE have been at the foundation of contemporary research into the structure of market volatility. Unfortunately, they are in conflict with many empirical observations and with common experience of market participants. Indeed, the implications of this theory have been rejected in broad areas of economics. In order to discuss specific issues I note that there are several outstanding problems or paradoxes (sometimes called "anomalies") related to the functioning of financial markets which the REE theory failed to resolve and current academic research has aimed to develop special theories to explain each one of these paradoxes. Since I will offer a *unified* view of market volatility, such a *single* theory would be more convincing if it could solve simultaneously many of these problems. Here I focus on four central such problems:

- **Problem A:** Why are asset prices and foreign exchange rates much more volatile than their underlying fundamentals?
- **Problem B:** Why do models based on REE predict an equity risk premium over the riskless rate of around .5% and a rate of return on riskless short term debt of around 5.5% while over the last hundred years the average equity risk premium in the U.S. has been around 7% and the riskless rate has been around 1%?
- **Problem C:** Why do asset prices and returns exhibit the "GARCH" behavior of time varying variances when there are no fundamental factors to explain it?
- **Problem D:** Why have interest rate differentials (between two countries) been such poor predictors of future changes in foreign exchange differentials in contrast with rational expectations, giving rise to the "Forward Discount Bias"?

Those who rejected the theory of rational expectations have tended to drift in diverse directions. Some have concluded that financial markets are dominated by investors who perceive probabilities incorrectly or are vulnerable to the impact of fads and mass psychology. Others have concluded that for some unexplained reason the market can be irrational *sometimes* and each failed prediction of the theory has been ascribed to a corresponding incident of such irrationality. As a result, it is common to find in the investment community the argument that each instant of such *presumed* irrationality offers an opportunity for excess returns (i.e. when an investment opportunity is viewed as "excellent" and inexpensive). These perspectives are in conflict both with principles of rationality as well as with the hope of finding *one* explanation for all these phenomena.<sup>2</sup> This is my motivation for seeking a *unified* theory for market volatility.

I proceed by reviewing in Section II the basic premises of my theory of Rational Beliefs and the allied concept of "Endogenous Uncertainty" which are the cornerstones of my approach. Section III, which is the main section of this paper, is devoted to showing via simulation results how the theory which I propose resolves the four Problems outlined. Most of the material presented here is based on papers published in a volume by Kurz (ed)[1997]<sup>3</sup> and on Kurz and Motolese [1999].

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<sup>2</sup> Models of "Noisy" rational expectation equilibria have also attempted to address this problem within the rational expectations paradigm. In these models the noise in prices is assumed to be generated by the erratic trades of "noise traders" who are uninformed and irrational traders constituting a significant proportion of all traders in the market. I do not review this work in the present paper since it stands in sharp contrast to the basic rationality postulates of that paradigm. That is, since all the conclusions of a model of noisy rational expectations are driven by the arbitrary market actions of irrational traders, such a model should be viewed as a theory of irrational behavior with which one can prove anything. Also, from the empirical perspective it is hard to see who these noise traders are and since on average they lose money it is not clear what makes such traders survive.

<sup>3</sup> Kurz (ed) [1997] *Endogenous Economic Fluctuations: Studies in the Theory of Rational Beliefs*. Studies in Economic Theory No. 6, Berlin and New York: Springer-Verlag. The introductory Chapter 1 (Kurz [1997a]) and the "Applications" Part B consisting of Chapters 9, 10, 11 and 12 contain the details which explain the ideas and support the results reported in the present paper.

## II. Endogenous Uncertainty and Rational Beliefs

### II(A) *Rational Beliefs*

My theory of Rational Belief Equilibrium (RBE) developed in Kurz [1994a], [1994b] is based on the following alternative assumptions:

(AA.1) Despite the fact that the economy may undergo structural changes yielding non-stationarity, the economic universe is *stable* in the sense that statistical and quantitative analysis can be successfully carried out in it. In such a system the concept of "normal" patterns makes empirical sense and provides useful knowledge. It is represented by the long-term averages of economic variables. Thus, although our economy experiences technological and economic changes, the price/earning ratios of major indices have well known "normal" ranges and long-term (i.e. asymptotic) means, variances and covariances. Interest rates, growth rates, capital/output ratios etc. all have well known long-run average behavior which reveals some important dimensions of the true structure of the economy.

(AA.2) Economic agents do not know the true probability law underlying equilibrium magnitudes. This is the first component of structural knowledge which agents are assumed to lack.

(AA.3) Agents do not know the map from exogenous variables to equilibrium quantities in general and prices in particular. They have, however, access to the very large volume of all *past* data on the performance of the economy. This data they can use to statistically test any theory which they may develop about the functioning of the economy and of the financial markets. In this sense agents may learn something about structural relationships in the economy.

In formal terms, let  $x_t \in X \subseteq \mathbb{R}^N$  be a vector of all observables at date  $t$  and assume it to be  $N$ , finite. The sequence  $\{x_t, t = 0, 1, \dots\}$  is a stochastic process with true probability  $\Pi$ . I use the notation  $x = (x_0, x_1, \dots)$  for members of  $(X)^\infty$  and denote by  $\mathcal{B}((X)^\infty)$  the Borel  $\sigma$ -field of  $(X)^\infty$ . The space  $((X)^\infty, \mathcal{B}((X)^\infty), \Pi)$  is *the true* probability space and the dynamical system  $((X)^\infty, \mathcal{B}((X)^\infty), \Pi, T)$  represents the true economy.  $T$  is the shift transformation defined as follows: let  $x^t = (x_t, x_{t+1}, x_{t+2}, \dots)$  then  $x^{t+1} = Tx^t$ ,  $t = 0, 1, 2, 3, \dots$ . A belief of an agent is a probability  $Q$ ; the agent is adopting the theory that the probability space is  $((X)^\infty, \mathcal{B}((X)^\infty), Q)$ .

An agent who observes the data does not know  $\Pi$  and using past data he tries to learn this probability. I assume that date 0 has occurred "a long time ago" and at date  $t$ , when agents form their beliefs about the future beyond  $t$ , they have an *ample* supply of past data. Denote by  $x = (x_0, x_1, x_2, x_3, \dots)$  the vector of observations generated by the economy. In studying joint distributions among observables, one considers *blocks* rather than individual observations. For example, to study the distribution of  $(x_{\text{today}}, x_{\text{today}+1})$  one uses the blocks  $(x_0, x_1), (x_1, x_2), (x_2, x_3), \dots$ . Hence, for any  $B \in \mathcal{B}((X)^\infty)$  let the set  $T^{-n}B$ , which is the preimage of  $B$  under  $T^n$ , be defined

by

$$T^{-k}B = \{x \in X^\infty : T^k x \in B\} .$$

$T^{-k}B$  is the event  $B$  occurring  $k$  dates later. A dynamical system  $((X)^\infty, \mathcal{B}((X)^\infty), \Pi, T)$  is *stationary* if  $\Pi(B) = \Pi(T^{-1}B)$  for all  $B \in \mathcal{B}((X)^\infty)$ . A set  $S \in \mathcal{B}((X)^\infty)$  is *invariant* if  $S = T^{-1}S$ . A dynamical system is *ergodic* if  $\Pi(S) = 1$  or  $\Pi(S) = 0$  for any invariant set  $S$ . For simplicity I assume that  $((X)^\infty, \mathcal{B}((X)^\infty), \Pi, T)$  is ergodic although this is not needed (see Kurz [1994a] where this assumption is not made). In order to learn probabilities agents study the *frequencies* of all economic events. For example, consider the event  $B$

$$B = \left\{ \begin{array}{l} \text{price of commodity 1 today} \leq \$1, \text{ price of commodity 6 tomorrow} \geq \$3, \\ 2 \leq \text{quantity of commodity 14 consumed two months later} \leq 5 \end{array} \right\} .$$

Now using past data agents can compute for any finite dimensional set  $B$  the expression

$$m_n(B)(x) = \frac{1}{n} \sum_{k=0}^{n-1} 1_B(T^k x) = \left\{ \begin{array}{l} \text{The relative frequency that } B \text{ occurred among} \\ n \text{ observations since date } 0 \end{array} \right\}$$

where

$$1_B(y) = \begin{cases} 1 & \text{if } y \in B \\ 0 & \text{if } y \notin B \end{cases} .$$

This leads to a definition of a basic property which  $((X)^\infty, \mathcal{B}((X)^\infty), \Pi, T)$  has:

**Definition 1:** A dynamical system is called *stable* if for any finite dimensional set (i.e. cylinder)  $B$

$$\lim_{n \rightarrow \infty} m_n(B)(x) = \overset{\circ}{m} \text{ exists } \quad \Pi \text{ a.e.}$$

The assumption of ergodicity ensures that the limit in Definition 1 is independent of  $x$ . In Kurz [1994a] it is shown that the set function  $\overset{\circ}{m}$  can be uniquely extended to a probability  $m$  on  $((X)^\infty, \mathcal{B}((X)^\infty))$ . Moreover, relative to  $m$  the dynamical system  $((X)^\infty, \mathcal{B}((X)^\infty), m, T)$  is stationary. There are two observations to be made.

- (a) Given the property of stability, in trying to learn  $\Pi$  all agents end up learning  $m$  which is a *stationary* probability. In general  $m \neq \Pi$ : the true dynamical system  $((X)^\infty, \mathcal{B}((X)^\infty), \Pi, T)$  may not be stationary.  $\Pi$  *cannot be learned*.
- (b) Agents know that  $m$  may not be  $\Pi$  but with the data at hand  $m$  is the only thing that they can learn and agree upon.

These conclusions mean that although agents have no structural knowledge *they do have a*

common empirical knowledge. I have noted that a *stationary* economy is one in which all the joint probabilities of economic variables remain the same over time. *Stationary systems are stable but stable systems are not necessarily stationary.* A system which experiences new technologies and new social organizations is not likely to be stationary but may be stable. This distinction is the central motive for the above assumptions and for this reason requires a some explanation.

Our economy is driven by a process of technological and organizational change which dominates every aspect of life in human history. This process is very complex but has a distinct character: once a new technology or organizational structure is established, it remains in place for some time until a new one is developed to replace it. While a technology or social organization is in place, the economy appears to have a fixed structure (i.e. it is stationary) until the next change. For simplicity I use the term "regime" to refer to such episodes in which the structure of the economy and the market are relatively fixed. Note that a regime in which steam ships dominate the technological frontier is very long and will have within it many, much shorter, sub-regimes. Moreover, the term may be used for the description of short periods in which a market may be dominated by a fixed configuration of factors, some fundamental and others involving the beliefs and perceptions of investors. In Figure 1a I give an example of such a sequence of regimes and

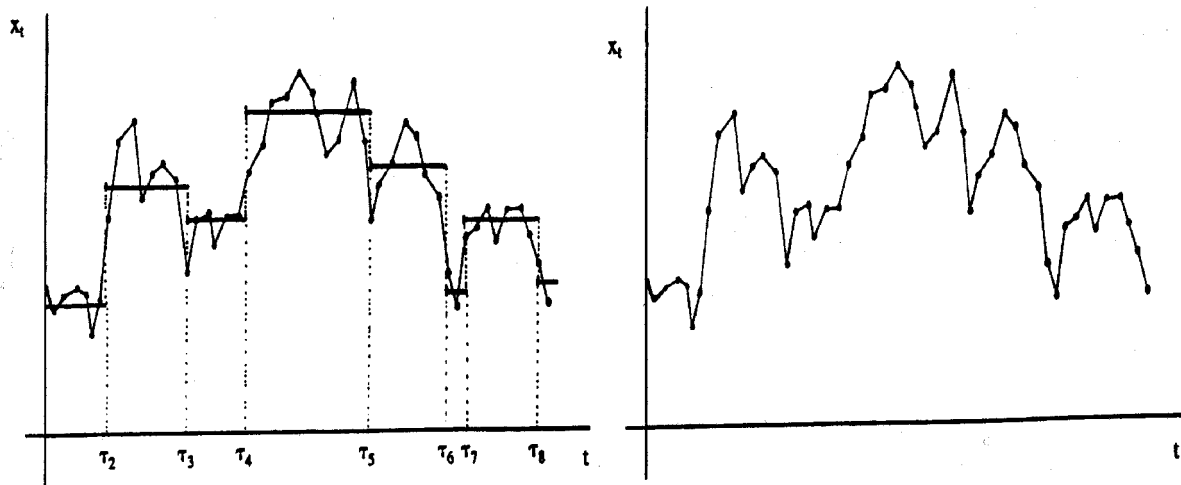


Figure 1a

Figure 1b

Legend

- (i)  $\tau_j$  are dates of regime change
- (ii) horizontal bars are mean value functions
- (iii) data seen with parameters of structural change

the data which they generate. The horizontal bars represent the mean value functions which are constructed as constant within each regime. Figure 1b shows how we see the data without the knowledge of either the start and end dates of each regime or the mean value function prevailing



within it.

The important feature of a market characterized as a sequence of regimes is that in real time no one knows exactly the parameters of the prevailing regime or its starting and ending dates. Assumptions (AA.1)-(AA.3) aim to capture this reality. They do not deny the fact that if a regime lasts long enough investors will figure out the character of the regime. Unfortunately, the fact that we can find out *in retrospect* the nature of the last regime does not mean that we learn the probability law of future observations or that we can correctly predict the next regime. This explains conclusion (a) above:

R(1) The true probability underlying the system cannot be learned and even if an agent discovers it, he cannot be sure that it is the true probability. Equally so, economic agents cannot learn the equilibrium map between market prices and those variables which determine prices. Such a map may change over time.

Assumptions (AA.1)-(AA.3) also specify *what the agents do know* and this fact is the basis for the next development. Specifically, assumption (AA.3) means that all agents know the empirical distribution of past data from which they all deduce the same stationary probability  $m$  specified in conclusion (b). Observe that  $m$  summarizes the entire collection of *asymptotic restrictions* imposed by  $((X)^\infty, \mathcal{B}((X)^\infty))$ ,  $\Pi$ ,  $T$ ) on the empirical distribution of all variables. This common empirical knowledge provides the basis for a new definition of the *rationality* of beliefs. I now proceed formally.

It is shown in Kurz [1994a] that for each stable dynamical system with probability  $\Pi$  there is a set  $B(\Pi)$  of stable probabilities  $Q$  with dynamical systems which generate the same stationary probability  $m$  and hence *impose the same asymptotic restrictions on the data as the true system with  $\Pi$* . The question is how to determine *analytically* if any proposed belief  $((X)^\infty, \mathcal{B}((X)^\infty))$ ,  $Q$ ,  $T$ ) generates  $m$  as a stationary measure. To examine this question consider, for any cylinder  $B$  the set function

$$m_n^\Pi(B) = \frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}B).$$

I note that  $m_n^\Pi(B)$  has nothing to do with data: it is an analytical expression derived from  $((X)^\infty, \mathcal{B}((X)^\infty))$ ,  $\Pi$ ,  $T$ ).

**Definition 2:** A dynamical system  $((X)^\infty, \mathcal{B}((X)^\infty))$ ,  $\Pi$ ,  $T$ ) is said to be *weak asymptotically mean stationary* (WAMS) if for all cylinders  $S \in \mathcal{B}((X)^\infty)$  the limit

$$\overset{\circ}{m}^\Pi(S) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}S) \text{ exists.}$$

It is shown in Kurz [1994a] that  $\overset{\circ}{m}^\Pi$  can be uniquely extended to a probability measure  $m^\Pi$  on

$((X)^\infty, \mathcal{B}((X)^\infty))$  and  $((X)^\infty, \mathcal{B}((X)^\infty)), m^\Pi, T$  is stationary. I then have a theorem which is the main tool in Kurz [1994a]:

**Theorem 1:**  $((X)^\infty, \mathcal{B}((X)^\infty)), \Pi, T$  is stable if and only if it is WAMS. If  $m$  is the stationary measure calculated from the data, then  $m(S) = m^\Pi(S)$  for all  $S \in \mathcal{B}((X)^\infty)$ .

The implication of Theorem 1 is that every stable system  $((X)^\infty, \mathcal{B}((X)^\infty)), Q, T$  generates a *unique* stationary probability  $m^Q$  which is *calculated analytically* from  $Q$ . This last fact is the foundation of the following:

**Definition 3:** A selection of belief  $Q$  cannot be contradicted by the data  $m$  if

- (i) the system  $((X)^\infty, \mathcal{B}((X)^\infty)), Q, T$  is stable ,
- (ii) the system  $((X)^\infty, \mathcal{B}((X)^\infty)), Q, T$  generates  $m$  and hence  $m^Q = m$  .

**Rationality Axiom:** A belief  $Q$  by an agent is a *Rational Belief* if it satisfies

- (I) *Compatibility with the Data:*  $Q$  cannot be contradicted by the data.
- (II) *Non-Degeneracy:* if  $m(S) > 0$  , then  $Q(S) > 0$  .

To express a belief in the non-stationarity of the process, an agent selects a probability  $Q^\perp$ . This probability is said to be *orthogonal with  $m$*  if there are events  $S$  and  $S^c$  such that

- (i)  $S \cup S^c = (X)^\infty$  ,  $S \cap S^c = \emptyset$  ,
- (ii)  $m(S) = 1$  ,  $m(S^c) = 0$  ,
- (iii)  $Q^\perp(S) = 0$  ,  $Q^\perp(S^c) = 1$  .

I want to characterize the set  $B(\Pi)$  of Rational Beliefs when the empirical distribution implies a stationary measure  $m$  induced by  $((X)^\infty, \mathcal{B}((X)^\infty)), \Pi, T$  .

**Theorem 2 (Kurz [1994a]):** Every Rational Belief must satisfy  $Q = \lambda Q_a + (1 - \lambda)Q^\perp$  where  $0 < \lambda \leq 1$ ,  $Q_a$  and  $m$  are probabilities which are mutually absolutely continuous (i.e. they are equivalent) and  $Q^\perp$  is orthogonal with  $m$  such that

- (i)  $((X)^\infty, \mathcal{B}((X)^\infty)), Q_a, T$  and  $((X)^\infty, \mathcal{B}((X)^\infty)), Q^\perp, T$  are both stable,
- (ii)  $m^{Q_a} = m^{Q^\perp} = m$  .

Moreover, any  $Q$  such that  $\lambda, Q_a$  and  $Q^\perp$  satisfy the above is a Rational Belief.

A rational belief must then have the property that if one simulates the model, over time it will generate the same empirical distribution as the one that was generated by the historical record of the market. Thus the concept of a rational belief isolates that subset of all possible theories or models that cannot be contradicted *by* the available data.

In my approach, the rationality of beliefs rests on the premise that the economic universe (or some transformation of it, in case of a growing economy) is stable so that two rational agents holding two different theories cannot disagree about the long run statistics which both of their individual theories are required to "reproduce". If any model generates long term statistics which differ from the empirical evidence, it is judged wrong and the underlying belief judged irrational. I will now explain other important interpretations and implications of Theorem 2.

## II(B) *Diversity of Beliefs and Mistakes in a Rational Belief Equilibrium*

A dynamically changing but stable economy is one in which economic variables may be transformed (e.g. into logs or into growth rates rather than absolute values if needed) so that although structural changes take place, all long term frequencies and averages converge. These frequencies and averages are learned by all agents and represent the "normal" probabilities of events. Investors often consult such information when they describe how frequently a certain pattern of events happened over the last two hundred years! An agent who believes that the world is stationary would adopt these normal frequencies as his belief. This result can be summed up by:

R(2) The theory holds that an agent who adopts the normal frequencies as his belief is rational since his belief is compatible with the empirical distributions.

Note, however, that such a person must also believe that the joint probability distributions of economic and financial variables in the 1990's are the same as the joint distributions in the 1890's and both are equal, according to him, to the joint distributions computed as averages over many past years. That is, he believes that no structural changes ever take place or that technological or structural changes in the real economy have a *neutral* effect on financial markets and thus have no effect on the structure of market performance.

If the economic system is stationary and if all the agents *knew for sure* that it is stationary, then they will all learn the true probability law of motion and will know that this true law of motion is the one calculated from the empirical distributions of past events. Under such circumstances there will be no disagreement in that economy.

In contrast, I have already expressed my view that the process of structural change (i.e. non-stationarity) in our society is *the* central building block of its complexity and the root cause of the diversity of beliefs about it. In such a system the past is not an entirely satisfactory basis for assessment of risks in the future and at every date many agents hold the view that the market may be *similar* to the past but yet very different. Hence, an agent who forms a forecast which is different from the historical statistical average is adopting a sharper view of the future than can be deduced from the statistics of the past. Such a theory may not be contradicted by past data but past data is not required to support it either. That is, an agent who holds a theory of the market which insists that the situation today is *different* from the past does not support his theory by the statistics of the past. He may offer some statistical evidence of *recent* developments to bolster his model but such evidence would lack high statistical reliability and thus may not be acceptable to

other agents. His theory may sometimes be right and sometimes be wrong.

What is the patterns of disagreement among these rational agents? Motivated by the observations above, Theorem 2 shows that:

R(3) The main source of disagreement among agents derives from the fact that they can hold different theories both about the *nature* and *intensity* of changes in the economy as well as their *timing*. As a result, given commonly observed news at any date, agents can have very different opinions regarding the significance of the news to future market performance. For example, some may be optimistic while others are pessimistic about it.

The mere fact that agents disagree has an immediate and very important implication.

R(4) A group of economic agents who hold rational beliefs and *pairwise* disagree forever (at all times and in the limit rather than have a one-time disagreement) must also experience variations in the probabilities with which they forecast future economic events at different dates. *This means that in a world of disagreement the states of belief of these agents must fluctuate over time.*

I stress that conclusion R(4) is a *consequence* of the theory of rational beliefs together with the observations that agents disagree. To understand why this conclusion holds note that if a group of agents disagree pairwise forever then at least all but one of them must not believe that the economy is stationary and hence they do not permanently adopt the normal frequencies as their beliefs. However, their beliefs must be compatible with the normal frequencies in the exact sense that *deviations of their one period probability beliefs from the normal frequencies must average to zero*. That is, if you are optimistic relative to the normal frequencies in some dates you must be pessimistic relative to those frequencies in other dates so that on average you expect your deviations from the normal frequencies to average to zero. But then it follows *that all permanent disagreements imply variability in probability beliefs around the normal frequencies.*

Let me examine the implication of R(4). It says that if we observe a market in which there is always some disagreement among agents who hold rational beliefs then their disagreements are not fixed. If we study those disagreements we shall find that they are the result of on-going reassessment and the states of beliefs of the disagreeing individuals are changing over time. Note that this does not mean that the distribution of beliefs in the market as a whole will be changing over time as well. I return to this important subject when I discuss in IV(iv) the results regarding the equity risk premium.

The dual requirement of stability and of compatibility with the empirical distributions impose restrictions on the models of the economy which a rational agent can adopt as his belief. Nevertheless, the theory allows sufficient heterogeneity of beliefs to persist over time so that the subjective models used by the agents may imply forecast functions which can be different for different agents *at all dates*. In short, my theory permits two intelligent investors who observe

the same vast information about the past to have different opinions and hence to make different forecasts of the future.

If there is a *true and unknown* equilibrium probabilistic law of motion underlying the dynamics of the market, and if there are substantial differences in probability beliefs among the agents about the future, then, although all the agents are rational, most may be holding wrong beliefs. This leads them to make *forecasting mistakes*. To clarify this point recall Figures 1a-1b which reveal the problem of an agent who forms a belief about the market. Suppose that the price/earnings ratio of an index of his interest is the highest in 40 years. If he follows the statistics of the long past he will compute the fact that, say, only in 7.8% of past cases the price/earnings ratio went higher than the observed level and hence the probability of capital gains is 7.8%. With such probability the investor decides that the index is too high and his portfolio decision is to sell. Another investor, observing the identical information about prices and earnings, formulates a model about the future productivity of the firms in the index on the basis of which he concludes that the statistical record of the past is not entirely applicable. Based on his model, he believes that the probability of higher prices is 60% on the basis of which his portfolio decision is to buy.

I suggest that one or both of the two investors hold wrong beliefs and are thus making a mistake. More formally, the *mistake* of an agent at date  $t$  is defined as the difference between the collection of his forecasts at date  $t$  conditional upon the information at that date and the forecasts that would be made with the correct model, were it known. Since an agent selects his decisions based on his beliefs, these mistakes in beliefs get translated into mistaken *actions*. In equilibrium, quantities and prices will reflect those mistakes. Thus, the economic variables which we observe at each date contain the mistakes of the agents and this fact will be the foundation of the concept of Endogenous Uncertainty.

I caution against a simplistic interpretation of the term "mistake". In its daily use this term usually refers to acts or thoughts which are wrong but *which could have been avoided*. Here a "mistake" is a rule by which a rational agent utilizes information efficiently but fails to make the correct forecast. In fact, it is essential that there is no statistical way through which an agent can be assured of avoiding making a "mistake" in my sense. Thus, in the context of this theory *rational agents make mistakes*. The theory does not say that agents who form an opinion which deviates from the statistical norm be "certain" or sure of the truth of their model. What the agents do know is that without committing to an investment program that will take advantage of the changing conditions of the market, they cannot make excess returns.

My approach implies, therefore, that the nature of "risk assessment" by the agents is quite different from the usual analysis of the covariance structure among asset returns. For these agents the market is an arena for the competition among theories that seek to capture future excess returns. In such a market the risky nature of a decision is tied to a commitment to a theory of the market without having *statistically* reliable evidence in support of such a theory. "Assessment" of such risks has something to do with the way we *interpret* existing information rather than with a utilization of past covariances. This is particularly true in an environment of changing regimes

where advanced signals about the coming regime may be available, but agents have insufficient evidence to be able to interpret such information with a high level of statistical reliability.

An economic equilibrium in which all agents hold rational beliefs is called a *Rational Belief Equilibrium* (RBE). In such an equilibrium the investment, consumption and portfolio decisions are, in part, determined by the mistakes of the agents and these effects can be substantial. Hence the mistakes of agents have an effect on equilibrium prices and on the real allocations in the economy. Alternatively, in an RBE the beliefs of agents have *real effects* on the performance of the economy; they influence the volatility of economic variables such as output, investment and prices. This leads to the fifth result:

R(5) If individual agents can make mistakes in the assessment of market values, then the market as a whole can also evaluate assets "incorrectly". This conclusion should be understood in the sense that such pricing can be different from that pricing that would be justified by the true market forecast. Equilibrium market prices may overshoot above "fundamental values" when asset prices rise and may overshoot downward, below "fundamental values" when asset prices decline.

This conclusion shows that an important component of the volatility of economic variables is generated by the mistakes of agents. To see why this could be important, suppose for example that some investors develop a theory according to which a particular imminent development may adversely affect the profits of some firm. The actions of these investors will induce a fall in the price of the shares of the firm with no exogenous event to "justify" it. Moreover, if the theory of these agents is wrong, prices will ultimately return to their original position and the entire move would have been induced only by the forecasting mistakes of the agents. Similar arguments apply to other variables such as an investment by a firm or a purchase of foreign currency by a trader: beliefs have real effects on the fluctuations of economic variables. That component of volatility beyond the level that is justified by the exogenous variables is therefore said to be *internally propagated*. I call this type of uncertainty *Endogenous Uncertainty*<sup>4</sup>.

## II(C) *Components of Endogenous Uncertainty*

Anticipating the developments in Section III below I briefly evaluate the specific factors which contribute to this component of market volatility. Think of a market in which, at any date or over a period, an agent holds a probability belief about future economic events which deviates from the normal pattern. For example, the agent may sometimes be relatively optimistic and sometimes relatively pessimistic about future increases of price/earnings ratios relative to the probability  $m$ . It turns out that in order to assess how these levels of *relative* optimism and pessimism contribute to market volatility over time we need to focus on the fluctuations in the distribution of beliefs. For example, Compare a distribution in which 5% of the agents are optimistic, 5% are pessimistic and 90% are neutral with a distribution in which 50% are optimists and 50% are pessimists. Although both distributions are "balanced," it is a fact that the latter

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<sup>4</sup> This component of market uncertainty is called *Endogenous Uncertainty* in Kurz [1974].

*contributes to market volatility much more than the former.* It is important to understand the two components of endogenous volatility:

1. *Amplification of exogenous shocks (Overshooting).* In an REE, Markov exogenous shocks, which alter the profit stream of a firm have an effect on price volatility. I define the price fluctuations generated by such exogenous shocks as those fluctuations which are "justified by dividends." The impact of endogenous amplification is simply to increase the effect of these exogenous shocks so that the price fluctuations could be much higher than those justified by the dividends. This is what is commonly known as the "overshooting" phenomenon in stock prices. In the models that will be discussed later, the degree of overshooting is very large.

2. *Pure endogenous volatility.* The second component of endogenous volatility is pure volatility. In models that have finite number of possible equilibrium prices this component simply generates new price states. That is, there are more possible prices in an RBE than in an REE. Indeed, in the typical model that will be discussed later, there are two exogenous dividend shocks leading to two prices under REE. Under RBE there are 8 possible prices generated by the exogenous shocks *and by the states of beliefs*. Hence, the pure effect is represented by the additional prices. In REE with infinite number of prices this distinction is more complicated and can be defined by regressing prices on the exogenous shocks: the component of price variability over the REE level which is explained by the exogenous shocks is defined as *amplification* and the component that cannot be explained by exogenous shocks is defined as *pure endogenous volatility*.

### III Explaining the Paradoxes: Simulation Analysis

I have suggested to the reader that my theory offers a unified paradigm to solve the four problems formulated in Section I. Here I review these solutions in the form of simulation results of models with endogenous uncertainty. Since the questions span issues related not only to the domestic but also to the international economy, I present the results of two slightly different models: one of the domestic economy and a second of the international economy<sup>5</sup>. The two models have the same basic structure which I shall review first. After this review I present the results and interpret them.

#### III(A) *The Basic OLG Models*

I will review the domestic component of the OLG model and then comment on the multi country version.

III(A.1) *The Economy.* I employ a two-agent, OLG, economy with a homogenous consumption

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<sup>5</sup> All numerical results for the domestic economy are developed in Kurz and Schneider [1996] in Kurz and Beltratti [1997] and Kurz and Motolese [1999] who utilize similar models. The results for the international economy are in Kurz [1997c] and Black [1997].

good. Each agent lives two periods, the first when he is "young" and the second when he is "old." A young agent is a replica of the old agent who preceded him, where the term "replica" refers to *utilities* and *beliefs*, and hence this is a model of two infinitely lived "dynasties" denoted by  $k = 1, 2$ . Only young agents receive an endowment  $\Omega_t^k$ ,  $t = 1, 2, \dots$  of the consumption good. This endowment is viewed as the labor income of the agents; the stochastic processes  $\{\Omega_t^k, t = 1, 2, \dots\}$  for  $k = 1, 2$  will be specified below. Additional net output is supplied by a firm which produces exogenously, as in Lucas [1978], the strictly positive dividend process  $\{D_t, t = 1, 2, \dots\}$  with no input.  $D_t$  is paid out as dividends at date  $t$  to the owners of the shares of the firm. The shares are traded on a public stock market and their aggregate supply is 1.

The economy has (i) a market for the single consumption good, (ii) a stock market as specified above and (iii) a market for a zero net supply, short term debt instrument which we call a "bill". Since  $\{d_t, t = 1, 2, \dots\}$  is a Markov process with two states, *the economy has a complete financial structure*: the number of assets equals the number of states. To ensure intergenerational efficiency, the financial sector is initiated at date 1 by distributing the supply of ownership shares among the old of that date. My notation is as follows: for  $k = 1, 2$

- $C_t^{1k}$  - consumption of  $k$  when young at  $t$ ;
- $C_{t+1}^{2k}$  - consumption of  $k$  when old at  $t + 1$  (implying that the agent was born at  $t$ );
- $D_t$  - total amount of profits or dividends produced exogenously at  $t$ ;
- $\theta_t^k$  - amount of stock purchases by young agent  $k$  at  $t$ ;
- $B_t^k$  - amount of one period bill purchased by young agent  $k$  at  $t$ ;
- $\Omega_t^k$  - endowment of young agent  $k$  at  $t$ ;
- $P_t$  - the price of the common stock at  $t$ ;
- $q_t$  - the price of a one period bill at  $t$ . This is a discount price;
- $I_t$  - history of all observables up to  $t$ ;
- $d_{t+1} = \frac{D_{t+1}}{D_t}$  - the random growth rate of profits or dividends;
- $p_t = \frac{P_t}{D_t}$  - the price/dividend ratio of the common stock at  $t$ ;

Consumption is used as a numeraire and hence the optimization problem of agent  $k$  at date  $t$  has the following structure where  $Q^k$  is a probability belief of agent  $k$ :

$$(1a) \quad \text{subject to} \quad \text{Max}_{(C_t^{1k}, \theta_t^k, B_t^k, C_{t+1}^{2k})} E_{Q^k} \left\{ u^k(C_t^{1k}, C_{t+1}^{2k}) \mid I_t \right\}$$

$$(1b) \quad C_t^{1k} + P_t \theta_t^k + q_t B_t^k = \Omega_t^k$$

$$(1c) \quad C_{t+1}^{2k} = \theta_t^k (P_{t+1} + D_{t+1}) + B_t^k.$$

To enable the computation of equilibria I take the utility function agent  $k$  to be



$$(2) \quad u^k(C_t^{1k}, C_{t+1}^{2k}) = \frac{1}{1 - \gamma_k} (C_t^{1k})^{1 - \gamma_k} + \frac{\beta_k}{1 - \gamma_k} (C_{t+1}^{2k})^{1 - \gamma_k}, \quad \gamma_k > 0, \quad 0 < \beta_k < 1.$$

With this specification the Euler equations for agent  $k$  are

$$(3a) \quad -P_t (C_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((C_{t+1}^{2k})^{-\gamma_k} (P_{t+1} + D_{t+1}) | I_t) = 0$$

$$(3b) \quad -q_t (C_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((C_{t+1}^{2k})^{-\gamma_k} | I_t) = 0.$$

The dividend process is as specified in Mehra and Prescott [1985]. It takes the form

$$(4) \quad D_{t+1} = D_t d_{t+1}.$$

where  $\{d_t, t = 1, 2, \dots\}$  is a stationary and ergodic Markov process. The state space of the process is  $J_D = \{d^H, d^L\}$  with  $d^H = 1.054$  and  $d^L = .982$  and a transition matrix

$$(5) \quad \begin{bmatrix} \phi, & 1 - \phi \\ 1 - \phi, & \phi \end{bmatrix}$$

with  $\phi = .43$ . Hence, over time agents experience a secular rise of total dividends and it is therefore convenient to focus on growth rates. To do that let

$\omega_t^k = \frac{\Omega_t^k}{D_t}$  is the endowment/dividend ratio of agent  $k$  at date  $t$ ;

$b_t^k = \frac{B_t^k}{D_t}$  is the bill/dividend ratio of agent  $k$  at date  $t$ ;

$c_t^{1k} = \frac{C_t^{1k}}{D_t}$  is the ratio of consumption when young to aggregate capital income;

$c_{t+1}^{2k} = \frac{C_{t+1}^{2k}}{D_{t+1}}$  is the ratio of consumption when old to aggregate capital income.

In the domestic model  $\omega_t^k = \omega^k$  for  $k = 1, 2$  are constant. Now divide (1b) by  $D_t$ , (1c) by  $D_{t+1}$ , (3a) by  $D_t^{1 - \gamma_k}$  and equation (3b) by  $D_t^{-\gamma_k}$  to obtain for  $k = 1, 2$

$$(6a) \quad c_t^{1k} = -p_t \theta_t^k - q_t b_t^k + \omega^k,$$

$$(6b) \quad c_{t+1}^{2k} = \theta_t^k (p_{t+1} + 1) + \frac{b_t^k}{d_{t+1}},$$

$$(6c) \quad -p_t (c_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((c_{t+1}^{2k} d_{t+1})^{-\gamma_k} (p_{t+1} + 1) d_{t+1}) | I_t) = 0,$$

$$(6d) \quad -q_t (c_t^{1k})^{-\gamma_k} + \beta_k E_{Q_t^k}((c_{t+1}^{2k} d_{t+1})^{-\gamma_k} | I_t) = 0.$$

The optimum conditions (6a) - (6d) imply the following demand functions for  $k = 1, 2$

$$(7a) \quad b_t^k = b_t^k(p_t, q_t, d_t, I_t),$$

$$(7b) \quad \theta_t^k = \theta_t^k(p_t, q_t, d_t, I_t).$$

Equilibrium requires the market clearing conditions

$$(7c) \quad \theta_t^1 + \theta_t^2 = 1$$

$$(7d) \quad b_t^1 + b_t^2 = 0;$$

The equilibrium implied by (7a)-(7d) depends upon the beliefs of the agents. I study Markov equilibria with a finite number of prices. An equilibrium is characterized either by one Markov matrix or by a sequence of such matrices which describe the transition from a price state to another. However, the stationary measure  $m$  will be described by a *single* transition matrix from prices at  $t$  to prices at  $t + 1$  which I call  $\Gamma$ . The two agents hold rational beliefs  $Q^k$  which are stable Markov probabilities with stationary measures defined by  $\Gamma$ . It is clear that the rationality of belief conditions can be very complicated. The technique of "assessment variables" is the main technical development in Nielsen [1996] and Kurz and Schneider [1996]. It enables a simple description of a large family of rational beliefs.

III(A.2) *Rational Belief Equilibrium.* Assessment variables of agents are sequences of random variables  $\{y_t^k, t = 1, 2, \dots\}$  for  $k = 1, 2$  and here I assume that  $y_t^k \in Y = \{0, 1\}$ . The belief of agent  $k$  is a probability  $Q^k$  over the *joint* process  $\{(p_t, q_t, d_t, y_t^k), t = 1, 2, \dots\}$  which is a Markov process. The decision functions in (6a) - (6d) are selected based on the *conditional probability* of  $Q^k$  given the value of  $y_t^k$ .

As a matter of economic interpretation, assessment variables are parameters indicating how an agent perceives the state of the economy and are thus tools for the description of stable and non-stationary processes. In the model at hand they are the method of describing if an agent is optimistic or pessimistic at date  $t$  about capital gains at date  $t + 1$ . I thus need to clarify how assessment variables enter the decision mechanism of agent  $k$ . Note that in (6c) - (6d) agent  $k$  specifies the probability of  $(p_{t+1}, q_{t+1}, d_{t+1}, y_{t+1}^k)$  *conditional* on  $(p_t, q_t, d_t, y_t^k)$  - the value of his assessment variable jointly with the observed data. It then follows from the Markov assumptions that the demand functions of agent  $k$  for stocks and bills are functions of the form

$$(10a) \quad b_t^k = b^k(p_t, q_t, d_t, y_t^k)$$

$$(10b) \quad \theta_t^k = \theta^k(p_t, q_t, d_t, y_t^k).$$

Consequently the market clearing conditions are

$$(10c) \quad \theta^1(p_t, q_t, d_t, y_t^1) + \theta^2(p_t, q_t, d_t, y_t^2) = 1$$

$$(10d) \quad b^1(p_t, q_t, d_t, y_t^1) + b^2(p_t, q_t, d_t, y_t^2) = 0.$$

The system (10c)-(10d) implies that the equilibrium map of this economy takes the form

$$(11) \quad \begin{bmatrix} p_t \\ q_t \end{bmatrix} = \Phi^*(d_t, y_t^1, y_t^2).$$

The equilibrium map (11) shows that prices are determined by the exogenous shock  $d_t$  and by the endogenous "state of belief"  $(y_t^1, y_t^2)$ . Here  $y_t^k \in \{0, 1\}$  for  $k = 1, 2$  and  $y_t^k = 1$  means that  $k$  is in a state of optimism while  $y_t^k = 0$  means that  $k$  is in a state of pessimism. The appearance of the endogenous vector  $(y_t^1, y_t^2)$  in the equilibrium map is the precise way in which *Endogenous Uncertainty* is present in the equilibrium.

Condition (11) implies that there are at most 8 *distinct* price vectors  $(p_t, q_t)$  that may be observed corresponding to the 8 combinations of  $(d_t, y_t^1, y_t^2)$ . Moreover, the *true* equilibrium transition probability from the 8 prices  $(p_t, q_t)$  to the 8 prices  $(p_{t+1}, q_{t+1})$  is determined by the transition from  $(d_t, y_t^1, y_t^2)$  to  $(d_{t+1}, y_{t+1}^1, y_{t+1}^2)$ . I select the joint process  $\{(d_t, y_t^1, y_t^2), t = 1, 2, \dots\}$  to be a stationary Markov process with a transition matrix  $\Gamma$ . This choice<sup>6</sup> implies that the true equilibrium process of prices  $\{(p_t, q_t), t = 1, 2, \dots\}$ , has  $\Gamma$  as the *fixed* stationary transition probability from  $(p_t, q_t)$  to  $(p_{t+1}, q_{t+1})$ . Hence, the agents who compute the empirical distribution of the process will discover  $\Gamma$ . Although this matrix characterizes the empirical distribution of the equilibrium dynamics, the agents *do not know this fact and form rational beliefs relative to  $\Gamma$* . Indeed, the fact that they form rational beliefs in accord with their assessment variables is the reason why  $\Gamma$  is the equilibrium probability of the implied RBE.

An assessment variable  $y_t^k$  determines completely the transition matrix from  $(p_t, q_t)$  to  $(p_{t+1}, q_{t+1})$  perceived by agent  $k$  at date  $t$ . Moreover,  $y_t^k \in \{0, 1\}$  implies that the agent has at most two  $8 \times 8$  Markov matrices and at each date the value taken by his assessment variable determines which of the two the agent uses. This means that over time the empirical frequencies determine  $\Gamma$  while each agent uses two Markov matrices  $(F_1, F_2)$  for agent 1 and  $(G_1, G_2)$  for agent 2. What are then the *rationality of belief conditions*? This technical question is fully answered by the fundamental *Conditional Stability Theorem* of Kurz and Schneider [1996] (see page 494). It turns out that the rationality conditions depend upon the marginal distributions of  $y_t^k$  for  $k = 1, 2$ . For simplicity I assume that these distributions are i.i.d. with  $Q^k\{y_t^k = 1\} = \alpha_k$  for  $k = 1, 2$ . The Conditional Stability Theorem then implies that the beliefs  $Q^1$  and  $Q^2$  of the

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<sup>6</sup> The choice of the equilibrium dynamics being generated by a fixed, stationary matrix is a matter of convenience and simplicity. The process  $\{(d_t, y_t^1, y_t^2), t = 1, 2, \dots\}$  could have been selected to be any stable process with a Markov stationary measure induced by the empirical distribution. In such a case the fixed transition matrix  $\Gamma$  would characterize only the *stationary measure* of the equilibrium dynamics rather than be the matrix of the *true* probability of the equilibrium dynamics of prices.

two agents are described by the following rule:

$$(12a) \quad \begin{array}{l} Q^1 \text{ for agent 1: use } F_1 \text{ if } y_t^1 = 1 \\ \text{use } F_2 \text{ if } y_t^1 = 0 \end{array} \quad \begin{array}{l} Q^2 \text{ for agent 2: use } G_1 \text{ if } y_t^2 = 1 \\ \text{use } G_2 \text{ if } y_t^2 = 0. \end{array}$$

The rationality of belief conditions then require that

$$(12b) \quad \alpha_1 F_1 + (1 - \alpha_1) F_2 = \Gamma \quad , \quad \alpha_2 G_1 + (1 - \alpha_2) G_2 = \Gamma .$$

The rational agents believe that the price-dividend process is not stationary and their beliefs are parametrized by their private assessment variables  $(y_t^1, y_t^2)$ . (12b) requires that the sequence of matrices which they adopt generates the same empirical distribution as the Markov process with transition matrix  $\Gamma$ .  $\alpha_1$  is the frequency at which agent 1 uses Matrix  $F_1$  and  $\alpha_2$  is the frequency at which agent 2 uses matrix  $G_1$ .

I now specify the matrixes  $(F_1, F_2)$  and  $(G_1, G_2)$ . One needs to specify only  $F_1$  and  $G_1$  since (12b) determines  $F_2$  and  $G_2$ . I have already noted that the two agents can be optimistic or pessimistic about capital gains in the future. Since high prices are associated with  $d^H$  they are the first four prices in the matrix  $\Gamma$ . It follows that optimism in  $F_1$  and  $G_1$  is expressed by a proportion  $\lambda > 1$  by which entries in the first four rows of  $F_1$  and  $G_1$  are increased relative to the corresponding entries in  $\Gamma$ . Thus, optimism or pessimism *is always defined relative to the long run conditional frequencies defined by  $\Gamma$* . Think of  $\lambda$  as the *intensity of optimism*. The rationality conditions (12b) are linear; they specify that for any period of optimism there must be a corresponding period of pessimism. But pessimism is represented by lower conditional probabilities at  $t$  of capital gains at  $t + 1$  relative to  $\Gamma$ . Hence, as the intensity of optimism  $\lambda$  rises, the pessimistic probabilities approach 0. In the limit there is a finite intensity of optimism which brings the pessimist at  $t$  to the 0 probability of high prices in  $t + 1$ . I mention these facts here since they play a crucial role in the analysis below.

I note that an REE is an RBE with  $Q^1 = Q^2 = \Pi$  and, deduced from (5), the probabilities of  $(p_{t+1}, q_{t+1}, d_{t+1})$  in (6c) - (6d) are conditioned only on the realized value of  $d_t$ . Since the growth rate of dividends takes two values a Markov REE is, in fact, a stationary equilibrium with two prices and two optimal portfolios.

The international model includes money and allows for monetary policies of the two economies. Since it is not my aim to study different monetary policies, I fix the policies in the two countries. They are set so that each country responds to its own exogenous shock: the domestic central bank adjusts the money supply in response to the random changes in the growth rate of earnings and the foreign central bank adjusts the money supply to changes in the growth rate of wages. In either country the objective of the bank is to maintain price stability. The foreign economy is purely hypothetical.

### III(B) *On the Method of Simulations.*

What is the logic of a simulation model and why should we consider this method of analysis valid? To answer this question I note first that the parameters of the real economy are selected so as to conform to well known parameters of econometric models that were estimated for the U.S. economy. These include the long term growth rates of wages and earnings and the coefficients of risk aversion and discount rates of the agents. As a result, the real part of the economy is required to act in conformity with what we know about the long run tendencies of the U.S. economy. Hence, *the fundamentals of the economy are exactly the same as we know from the statistics of the real economy.* The parameters which I, as a model builder, will select are those that relate to the beliefs of the agents and their distributions. The simulation models then ask what would be the implications of alternative belief structures for price volatility, *holding the fundamentals fixed.* Since rational expectations are among the beliefs which can be examined in the model, the results below will provide a comparison between the implications of rational expectations and rational beliefs for price volatility, keeping the real economy the same.

It has been well documented that if one imposes on the real fundamentals of the simulation models the assumption of rational expectations by the agents, all the problems and paradoxes specified earlier will appear and I shall demonstrate that this remains true in the models at hand. However, if I can show that under the assumption that the agents hold rational beliefs the financial markets will not exhibit any of the paradoxes, then it follows that the belief structure of the agents does provide a unified paradigm to resolve the specified problems. It would then be useful to have an intuitive understanding of the structure of beliefs that generate the various conclusions and I will attempt to provide some interpretation in a later section.

The foreign economy is a purely hypothetical economy; it is not calibrated to any particular economy. The two economies will have a common ("world") stock market and the foreign economy will have an exogenous endowment shock.

### III(C) *Simulation Demonstration of the Solutions to the Four Problems*

In the Tables below I present comparisons between the simulation results under rational expectations and under rational beliefs. The aim is first to exhibit what are the problems which arise under REE and then to show that these problems are significantly resolved under the unified paradigm of the theory of rational beliefs. The sequence of the tables below correspond to the questions posed at the start.

#### III(C.1) Problem A: Asset Price Volatility in the Domestic Economy

Table 1 reports two measures of price volatility. The first is the interval in which the price/earnings ratio fluctuates 95% of the time. The long term mean of this variable is fixed at 13.9 which is the actual long term average of the price/earnings ratio of the S&P 500 index. This average has no significance in the table; it is used only as a reference for measuring the interval of fluctuations under each of the model assumptions. The second row reports the long term standard deviation of the real rate of return on equity.

Table 1: Long Run Volatility of the Price/Earnings Ratio  
and the Return on Equity

	Under Rational Expectations	Under Rational Beliefs	Actual
Interval in which the price/earnings ratio fluctuates 95% of the time	[ 13.8 , 14.0 ]	[ 9.4 , 18.4 ]	[ 5.5 , 22.3]
$\sigma_r$ - the long term standard deviation of the return on equity	4.1%	17.5%	18.4%

The table exhibits the problem which arises under rational expectations: if stock prices vary strictly in accord with fundamentals they would not change very much! The variance of the price/earning ratio is bigger *by an order of magnitude* under rational beliefs than under rational expectations. The table shows that under rational beliefs the index would have spent 95% of the time between 9.4 and 18.4 which is of the same order of magnitude as the historical record. This interval is somewhat smaller than the actual interval reported in the last column, a fact that may be explained by the generally agreed upon observation that the fluctuations of the reported price/earnings ratio are sensitive to tax and accounting practices. These tend to overstate the volatility of recorded earnings relative to the true economic earnings of the companies in the index. The actual long term standard deviation of the return on the S&P 500 index is 18.4% and the simulations under rational beliefs lead to a standard deviation of 17.5%. These two measures of volatility are very close.

III(C.2) Problem B: The Equity Premium and the Riskless Rate in the Domestic Economy

In Table 2 I record the long term averages of the riskless real rate of return on short term

Table 2: The Long Run Average Riskless Rate and the  
Equity Premium Over the Riskless Rate

	Under Rational Expectations	Under Rational Beliefs	Actual (Approx)
$r^F$ - the long term average of the riskless rate	5.72%	.71%	1.00%
$\rho$ - the long term average risk premium of equity	.49%	7.97%	7.00%

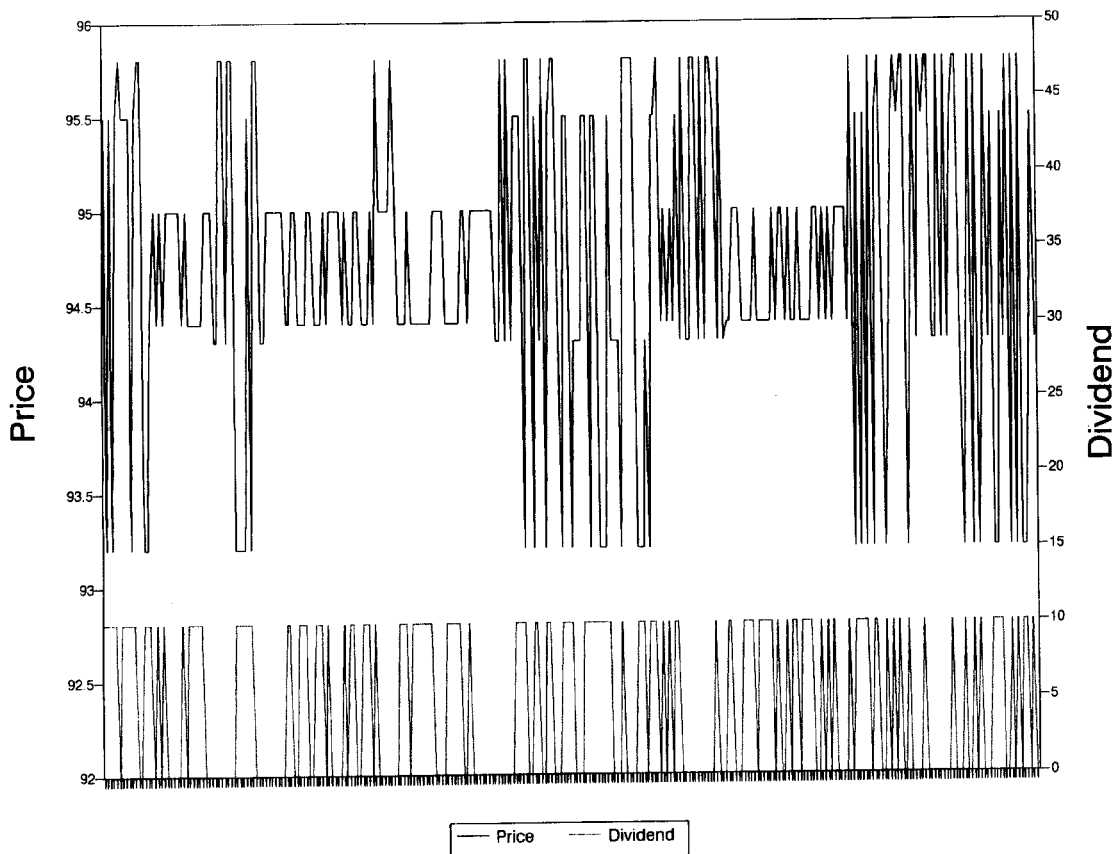
debt and of the equity risk premium. The table exhibits the problem which arises under rational expectations: the historical record over the last hundred years shows a riskless short term interest rate in the order of magnitude of 1% and an average risk premium of around 7%. Under rational

expectations the model fails to come close to these facts. Under rational beliefs the average equity premium is 7.97%, the average riskless rate is .71% and these figures correspond to the historical record.

III(C.3) Problem C: The GARCH Property of Stock Prices in the Domestic Economy

It has been observed both by experienced market traders as well as by academic researchers that over time, the variance of stock prices and returns changes *without* a corresponding change in fundamentals to account for it. This is known as "the GARCH property of stock prices" and this represents a problem for rational expectations since under such expectations prices change only in response to changes in fundamentals. In Figure 2 I exhibit a plot of the time series of 300 prices that were simulated in the domestic model. The growth rate

Figure 2



of dividends is assumed to take two values in these calculations and since these are also random, I plot them at the bottom. It is clear that over time the model exhibits drastic changes in price volatility but there are only two volatility regimes: one is a *high volatility regime* and the second is a *low volatility regime*. Both regimes exhibit substantial persistence. Variations in the growth rate of dividends has a slight effect on these regimes so that within the high and low volatility

regimes there are sub-regimes whose volatility depends to a small degree upon dividends.

III(C.4) Problem D: Volatility of the Foreign Exchange Rate and the Forward Discount Bias

Table 3 reports selected results of the international model which I now draw upon for the first time. Before discussing those let me define exactly the concept of "forward discount bias" which was mentioned in Problem D above. Suppose you estimate a regression of the form

$$\frac{ex_{t+1} - ex_t}{ex_t} = c + \beta(r_t^D - r_t^F) + \varepsilon_{t+1}$$

where  $(ex_{t+1} - ex_t)$  is the change of the exchange rate between date  $t$  and date  $t + 1$  while  $(r_t^D - r_t^F)$  is the difference between the short term nominal interest rates in the domestic and the foreign economies. Under rational expectations the differential of the interest rates between the two countries at date  $t$  should provide correct predictor of the *actual* depreciation of the currency that will occur between date  $t$  and date  $t + 1$ . This means that apart from a technical correction for risk aversion the parameter  $\beta$  should be close to 1. In 75 empirical studies in which equations like the above were estimated, the estimates of the parameter  $\beta$  are significantly less than 1. Indeed, Froot [1990] estimates that the average for all these studies is -.88! The failure of this parameter to exhibit estimated values close to 1 has come to be known as the "forward discount bias" (see Engel [1996] for an extensive recent survey and Froot and Thaler [1990] for a simple exposition of the problem).

Table 3 reports (i) an interval in which the exchange rate fluctuates 95% of the time and where the mean exchange rate has been *arbitrarily* calibrated to be 120, (ii) the value of the parameter  $\beta$  which the simulation models predict. The selection of 120 as the mean of the exchange rate has no significance to the volatility measures reported. It is only meant to establish a comparable frame of reference. The actual Yen/Dollar rate

Table 3: The Volatility of the Exchange Rate and the Forward Discount Bias

	Under Rational Expectations	Under Rational Beliefs	Actual Yen/Dollar
Interval in which the exchange rate fluctuates 95% of the time	[115 , 125]	[ 67 , 173]	[ 84 , 156]
$\beta$ - the forward discount bias parameter	.957	.152	Diverse

of exchange in the last column of Table 3 has fluctuated in part due to different inflation rates in the U.S. and Japan and I have thus computed the variance of the exchange rate based *on logarithmic detrending of the data*. The "actual" variability in the table is then that part of the variability of the Yen/Dollar exchange rate *around the average geometric trend*. The table exhibits the problems which arise under rational expectations: the variance of the foreign exchange rate is negligible and the parameter  $\beta$  has a value close to 1. Under rational beliefs the



results are drastically different: the variance of the foreign exchange is of the order of magnitude of observed fluctuations in the market. Finally, the forward discount bias parameter in the RBE reported in the table is .152 which is significantly less than 1. Within the class of models used here a negative parameter could not be predicted.

#### IV. Simple Explanations of How the Theory Resolves Each of the Four Problems

In Section III I demonstrated that the unified paradigm offered by the theory of rational belief equilibrium (RBE) goes a long way towards solving the four problems that could not be solved within the prevailing rational expectations paradigm. In this section I will offer a simple but systematic explanation of the results presented in Section III. In doing so I will also demonstrate the workings of the model of RBE.

(i) *Volatility of Prices and Exchange Rates.* The explanation of why the volatility of prices and exchange rates in an RBE exceed the level determined by the exogenous fundamentals of the economy is simple. Each agent forms his own theory of what the future will bring and the distribution of the private models in the economy constitutes the "social state of belief." Variability in the state of belief in the market is then an important factor, together with the exogenous shocks, in explaining price volatility. Since the social state of belief is not observable we need to seek proxies for it. Incomplete proxies can be seen in the distribution of price forecasts announced by different forecasters in the market. Interesting distributions of short term and long term interest rate forecasts by professional economists are also revealing since all use the same data. Thus the "state of belief" in the market is simply a "distribution of beliefs".

Endogenous uncertainty is then that component of price volatility which is caused by the distribution of beliefs of the agents. Therefore, equilibrium price volatility can be represented as a sum of the form

$$\text{Market Uncertainty} = \text{Exogenous Uncertainty} + \text{Endogenous Uncertainty}$$

Since exogenous uncertainty is that component of market volatility which is determined by the volatility of the exogenous fundamental conditions in the market, it is then clear why total market volatility exceeds the level justified by fundamentals.

Without introducing technical details I stress that endogenous uncertainty has a dual effect on market volatility. One component of endogenous uncertainty is the *amplification* of the effect of fluctuations of exogenous fundamentals on price volatility. This is the effect whereby the distribution of beliefs in the market can cause changes in the fundamental exogenous variables to have a larger effect on price volatility than would be true in a corresponding rational expectations equilibrium. The second component of endogenous uncertainty arises from the fact that variations in the distribution of beliefs cause additional *pure price volatility* which is uncorrelated with any fundamental information. This component of endogenous uncertainty may have

dramatic effects on the volatility of prices in an RBE since this component turns out to be affected by correlation and commonality of beliefs among traders. When a large number of agents become optimistic about capital gains, prices may rise. Conversely, when a large number of agents become pessimistic prices decline.

The amplification component of Endogenous Uncertainty provides a natural explanation of the phenomenon which is recognized as "*market overshooting*." This is usually a reference to the fact that when prices are high they often proceed to go higher than can be justified by fundamentals and when they go low, they go lower than can be justified by the exogenous variables. Naturally, excess volatility and overshooting is part of the historical record and is incorporated in the empirical distribution of any market. Hence it becomes part of the belief structure of agents: they expect the market to overshoot.

(ii) *The Forward Discount Bias in foreign exchange rates.* To see why this bias arises naturally in an RBE recall the rational expectations argument in favor of  $\beta = 1$  (apart from the correction for risk aversion which I ignore here). Hence, in such an equilibrium it is a *theoretical* conclusion that the difference between the one period nominal rates in the two countries at date  $t$  is exactly equal to the expected percentage depreciation of the exchange rate between the two currencies between dates  $t$  and  $t + 1$ . This *expectational arbitrage* argument implies that in the real economy the differential between the one period nominal rates in the two countries will be an unbiased *statistical* forecast of the one period depreciation of the exchange rate in the next period. Under this proposition one would expect to have a regression coefficient of 1 between the percentage differential of the nominal rates at date  $t$  and the *actual* percentage change of the exchange rate between dates  $t$  and  $t + 1$ .

The theory of RBE predicts that agents holding rational beliefs may make significant forecasting mistakes. This would result in a true, equilibrium, process of the exchange rate which would fluctuate excessively in part due to these mistaken forecasts. Hence, at almost no date would the nominal interest differential between the two countries be an unbiased estimate of the rate of depreciation of the exchange rate one period later and under such circumstances one should not expect the regression coefficient to be close to one. Agents who want to take advantage of such a regression, basing their investment strategy on a nominal rate differential which appears to offer an investment opportunity, will find that this is not arbitrage in the standard riskless sense of the term. At date  $t$  the exchange rate at date  $t + 1$  is a *random variable*. In an RBE any trade on the spread between the nominal interest rates of two currencies requires agents to take a foreign exchange risk which is valued differently by agents holding diverse beliefs.

Should we expect that under rational beliefs the parameter  $\beta$  satisfies  $\beta < 1$ ? The answer is yes for the following reason. Consider first a rational expectations equilibrium in which the difference between the domestic and foreign nominal interest rates is  $z\%$ . In that equilibrium you do not need to form expectations on the currency depreciation itself. It is sufficient for you to believe that other investors or currency arbitrageurs know the true probability of currency

depreciation and they have already induced the interest differential to be equal to the average rate of currency depreciation *which will be  $z\%$* . Now consider an RBE. All agents know that no one knows the true probability distribution of the exchange rate and therefore the exchange rate is subject to endogenous uncertainty. Being risk averse, agents who invest in foreign currency would demand a risk premium on endogenous uncertainty and over the long run the difference  $(1-\beta)$  is the premium received by currency speculators for being willing to carry foreign currency positions. For a positive premium it follows that  $\beta < 1$ .

(iii) *The GARCH Property of Asset Prices*<sup>6</sup>. As indicated earlier, the states of belief of different individual investors may be highly correlated and this is a consequence of the many modes of communication in our society. Investors talk to each other and this interaction causes them to influence each other; they all read the same newspapers, the same reports of the Wall Street analysts and watch the same television programs which feature expert views on the economic conditions in the future. Analysts and experts know each other, they talk to each other and attend the same conferences thus tend to correlate their views either in agreement or disagreement. The consequence of this correlation among the beliefs of agents is that the distribution of beliefs tends to switch across different "cognitive" centers of gravity. Indeed, each such center of gravity is a "belief regime". The important examples of such regimes of belief are regimes of "consensus" and "non-consensus." The persistence of the states of belief is an important element in the emergence of the GARCH property. In the models studied here the state of belief is a Markov process with degrees of persistence which depend upon the parametrization of each model.

The emergence of the GARCH property is a consequence of two different effects which the distribution of beliefs has on the market. These two effects are directly related to the relative strength of the two components of Endogenous Uncertainty discussed above: *amplification* and *pure endogenous volatility*. I will start with pure endogenous volatility since this effect is simpler. It turns out that what really matters for the effect of this component of endogenous volatility on the emergence of the GARCH phenomenon is the *persistence* of the regime of consensus vs. the regime of market non consensus. A regime of *market consensus* is formed when the models of the majority of traders generate similar predictions and if the regime persists, then over time, if the real economy remains in the same state ("high" for expansion and "low" for recession) the traders move *together* between states of optimism and states of pessimism. Such fluctuations between optimistic and pessimistic outlooks on market prices may occur on many different frequencies. *Non-consensus* is a belief regime in which the distribution of models used by the agents is relatively spread out and consequently their predictions vary widely across the different possible outcomes in the future. If the regime of non-consensus persists then, at a given state of the real economy, the diverse forecasts tend to cancel each other out over time.

I now observe that when the regime of consensus is formed the pure volatility component

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<sup>6</sup> For more details about the nature of GARCH and related processes see Bollerslev, Chou and Kroner [1992] and Bollerslev, Engle and Nelson [1994].

of security prices will be high. This is so since all agents are either optimistic or pessimistic *at the same time*; when optimistic they want to buy the same portfolio and when pessimistic they want to sell the same portfolio leading to price volatility. Conversely, when a non-consensus regime occurs the opposite is true: the distribution of beliefs is one in which the excess demand of the optimists matches the excess supply of the pessimists leading to low volatility. This component of endogenous volatility is not correlated with real exogenous shocks.

I turn now to the effect of "amplification" which is drastically different from the first effect. To understand this second effect consider two different states of belief each of which has a high degree of persistence. For example, let the first state be one of "universal optimism" and the second state be one of "non-consensus" or "disagreement". Given the state of optimism, prices will vary with the state of the dividend growth. However, assuming a strong endogenous amplification, prices will overreact to changes in the fundamental information *but the degree of amplification is not the same in the state of total optimism as in the state of non-consensus*. If the economy is in a state of optimism (and remains there) the variations in asset prices due to cyclical output fluctuations is usually relatively small so that the variance of prices in that state is relatively small. If the economy is in the state of non-consensus (and remains there) the price response to cyclical output fluctuations is very different depending upon the intensity of optimism and pessimism. As we shall see later, in all models presented here the pessimists are more intense than the optimists. Consequently, if in a state of non-consensus the net output state is "low" (i.e. recession) then the pessimists will dominate by making the price crash even further but if the state is "high" the pessimistic outlook has less force and prices will rebound sharply. As a result, the variance of prices in that state is very high. In short, as the economy moves among different states of belief the level of asset prices will change, but more importantly, the variance of prices will vary, giving rise to the GARCH property.

To generalize these conclusions beyond the simulation models, the theory of RBE shows that the variance of stock prices depends upon the *distribution of beliefs* in the market and since this distribution changes over time, so does the variance of stock prices. Also, in an RBE agents can utilize exogenous shocks and realized prices to determine their state of belief about the future. Consequently, the distribution of beliefs and hence the variance of prices may depend upon both the correlation among the beliefs of the agents as well as the exogenous shocks and realized prices. Both of these may change abruptly, and so can the induced regime of beliefs.

The models used in the simulations relate to events which occur over long stretches of time and hence the simulation results apply to low frequencies (i.e. months and years). These do not address the structure of volatility at high frequencies investigated by some papers of the GARCH literature (see for example Bollerslev, Chou and Kroner [1992] and Brock and LeBaron [1996]). This limitation of the results here should not obscure the main conclusion to which the theory of RBE leads: the GARCH phenomenon is caused both by the persistence as well as by the abrupt shifts in the distribution of beliefs. These forces hold over low or high frequencies.

(iv) *The Equity Premium and the Riskless Rate*<sup>7</sup>. Explaining the factors which determine the equity risk premium (i.e. "the" premium) in an RBE is ultimately simple but demands the review of the technical conditions which formulate the rationality of beliefs of the agents. A *direct* and simple explanation flows naturally from the resolution of Problem A. It proposes that in an RBE endogenous uncertainty causes the total level of uncertainty to exceed the level that would prevail under rational expectations. Risk averse investors would then demand a higher risk premium for holding equity which is more risky in an RBE than in a rational expectations equilibrium and for that reason the premium would be higher in an RBE. This explanation has a grain of truth but needs to be qualified by two additional considerations.

The first consideration suggests that due to the diversity of beliefs the equity premium arises in a world where optimists and pessimists reside together. The risk premium demanded by optimists is likely to be different from the premium demanded by pessimists and hence, the market premium must depend upon the *distribution* of beliefs. Indeed, there are proportions of optimists and pessimists which do not generate a higher equity risk premium than is generated under rational expectations. Second, an important component of the equity premium puzzle has been the question of why the riskless rate predicted by rational expectations models has been so much higher than the mean riskless rate realized over the last century and this question must be cleared as well. The *direct* explanation given above does not address the question of why the riskless rate is so much lower in the simulated RBE relative to rational expectations equilibria.

To gain intuition into the two issues above I must bring you into some of the more technical aspects of the theory and to do that I examine a very simple model (based on Kurz [1998]). Consider an economy with two types ( $\alpha$  and  $\beta$ ) of agents who are different only in their models of market price behavior (i.e. their beliefs). As part of their models, each of the type  $\alpha$  agents has an assessment and when the assessment takes the value 1 the agent uses probability distribution  $F_1$  of future prices and when it takes the value 2 the agent uses probability distribution  $F_2$ . These assessment variables are different for the two types. For this reason I denote the probabilities used by type  $\beta$  agents by  $G_1$  when the assessment of a type  $\beta$  agent takes the value 1 and by  $G_2$  when the assessment of a type  $\beta$  agent takes the value 2. However, I also assume that there is a very large number of agents of each type and each of them has his own separate assessment. Now, the assessments of the large number of agents of each of the types are, statistically speaking, the *same* random variables since these agents are of the *same* type but now comes the deeper question: are these assessments *independent random variables*? To address this question I must take an indirect route.

Some who object to models with heterogenous beliefs have suggested that in a large economy consisting of many agents with *independent* beliefs the law of large numbers would operate to average out the diversity of beliefs. Such averaging should render the model of diverse

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<sup>7</sup> The debate regarding "The Equity Premium Puzzle" was initiated by Mehra and Prescott [1985]. A sample of other papers on the subject include Mankiw [1986], Reitz [1988], Weil [1989], Epstein and Zin [1990], Constantinides [1990], Campbell and Cochrane [1999], Brennan and Xia [1998] and Abel [1999].

beliefs irrelevant, leading the model of a large economy to function like a model of the representative household with a *single*, rational expectations belief. This intuitive argument is misleading and the reasons are the key to understanding why a large equity premium and a low riskless rate can be generated in an RBE.

Let me then return to my simple model and make the strong assumption that all the assessments within each type are i.i.d. with the probability of assessment taking the value of 1 being, say, .60. The consequence of this assumption is that although the probability used by any one agent depends upon his assessment, the distribution of beliefs in the economy is fixed at  $((.60, .40), (.60, .40))$ . That is, *at all times* 60% of type  $\alpha$  agents use probability distribution  $F_1$  and 40% of them use  $F_2$ . A similar situation is assumed with respect to type  $\beta$  agents. If I now interpret  $F_1$  and  $G_1$  to mean "optimistic beliefs about higher returns next period" and  $F_2$  and  $G_2$  to mean "pessimistic beliefs about higher returns next period" then I have an economy where the law of large number holds as required. *At all times* the distribution of beliefs is constant with 60% of each type optimistic and 40% pessimistic. The terms "optimism" and "pessimism" are exactly the same proportions defined before and hence I will call these proportions (which are fixed for each type but may be different across the two types) the "intensities of optimism" or the "intensities of pessimism". I use the term "intensities" rather than "intensity" since these intensities of optimism or pessimism may vary depending upon current prices.

In this economy 60% of the agents are always optimistic, using  $F_1$  or  $G_1$ , and hence each agent fluctuates between optimistic and pessimistic outlooks with a frequency of .60 in the optimistic mode and a frequency of .40 in the pessimistic mode. This would make sense only when I consider the rationality of belief conditions which the agents satisfy. These stipulate that the beliefs may fluctuate over time but must average to  $\Gamma$ . The RBE is then established if type  $\alpha$  agents satisfy the condition  $.60F_1 + .40F_2 = \Gamma$  and type  $\beta$  agents satisfy condition  $.60G_1 + .40G_2 = \Gamma$ . But now I need to compare two equilibria: an REE in which *all the agents hold the belief  $\Gamma$*  and the RBE in which *60% are optimists and 40% are pessimists relative to  $\Gamma$* . Those claiming that independent beliefs do not matter would propose that these two equilibria are the same in the sense that prices and allocations are the same. I will show that these two are very different equilibria with drastically different equity premia and volatility characteristics.

To convince you of that fact suppose that the initial percentage of pessimists in the economy is  $x = .40$  and in equilibrium the rationality condition  $(1 - x)F_1 + xF_2 = \Gamma$  is satisfied. Now I lower the percentage  $x = .40$  to  $x'$ . Would the rationality of belief condition  $(1 - x')F_1 + x'F_2 = \Gamma$  be satisfied with  $x'$ ? The answer is no since my decrease of the percentage of pessimists from  $x = .40$  to  $x'$  without changing the matrices  $(F_1, F_2, G_1, G_2)$  means that I reduced the weight assigned to the pessimistic matrix  $F_2$  and increased the weight assigned to the optimistic matrix  $F_1$  leading to the result that  $(1 - x')F_1 + x'F_2 \neq \Gamma$ . Hence, as the number of pessimists in the market declines, I must adjust the intensity parameters in  $F_2$  and in  $G_2$  so that the intensity of their pessimism *increases*. Indeed, a point will be reached at which I could not lower the fraction of pessimists any further since the intensity of their pessimism has reached a point where, given some price, they are virtually certain that they will lose money between date t

and date  $t + 1$ . I will then have an economy with a reduced proportion of pessimists but who are so intensely pessimistic that they are willing to pay a very high price for the bill to secure their wealth for next period. What will happen to the interest rate and to the risky returns in the model under these circumstances? The price of the bill will rise, lowering the riskless rate, and the price of the stock will fall causing the equity risk premium to rise. In the section below I will provide an example that would apply to the situation under discussion. This concludes my demonstration that the RBE under discussion is very different from the REE with a representative agent.

The central observation is that the rationality of belief conditions are linear conditions of the form  $(1 - x)F_1 + xF_2 = \Gamma$  but variations in the percentage/intensity combinations of optimists and pessimists have a *non-linear impact on the demand functions for securities*. Hence, as these combinations vary over the feasible parameter space of the model, the riskless rate and the equity premium change. The reader may note that since the rationality condition  $(1 - x)F_1 + xF_2 = \Gamma$  is linear, any increase in probability in  $F_1$  must be compensated by a reduction in  $F_2$ . Moreover, if  $x = .5$  then the compensation in  $F_2$  must be exactly the same as the increase in  $F_1$  and this is entirely symmetric. So, if such changes have non-linear effect then the model at hand must imply some asymmetry. To understand it let me present a simple example which will clarify the issue.

**Example:** Consider the case of a  $3 \times 3$  matrix in which optimism is defined with respect to states 1 and 2. Assume  $\alpha = .50$  and that  $\lambda = 2.0$  is feasible. Both the matrix  $\Gamma$  and the optimistic matrix  $F_1$  are entirely regular. The rationality conditions imply that the pessimistic matrix is

The Matrix  $\Gamma$

	s = 1	s = 2	s = 3
s = 1	.25	.20	.55
s = 2	.20	.20	.60
s = 3	.30	.10	.60

The Matrix  $F_1$  Of Optimists

	s = 1	s = 2	s = 3
s = 1	.50	.40	.10
s = 2	.40	.40	.20
s = 3	.60	.20	.20

$F_2 = \frac{1}{.50}(\Gamma - .50F_1)$ . It can be checked that the matrix  $F_2$  is the one shown. I think it is

Matrix  $F_2$  of Pessimists

	s = 1	s = 2	s = 3
s = 1	0	0	1
s = 2	0	0	1
s = 3	0	0	1

reasonable to think of the pessimists using matrix  $F_2$  as being "more intense" in their pessimism than the intensity of the optimists which I quantified to be 2. I hasten to add that in the basic

model discussed in this paper  $\lambda = 2.0$  is not feasible and the matrix  $\Gamma$  needs to be compatible with (5) and for this reason does not have the simple structure as in the example. Indeed, in the model above  $\lambda = 1.7542$  and  $\alpha = .57$ .

Given the basic observation that at any date the risk premium is determined by the exogenous variables and by *the distribution of beliefs* in the market, I reexamine the assumptions made earlier. Recall that I have assumed that the assessments are i.i.d. in order to refute the criticism that heterogeneity of beliefs is irrelevant in a large economy with independent beliefs. Extensive research conducted in recent years has shown that it takes very little local interaction among agents in the market in order to remove the effect of the law of large numbers on equilibrium variables such as prices. More specifically, under small local interactions, equilibrium aggregate variables of a large economy *act as random variables* rather than as constants<sup>8</sup>. Given the natural interaction among the agents in financial markets there is ample theoretical justification for assuming that the beliefs of agents in the market are correlated and hence their assessments are not jointly i.i.d. On the empirical side, there is little data on the distribution of beliefs in the market. However, the little evidence which is available (such as the forecasts of analysts on Wall Street) suggests that individual beliefs are highly correlated. Hence, both theoretical as well as empirical considerations imply that we should study models where the distribution of beliefs is a random variable, jointly distributed with prices and other equilibrium variables.

The argument developed earlier (for an economy with i.i.d. assessments) regarding the belief intensity of the pessimists remains valid in an economy with correlation among the assessments of the agents. The only difference is that now the distribution of beliefs changes over time and the riskless rate and equity premium vary with the states of the economy. Hence, the RBE model's prediction of the long term averages of the riskless rate and of the equity premium depends now also upon the frequency at which the system visits those distributions of beliefs which generate low riskless rate and high premium. As we consider patterns of correlation among the beliefs of agents we may also expand the range of empirical evidence that needs to be explained. In this paper I considered only four variables which needed to be explained. In a complete analysis of the equity premium one may ask for the model prediction to match other empirical regularities. These would include:

- (i) the first and second moments of the price\earning or price\dividend ratio;
- (ii) the first and second moments of the riskless rate on short term debt;
- (iii)the first and second moments of the risky return of equities;
- (iv)the equity premium.

The remarkable fact is that the basic model presented in this paper can explain all these regularities *simultaneously*. In order to do that, the parametrization needs to be specified as follows:

- (i) the optimists need to constitute a majority of about 57%;
- (ii) the intensity of the optimists needs to be set at 1.7542 which is approximately the maximal rate feasible;

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<sup>8</sup> See, for example, the papers by Brock [1993],[1996], Durlauf [1993],[1994] and Föllmer [1974].



(iii) Condition (ii) ensures that the pessimists are more intense in that they act as if they are virtually certain at each date  $t$  that at date  $t + 1$  the economy will slide into a recession;  
 (iv) the dynamics of the state of belief has to be such that the market prices cannot rise directly from the crash-recession levels to the highest prices of the bull market but it can crash from the high prices to the crash prices.

In sum, the RBE theory presented here offers a very simple explanation for the observed low average riskless rate of around 1% and a high equity premium of about 7%. The theory proposes that such a pattern arises as a consequence of *the diversity of beliefs in our financial markets when the majority of traders are optimistic but where there is always a minority of intense pessimists*. The identity of these agents changes at all times since no rational agent is always optimistic or always pessimistic. This distribution of beliefs has two drastic consequences. First, it causes our financial markets to "overshoot" in the sense of experiencing much larger fluctuations of prices than could be explained by exogenous, fundamental, factors. Second, and this is the main conclusion of this Section, the high intensity of the pessimists is the decisive factor which, in the long run, dominates the market for short term debt instruments. These are the agents who push the riskless rate down and the equity premium up. This ability of the theory of Rational Beliefs to provide this explanation of the empirical evidence is a central dimension of the unified paradigm proposed in this paper. That is, our explanation of the empirical evidence flows directly from the conditions of rationality of the agents since the crucial asymmetry of the argument which grants the pessimists the greater intensity is a direct consequence of the rationality conditions.

A final observation regarding the historical record is of interest. There is some evidence that the riskless rate has exhibited a rather irregular pattern over the last 200 years. Table 4, drawn from data provided in Siegel [1994], shows that the very low average rate of return of less than 1% on riskless debt instruments is a phenomenon which occurred mostly after the great depression. Indeed, Siegel [1994] shows that the large spread between rates of return on long

Table 4: Real Rates of Return on Debt Instruments (Segal [1994])

	on Short Term Government	on Long Term Government
1802-1870	5.1%	4.8%
1871-1925	3.2%	3.7%
1926-1997	0.6%	2.0%

and short term government debt instruments opened up exactly around 1930 and remained high until 1997. I might caution the reader that historical data prior to World War I are subject to large errors and could be interpreted in many different ways. Suppose, however, that Siegel [1994] is correct in identifying the data on the riskless rate. In that case, it appears that the 1930

depression has something to do with the low riskless rate. But such a fact provides further support for the theory offered in this paper since this may establish the fact that the pessimists in my RBE model based their pessimism on the experience of the 1930's. This does not mean that the probability which the pessimists attached to capital losses are exactly the empirical frequencies of the great depressions. These empirical frequencies are part of the average historical record in the matrix  $\Gamma$ . Rationality of belief permits the pessimists to hold a probability  $F_2$  or  $G_2$  which do not correspond to any specific empirical frequency. However, it is the great depression that may have been responsible for the *nature* of the RBE which we have been discussing all along.

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