Economic Theory 8, 383–397 (1996)

Economic Theory

© Springer-Verlag 1996

Symposium

Rational beliefs and endogenous uncertainty*

Mordecai Kurz
Department of Economics, Serra Street at Galvez, Stanford University, Stanford, CA 94305-6072, USA

1 On the diversity of probability beliefs

Sharp differences of opinions and probability beliefs among economic agents are very commonly observed phenomena. Both the Arrow-Debreu [1954] model as well as Savage's [1954] theory accommodate these empirical observations and permit agents to hold diverse probability belief about the exogenous states. However, within the Arrow-Debreu model this diversity has very limited implications. The diversity of probability beliefs becomes a very significant question in any model in which trades occur *sequentially* and securities rather than contingent claims are the means of trading uncertainty. Arrow [1953] and Radner [1972], [1979] show that in order to permit agents to hold diverse probability beliefs about exogenous states and still attain the Arrow-Debreu [1954] allocations via securities, one needs to assume that the agents hold rational expectations which take the form of "conditional perfect foresight". It requires the agents to know the equilibrium map between future exogenous states and future spot prices.

In most dynamic applications in economics and finance, the requirement that agents know the equilibrium map is replaced by the requirement that agents know the *true* equilibrium probability distribution of all variables. We refer to both forms of knowledge as "structural knowledge". Most economists agree that both conditions impose unreasonable requirements on what an agent must know in order to act "rationally". But then, if we assume that agents do not posses structural knowledge, current equilibrium concepts need to be revised and the important question is how to extend the theory of expectations to such circumstances.

The above considerations have generated, over the last 50 years, an interest in the market impact of heterogeneity of beliefs. A large body of empirical work has studied the way agents *actually* form expectations. Brennscheidt [1993] provides a survey of experimental and empirical work in general areas of economics while Takagi [1991] reviews the results reported in surveys about exchange rate expectations. An influential line of research in international economics has focused

* This work was supported by Fondazione Eni Enrico Mattei, Milano, Italy. The author thanks Andrea Beltratti, Carsten Nielsen, Martin Schneider and Ho-Mou Wu for helpful comments on an earlier draft of this paper. He thanks Carsten Nielsen for extensive editorial assistance in the preparation of this volume.

Copyright © 2001 All Rights Reserved

on explaining the observed volatility of foreign exchange rates by the heterogeneity of beliefs and the reader may consult such papers as Frankel and Froot [1987], [1990], Froot and Frankel [1989] and the recent major surveys by Frankel and Rose [1995] and by Taylor [1995] for details. Kandel and Pearson [1995] conduct an empirical investigation of investor's heterogeneity of beliefs. They show that agents in speculative markets interpret new *public* information in a differential way.

A constant stream of theoretical work has analyzed the effects of heterogeneity of beliefs on the behavior of markets. In some papers beliefs are assumed arbitrary and in others agents are assumed to be Bayesians with arbitrary priors. These papers vary in their assumptions regarding the knowledge of the equilibrium map by the agents. This literature is large and cannot be reviewed here. Some recent examples include De Long, Shleifer, Summers and Waldman [1990]; Harris and Raviv [1993]; Cabrales and Hoshi [1993]; Detemple and Murthy [1994] and Morris [1994]. In the area of general equilibrium Kurz [1974] introduces the concept of *Endogenous Uncertainty* defined as the uncertainty which is propagated internally by the beliefs and actions of agents rather than by exogenous forces. He then proposes an agenda for the reformulation of general equilibrium theory which treats endogenous uncertainty as the primary uncertainty and considers price contingent contracts as the main vehicle for trading such uncertainty. Svensson [1981] is the first to formulate a general equilibrium model with price uncertainty but without restrictions on beliefs.

The hypothesis that agents have diverse beliefs remains, however, controversial and the present volume is part of the debate on this question. The most common explanation given to the observed *persistent* heterogeneity of beliefs is *the diversity of private information.* Undoubtedly, asymmetric information is a phenomenon with important consequences. However, there is ample evidence that equally informed agents *often interpret differently the same information* (see Frankel and Froot [1990], Frankel and Rose [1995] and Kandel and Pearson [1995]).

The Bayesian work in Game Theory have generally followed Harsanyi's "common prior" doctrine which is in conflict with the Axioms of Savage [1954]. Aumann [1987] takes the view that the common prior doctrine must be employed as a matter of scientific discipline since without it the"...equilibrium places very few restrictions on the possible outcomes" (page 15). Aumann's comment is significant in that it reflects the fact that most objections to heterogeneity of beliefs are based on the erroneous premise that if heterogeneity is introduced then *all beliefs should be permissible.* Morris [1995] questions the validity of the Harsanyi doctrine. The papers in this issue propose that reasonable criteria of rationality of beliefs can lead to a scientific middle ground between a theory which permits *all possible beliefs* and a theory which insists on *a single common belief.*

It is clear that in any sequential trading model the distribution of beliefs can have an important effect on the time series generated by the economy. On the more fundamental level, the hypotheses of diverse beliefs without structural knowledge suggests that both economic fluctuations as well as uncertainty have a large endogenous component which is propagated within the economy rather than "caused" by exogenous factors. This endogenous uncertainty is *indirectly* the

Copyright © 2001 All Rights Reserved

uncertainty about the beliefs and actions of other agents and hence price uncertainty is a central form of uncertainty in a sequential economy. These ideas have important implications to the way we need to think about general equilibrium and about the financial institutions used to trade and reallocate endogenous uncertainty. They also have important implications to public policy since what is endogenously propagated may be affected by collective action. We think, therefore, that the debate on the scientific usefulness of theories with diverse beliefs is an important debate. For this reason we feel that it would be constructive to review some arguments against the critics of models with diverse beliefs.

As explained above, those who object to the introduction of *any* diversity of beliefs insist that it enlarges the set of individual actions which are considered optimal. One can respond to this criticism in several ways. Note first that the *primitive* assumption is that agents do not have structural knowledge; it is easy to construct procedures for individual formation of beliefs in which the diversity of beliefs emerges as a *logical* consequence of lack of structural knowledge. Secondly, the starting point of the debate is the empirical observation that current models, which focus only on *exogenous* sources of risk and fluctuations, fail to account for a large component of market fluctuations. The hypothesis of diversity of beliefs without structural knowledge leads to a theory which points to a missing component of internally propageted uncertainty and economic fluctuations. It is then natural that if the theory is to allow for *added fluctuations* on a micro-economic level, then *a larger set of individual actions must be permitted*. The point is that the enlarged set of outcomes is exactly why endogenous uncertainty matters!

Next, focusing on the enlarged set of *individual* actions sidesteps the fact that replacing the common belief assumption with diverse beliefs does not necessarily lead to a theory of aggregate fluctuations. This is so since if beliefs are "independent" across agents then aggregation in large markets acts as a market "law of large numbers" rendering the belief of any one agents irrelevant to a theory of market performance. Hence, the enlarged set of individual actions may, *by itself*, be irrelevant to market risk. One of the ideas developed in this volume is that heterogeneity of beliefs enables the emergence of a complex "externality" or interaction among agents. Such interaction takes the form of correlation of beliefs which affects aggregate market fluctuations and is then an essential cause of endogenous uncertainty. An added implication is that endogenous uncertainty opens up a new arena for public policy since collective action can have an impact. We briefly discuss this issue in the concluding section 5.

Our last point is methodological. Given the *empirical evidence* for the presence of heterogenous beliefs, it is a sound scientifc procedure to explore the implications of alternative theories which are compatible with these facts. Indeed, any such theory should have the model of common belief with full structural knowledge as a special case; comparisons with this case should be important in determining when is each approach most useful.

The theory of Rational Beliefs (see Kurz [1994a], [1994b]) is the scientific "middle ground" which is the basis of most of the papers in this volume. To make this volume self-contained, we briefly review this theory in the next section. We also aim to avoid repetition of definitions in the papers of this volume: authors will refer

Copyright © 2001 All Rights Reserved

to the "Editor's Introduction" for terms and concepts which are developed in this review.

2 A brief review of the theory of rational beliefs

We start with some notation. $x_t \in \mathbf{R}^N$ is a vector of N observables at date t and the sequence $\{x_t, t = 0, 1, \ldots\}$ is a stochastic process with true probability Π. Since every $x = (x_0, x_1, \ldots)$ is a sequence in $(\mathbf{R}^N)^\infty$ we use the notation $\Omega = (\mathbf{R}^N)^\infty$ and denote by $\mathscr{B}$ the Borel σ-field of Ω. The space $(\Omega, \mathscr{B}, \Pi)$ is *the true* probability space. A belief of an agent is a probability Q; the agent is adopting the theory that the probability space is $(\Omega, \mathscr{B}, Q)$. An agent who observes the data takes $(\Omega, \mathscr{B}, \Pi)$ as fixed but does not know Π. Using past data he will try to learn as much as possible about Π. The theory of Rational Beliefs aims to characterize the set of all beliefs which are compatible with the available data.

The assumption made is that date 1 has occurred "a long time ago" and at date t, when agents form their beliefs about the future beyond t, they have an *ample* supply of past data. We think of $x = (x_0, x_1, x_2, x_3, \ldots)$ as the vector of observations generated by the economy. In studying complex joint distributions among the observables, one considers *blocks* rather than individual observations. For example, to study the distribution of $(x_{\text{today}}, x_{\text{today}+1})$ we would consider the sequence of blocks $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots$ It is thus useful to think of the data *from the perspective of date 0* as the infinite vector $x = (x_0, x_1, x_2, \ldots)$ and the data *from the perspective of date n* as $x^t = (x_t, x_{t+1}, \ldots)$ where $x \equiv x^0$ and $x^t = Tx^{t-1}$, $t = 1, 2, 3, \ldots$. The map T is the *shift transformation* and the stochastic dynamical system is denoted by $(\Omega, \mathscr{B}, \Pi, T)$. For any $B \in \mathscr{F}$ consider the set $T^{-n}B$ which is the preimage of B under T^n defined by

$$T^{-n}B = \{x \in \Omega : T^n x \in B\}.$$

$T^{-n}B$ is the event B occurring n dates later. A dynamical system $(\Omega, \mathscr{B}, \Pi, T)$ is said to be *stationary* if $\Pi(B) = \Pi(T^{-1}B)$ for all $B \in \mathscr{B}$. A set $S \in \mathscr{B}$ is said to be *invariant* if $S = T^{-1}S$. We say that a set S is *invariant Π a.e.* if $\Pi(S \Delta T^{-1}S) = 0$ (where $S \Delta T^{-1}S = (S \cup T^{-1}S) \backslash (S \cap T^{-1}S)$). The distinction between these two concepts of invariance are minimal and will be disregarded here. A dynamical system is said to be *ergodic* if $\Pi(S) = 1$ or $\Pi(S) = 0$ for any invariant set S. Here we assume for simplicity that $(\Omega, \mathscr{B}, \Pi, T)$ is ergodic but this assumption is not needed (see Kurz [1994a] where this assumption is not made).

In order to learn probabilities agents adopt the natural way of studying the *frequencies* of all possible economic events. For example, consider the event B

$$B = \left\{ \begin{array}{l} \text{price of commodity 1 today} \leqq \$1, \text{ price of commodity 6 tomorrow} \geqq \$3, \\ 2 \leqq \text{quantity of commodity 14 consumed two months later} \leqq 5 \end{array} \right\}.$$

Now using past data agents can compute for any finite dimensional set B the expression

$$m_n(B)(x) = \frac{1}{n} \sum_{k=0}^{n-1} 1_B(T^k x) = \left\{ \begin{array}{l} \text{The relative frequency that } B \text{ occurred among} \\ n \text{ observations since date } 0 \end{array} \right\}$$

Copyright © 2001 All Rights Reserved

where

$$1_B(y) = \begin{cases} 1 & \text{if } y \in B \\ 0 & \text{if } y \notin B. \end{cases}$$

This leads to a definition of the basic property which the system $(\Omega, \mathscr{B}, \Pi, T)$ is assumed to have:

Definition 1. *A dynamical system is called stable if for any finite dimensional set (i.e. cylinder) B*

$$\lim_{n \to \infty} m_n(B)(x) = \mathring{m} \quad \text{exists} \qquad \Pi \text{ a.e.}$$

The assumption of ergodicity ensures that the limit in Definition 1 is independent of x. In Kurz [1994a] it is shown that the set function $\mathring{m}$ can be uniquely extended to a probability m on $(\Omega, \mathscr{B})$. Moreover, relative to m the dynamical system $(\Omega, \mathscr{B}, m, T)$ is stationary. There are two observations to be made.

(a) Given the property of stability, in trying to learn Π all agents end up learning m which is a *stationary* probability. In general $m \neq \Pi$: the true dynamical system $(\Omega, \mathscr{B}, \Pi, T)$ may not be stationary. *Π cannot be learned.*
(b) Agents know that m may not be Π but with the data at hand m is the only thing that they can learn and agree upon.

Non-stationarity is a term which we employ to represent the process of *structural change*. Hence, a stable but non-stationary system is a model of an economy with structural change but in which econometric work can still be successfully carried out. If all agents knew that the true system is stationary they would adopt m as their belief. Hence, even if it was stationary, agents may still not adopt m as their belief. Note that m summarizes the entire collection of *asymptotic restrictions* imposed by the true system with probability Π on the empirical distribution of all the observed variables. It is shown in Kurz [1994a] that for each stable system with probability Π there is a set $B(\Pi)$ of stable probabilities Q with dynamical systems which generate the same stationary probability m and consequently *impose the same asymptotic restrictions on the data as the true system with Π.* The question is how to determine *analytically* if any proposed dynamical system $(\Omega, \mathscr{B}, Q, T)$ generates m as a stationary measure. To examine this question let us return to $(\Omega, \mathscr{B}, \Pi, T)$ and consider, for any cylinder B the set function $m_n^{\Pi}(B) = \frac{1}{n}\sum_{k=0}^{n-1} \Pi(T^{-k}B)$. Note that $m_n^{\Pi}(B)$ has nothing to do with data: it is an analytical expression derived from $(\Omega, \mathscr{B}, \Pi, T)$.

Definition 2. *A dynamical system $(\Omega, \mathscr{B}, \Pi, T)$ is said to be weak asymptotically mean stationary (WAMS) if for all cylinders $S \in \mathscr{B}$ the limit*

$$\mathring{m}^{\Pi}(S) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \Pi(T^{-k}S) \text{ exists.}$$

It is *strong asymptotically mean stationary* if the limit above holds for all $S \in \mathscr{B}$.

Copyright © 2001 All Rights Reserved

It is shown in Kurz [1994a] that $\mathring{m}^{\Pi}$ can be uniquely extended to a probability measure on $(\Omega, \mathscr{B})$. We then have the important theorem which is the main tool in Kurz [1994a]:

Theorem 1. *$(\Omega, \mathscr{B}, \Pi, T)$ is stable if and only if it is WAMS. If m is the stationary measure calculated from the data, then $m(S) = m^{\Pi}(S)$ for all $S \in \mathscr{B}$.*

The implication of Theorem 1 is that every stable system $(\Omega, \mathscr{B}, \Pi, T)$ generates a *unique* stationary probability m^{Π} which is calculated analytically from Π. This last fact is the foundation of the following:

Definition 3. *A selection of belief Q cannot be contradicted by the data m if*

(i) the system $(\Omega, \mathscr{B}, \Pi, T)$ is stable,
(ii) the system $(\Omega, \mathscr{B}, \Pi, T)$ generates m and hence $m^Q = m$.

Rationality Axioms: A selection Q by an agent is a *Rational Belief* if it satisfies

(I) *Compatibility with the Data:* Q cannot be contradicted by the data.
(II) *Non-Degeneracy:* if $m(S) > 0$, then $Q(S) > 0$.

Now, to express a belief in the non-stationarity of the environment, an agent may select a probability $Q^{\perp}$. The probability is said to be *orthogonal with m* if there are events S and S^c such that

(i) $S \cup S^c = \Omega$, $S \cap S^c = \varnothing$, (ii) $m(S) = 1$, $m(S^c) = 0$, (iii) $Q^{\perp}(S) = 0$, $Q^{\perp}(S^c) = 1$.

We aim to characterize the set $B(\Pi)$ of Rational Beliefs when $(\Omega, \mathscr{B}, \Pi, T)$ generates the data.

Theorem 2 (Kurz [1994a]). Every Rational Belief must satisfy $Q = \lambda Q_a + (1 - \lambda) Q^{\perp}$ where $0 < \lambda \leqq 1$, Q_a and m are probabilities which are mutually absolutely continuous (i.e. they are equivalent) and $Q^{\perp}$ is orthogonal with m such that

(i) $(\Omega, \mathscr{B}, Q_a, T)$ and $(\Omega, \mathscr{B}, Q^{\perp}, T)$ are both stable,
(ii) $m^{Q_a} = m^{Q^{\perp}} = m$.

Moreover, any Q such that λ, Q_a and $Q^{\perp}$ satisfy the above is a Rational Belief.

The probability $Q^{\perp}$ is central since it represents the theory of the agent of how the probability of an event at any date differs from the stationary probability at that date. This reveals a crucial characteristic of non-stationary systems: the timing of events matters in terms of the probabilities which are attached to them. Thus, the probability $Q^{\perp}$ permits an agent to assign to a given event different probabilities at different dates at which it may occur. Rationality of belief requires, however, that averaging the probabilities assigned to this event over all dates must yield the stationary probability assigned to it by m. Consequently, a Rational Belief Q may induce forecasts which are different from the forecasts of m at all dates and the difference between the forecasts of Q and m need not converge to zero. $Q^{\perp}$ may also place positive probabilities on events on which m places zero probability.

We now turn to the review of the papers in this issue. In order to assist the reader in sorting out the results reported we think it useful to review the papers in two

Copyright © 2001 All Rights Reserved

groups. Group I includes *Nielsen [1996a], Henrotte [1996], Kurz and Wu [1996] and Kurz and Schneider [1996]*. These papers study the structure of equilibria with endogenous uncertainty in economies in which agents have diverse beliefs. It is a common property of such economies that the state space itself is endogenous and consequently they face the common problem of constructing the state space in a manner compatible with the postulated economic and financial structure and with the rationality of belief conditions. Group II includes *Chuang [1996] and Nielsen [1996b]*. These papers study the properties of the set of rational beliefs.

3 Group I: Nielsen [1996a], Henrotte [1996], Kurz and Wu [1996], and Kurz and Schneider [1996]

In order to use the theory of rational belief one needs to work with processes which are non-stationary but stable. Nielsen's [1996a] paper fully characterizes a class of such processes called Simple Independently Distributed Stable (SIDS) processes. This class is analogous to i.i.d. processes and hence extremely useful in applications. A simple example will help clarify the nature of this class. Pick a process $\{y_j, j=0,1,\ldots\}$ of i.i.d. random variables taking values in $\{0,1\}$ with probability of 1 being, say, 1/4 and generate an infinite sequence of observations $y^* = (y_0^*, y_1^*,\ldots)$ of the process. The realizations $y_j^* = 1$ or $y_j^* = 0$ are now treated as *parameters* of a non-stationary SIDS process $\{x_t, t=0,1,\ldots\}$ to be defined. Thus, select a set of parameters $\{\alpha, \beta\}$ and a map associating the value $y_j^* = 1$ with the value α and the value $y_j^* = 0$ with β such that the process $\{x_t, t=0,1,\ldots\}$, where $x_t \in X = \{0,1\}$, is a sequence of *independent* random variables satisfying

$$P\{x_t = 1\} = \begin{cases} \alpha & \text{if } y_t^* = 1 \\ \beta & \text{if } y_t^* = 0. \end{cases} \tag{1}$$

Suppose that α and β satisfy $(1/4)\alpha + (3/4)\beta = .35$ then the time average of the x_t's is .35. For the specified y^* and given the assumption of independence, (1) determines Π_{y^*}, the probability of sequences $x \in \{0,1\}^\infty \equiv X^\infty$ given y^*. Hence the dynamical system of the x_t's is $(X^\infty, \mathscr{B}(X^\infty), \Pi_{y^*}, T)$. This SIDS system is stable and has a stationary measure m represented by the i.i.d. process $\{w_t, t=0,1,\ldots\}$, $w_t \in X$ with $P\{w_t = 1\} = .35$. The measure m is independent of the realization of the generating parameters y^*.

Theorem 2 above (Kurz [1994a]) implies that the stationary probability m is a rational belief. At each t there are, however, other rational beliefs Q about $\{x_\tau, \tau = t+1, t+2,\ldots\}$ such that $m^Q = m$. For example, an agent may have a private generating process $\{z_t, t=0,1,\ldots\}$ of i.i.d. coin tossing with probability of 1 being ξ and a generating sequence of parameters z^* in the same way as above with the frequency of $\{z_t^* = 1\}$ being ξ. Define now the perceived process $\{x_t', t=0,1,\ldots\}$ as in (1) but with z_t^* replacing y_t^* and (α', β') replacing (α, β). This independent but non-stationary sequence of random variables defines a stable SIDS measure P with m as its stationary measure if $\xi\alpha' + (1-\xi)\beta' = .35$.

This example is fully generalized by Nielsen [1996a] as follows. The space of observable economic variables is the measurable space $(X, \mathscr{B}(X))$ where $X \subseteq \mathbf{R}^N$ and $\mathscr{B}$ denotes the Borel σ-field of the specified set. Let $Y = \mathbf{N}$ and let $\mathscr{P} = \{P_1, P_2,\ldots\}$ be

Copyright © 2001 All Rights Reserved

a countable collection of probability measures on $(X, \mathscr{B}(X))$. Now pick any Q such that the dynamical system $(Y^\infty, \mathscr{B}(Y^\infty), Q, T)$ is ergodic and stationary and select a realization y^* of this system. Let $(P_{y_0^*}, P_{y_1^*}, \ldots)$ be the corresponding sequence of one period probabilities on $(X, \mathscr{B}(X))$. The product measure $\mu = \bigtimes_{t=0}^{\infty} P_{y_t^*}$ on $(X^\infty, \mathscr{B}(X^\infty))$ is an SIDS measure. The generating sequence y^* is then a sequence of parameters of the non-stationary SIDS dynamical system $(X^\infty, \mathscr{B}(X^\infty), \mu, T)$ in the sense that if at date t $(Y^\infty, \mathscr{B}(Y^\infty), Q, T)$ selects y_t^* then at date t the probability P_{y^*} is the operative one. Nielsen [1996a] studies the structure of such systems and shows that for Q almost all realizations they are stable and the stationary measure is independent of the particular realization. The method of generating processes as a tool in the study of non-stationary and stable processes is taken up by Kurz and Schneider [1996] and is generalized to any joint system $((X \times Y)^\infty, \mathscr{B}((X \times Y)^\infty), \Pi, T)$ where the data x and the parameters y are interrelated. Their "Conditional Stability Theorem" then states that if the joint system is ergodic and stable then the conditional system on the data $(X^\infty, \mathscr{B}(X^\infty), \Pi_{y^*}, T)$, given any realized sequence y^* of parameters, is stable for Π almost all realizations y^*. Being stable, it has a stationary measure and they show that it is derived as follows. Let Π_X be the marginal measure of Π on $(X^\infty, \mathscr{B}(X^\infty))$. Then the stationary measure of Π_X is equal to the stationary measure of the conditional measure Π_{y^*}. When the join system is stationary, then Π_X is stationary and hence the stationary measure of Π_X is Π_X itself. Kurz and Schneider [1996] apply these ideas to systems in which economic variables exhibit dependency over time and interdependency between economic variables and generating variables.

The method of generating variables is a device used in this volume for the analysis of Rational Belief Equilibria (RBE) which are equilibria in which the agents hold rational beliefs. For the description of non-stationary but stable systems such variables have the simple interpretation of parameters. However, for the description of rational beliefs of agents, generating variables can be interpreted either as parameters of a perceived probability or as private signals (such as a private report of a firm's research department) which then leads to a selection of a distribution parameter. The use of such variables for the construction of the state space is due to Nielsen [1994] who uses the term "Rational Belief Structure" rather than a "price state space" which we use here. We outline here an intuitive explanation of the procedure employed.

Start by noting that at date t the generating variable of each agent is a parameter in his demand function and hence the market clearing conditions of a market with K agents depend upon the vector $y_t^* = (y_t^{1*}, \ldots, y_t^{K*})$ of these K variables. If s_t is a vector of exogenous variables and p are market clearing prices at t, then consider the class of market clearing functions of the form

$$p_t = \Phi(s_t, y_t^*). \tag{2}$$

Assuming that all exogenous variables and generating variables take finite number of values implies that only a finite number of prices, M, will ever be observed. The rationality conditions therefore require that a rational belief must place positive stationary probability only on the finite number of M prices which are ever

Copyright © 2001 All Rights Reserved

observed. Now, since placing probabilities on $(p_1, p_2, \ldots, p_M)$ is equivalent to placing these probabilities on the price state space $\{1, 2, \ldots, M\}$ one can then state all the rationality conditions on the price state space together with the state space of the process of exogenous variables without specifying the *values* of the exogenous variables and equilibrium prices. Hence, the deeper part of the problem of constructing a consistent price state space is *the statement of the rationality of beliefs conditions of the agents* in such a way that they are stochastically compatible with the market clearing conditions. This is where Nielsen's [1996a] SIDS construction and the conditional stability theorem of Kurz and Schneider [1996] are used. Given the specified equilibrium functions (3) one shows that under some technical conditions if the exogenous variables and the generating variables are either jointly SIDS or Markov and if the generating variables used by the agents to form beliefs are SIDS or Markov, then the equilibrium prices and the exogenous variables become jointly SIDS or Markov. In this sense the SIDS and Markov classes of processes are "closed" classes. This argument can easily be extended to the case where all variables take countable rather than finite values.

Henrotte's [1996] paper demonstrates some of the difficulties which one confronts if one allows, at the start, a continuum of possible distinct equilibrium prices. The paper assumes that agents have heterogenous probability beliefs on the space of possible prices. It sets up a complex structure of interrelated securities: common stocks and derivative securities where the return on an investment in a derivative security is a known function of future *prices* of other securities. Henrotte [1996] studies the construction of a consistent price state space by defining it to be the set of non-arbitrage prices. To define this term let a positive dynamic portfolio be an initial portfolio together with a reinvestment strategy which ensures non-negative payoff *in all states of the world*. The condition of non-arbitrage requires that the price of any positive dynamic portfolio be non-negative. However, since the prices of all securities are part of the state space on which derivative securities are defined, we have a self referential definition: the price states determine the positive dynamic portfolios which, in turn, define the non-arbitrage prices. The main result establishes under some technical conditions the existence of a non-arbitrage price state space.

Henrotte's [1996] paper is different from the other three in this group since he does not incorporate any rationality restrictions. However, the interest in non-arbitrage prices arises from the fact that it is a "minimal" rationality condition which should be imposed on the supports of the probability beliefs of the agents. This leaves a continuum of configurations which constitute the price state space.

All four papers in this group attack equilibrium problems. Nielsen [1996a] uses SIDS processes to demonstrate the existence of an RBE of an OLG economy with K agents, a homogeneous consumption good and money. However, the model allows agents only very limited opportunities to trade endogenous uncertainty. This brings up the important general question of how we should reformulate general equilibrium theory to allow agents to trade endogenous uncertainty in general and price uncertainty in particular. We mentioned earlier the idea of replacing the Arrow-Debreu system of *exogenous state* contingent contracts by a system of *price* contingent contracts (in short, PCC). Utilizing the Bewley [1972] fixed point

Copyright © 2001 All Rights Reserved

theorem, Svensson [1981] formulates a temporary equilibrium model with endogenous uncertainty in which agents can trade PCC.

The paper by Kurz and Wu [1996] aims to integrate the financial structure of PCC with the rationality of belief conditions. Although it is cast in the context of a relatively simple OLG economy, it studies the full problem of general equilibrium with endogenous uncertainty in which the ownership shares of a firm are traded in a stock market, agents hold rational beliefs and can trade uncertainty using a full set of PCC. Such PCC are contracts traded at date t and permit an agent to receive or deliver units of the common stock in period $t+1$. The quantity received or delivered is specified in the contract to be contingent upon *the price of the common stock in period* $t+1$. Since these PCC are "derivative" securities Kurz and Wu [1996] introduce non-arbitrage pricing restrictions which, in their case, can be directly imposed on the set of allowable prices. They then assume that the probability of the exogenous endowments and the beliefs of the agents are jointly SIDS and this assumption enables them to construct a consistent price states space. The paper considers in detail a simple special case aiming to explain the construction of the state space and to demonstrate the exact restrictions which the rationality conditions impose on the beliefs of agents. Having constructed the state space, Kurz and Wu [1996] give a definition of "endogenous uncertainty" *in terms of the price states.* First time readers may find Section (III.a) of this paper helpful for understanding some of the basic ideas of the volume.

The general result of all four papers in Group I is that an equilibrium exists whenever a consistent price state space can be constructed. However, the trading of PCC presents new technical difficulties and an important aspect of the Kurz and Wu [1996] paper is the existence proof. It demonstrates that contrary to Svensson [1981], the utilization of the rationality conditions for the OLG economy enables a *reduction* in the dimension of the price state space and this simplifies the structure of the PCC so that finite dimensional methods of proof become applicable. Kurz and Wu [1996] also argue that endogenous uncertainty is generic in an RBE and an RBE is constrained Pareto Optimal under the restriction that the endowment risk of the unborn young cannot be reallocated.

Kurz and Schneider [1996] utilize the technique of generating variables to study the RBE of a multi-agent OLG economy with a single consumption good but multiple firms whose shares are traded on a stock market. The paper studies the effects of correlation among agents on the volatility of stock prices. For that purpose, Kurz and Schneider [1996] construct a simulation model for which equilibrium prices and their stationary distribution can be calculated. The model assumes an economy with one firm, two agents, a dividend process taking two values, each generating variable taking two values and a Markov joint structure with Markov marginals. To be able to discuss "correlation" between the agents the construction of the generating variables is given a specific meaning: given each of the two values that a variable can take, the agent selects a different Markov transition matrix over the observables (i.e. prices and dividends). In each matrix an agent can become either optimistic or pessimistic about the state of "high" dividends in the subsequent period. The *intensity* of such optimism or pessimism is measured by the displacement of the probabilities of the high or low dividend states relative to the corre-

Copyright © 2001 All Rights Reserved

sponding values of the stationary measure. A second measure of correlation are parameters which directly influence the correlation between the generating variables. Kurz and Schneider [1996] view both forms of correlation as consequences of the process of communication in society. The results of Kurz and Schneider [1996] show that the presence of correlation has a dramatic effect on asset price volatility. Moreover, correlation among the generating variables of the agents tend to endogenously create non-stationarity in the time series of prices which takes the form of multiple "regimes" of high or low price volatility. The high volatility regimes consist of those states in which the agents "agree" and their agreement (in optimism or pessimism) induce increased price volatility. The low volatility regimes are the "disagreement" states in which the disagreements of the agents tend to cancel each other out and consequently reduce price volatility.

4 Group II: Chuang [1996] and Nielsen [1996b]

These papers study the structure of the set $B(\Pi)$ where $(\Omega, \mathscr{B}, \Pi, T)$ generates the data. Kurz [1994a] shows that $B(\Pi)$ is a convex set and asserts (page 886), in error, that $B(\Pi)$ is also compact in the topology of weak convergence. Nielsen [1996b] provides a counter-example and we retract the previous claim. In the present study Nielsen [1996b] shows that both the set of stable measures as well as the set of rational beliefs are closed in the topology of strong (sup norm) convergence but this is too restrictive for economic applications. The interest in the closedness properties of $B(\Pi)$ is motivated both by a desire to clarify the technical foundations of the theory as well as by the economic significance of the results. To explain this we note that the theory of rational beliefs does not propose to explain why agents select any particular member of $B(\Pi)$. Agents must, therefore, employ additional criteria which lead to the selection of the beliefs which they hold. The requirement of rationality implies that with his one very long (but of finite length) time series an agent must calculate the relative frequencies of a generating collection of cylinders. This collection may be the class of all cylinders or, for an agent who maximizes over a future of length N, the class of cylinders of dimension less or equal to N. It is then plausible for the agent to subjectively restrict his choice to probability measures in $B(\Pi)$ for which *the speed of convergence of his data is uniformly high on the specific class of cylinders with which he is concerned.* These considerations lead Nielsen [1996b] to examine various collections of probability measures which are *uniformly* stable in the sense just explained. In essence, Nielsen's [1996b] result is that sets of probability measures which are uniformly stable are closed in the topology of weak convergence.

Chuang [1996] also studies the properties of $B(\Pi)$ but in terms of the long term average of the *conditional forecast* function. Thus suppose that Q is a rational belief with respect to Π and a bounded function $f(x_t, x_{t+1}, \ldots, x_{t+L})$ is given. The problem is to characterize the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} E_Q(f(x_t, x_{t+1}, \ldots, x_{t+L}) | I_t) \tag{3}$$

Copyright © 2001 All Rights Reserved

if it exists. Chuang's [1996] main theorem says that if the process $\{x_t, t = 1, 2, \ldots\}$ is stable and if f is bounded, then the limit in (3) exists for Q almost all histories and is equal to the unconditional expectations of f under the stationary measure m.

To see why the above problem has important empirical implications consider the typical portfolio problem of an agent who maximizes an expected utility over consumption streams (c_t) given a rational belief Q and information I_t. His first order conditions are of the well known form

$$E_Q\left[\frac{u'(c_{t+1})}{u'(c_t)}\frac{P_{t+1}+r_{t+1}}{P_t}\,|\,I_t\right]\left(\frac{1}{1+\delta}\right)=1 \tag{4}$$

where P_t is the price of the asset, r_t is the dividend and δ is the discount rate. It is shown in Kurz [1996] that when there are diverse beliefs in asset markets the standard orthogonality conditions do not hold. Hence (4) implies that there exist functions $\eta_t \neq 0$ which are not orthogonal to the subspace spanned by I_t such that

$$\frac{u'(c_{t+1})}{u'(c_t)}\left(\frac{P_{t+1}+r_{t+1}}{P_t}\right)\frac{1}{1+\delta}-1=\eta_t(I_t, \varepsilon_{t+1}). \tag{5}$$

The functions η_t which depend upon the pure noise ε_{t+1} and on I_t, measure the mistakes of rational agents who hold beliefs which are rational but are not equal to the true equilibrium probability Π. We then want to know the limit of the long term average of the η_t. Chuang's [1996] main theorem implies that if consumption is a time invariant function which depends on information variables of bounded length so that the expression in (5) can be written as the function $f(\cdot)$ in (3), then the long term average of the η_t tends to 0 for Q almost all histories $\{I_t, t = 1, 2, \ldots\}$. This result does not assert the conclusion for Π almost all histories. When agents hold non-stationary beliefs then consumption is not time invariant and the problem of convergence of the time average of a sequence of functions $f_t(\cdot)$ (as in (5)) is an open problem.

5 Some remarks on empirical implications and public policy under rational beliefs

The question whether variations in equilibrium prices and quantities are due only to variations in exogenous "fundamentals" has been a long standing controversial issue. The theory advanced in this volume derives an internally propagated component of economic fluctuations from principles of individual rationality. The usefulness of these principles must be judged either by the positive restrictions which the theory places on the data generated by the economy and/or by their policy implications. Focusing only on the theory, the papers in this volume do not address these issues directly. For readers who are interested in applications of the theory we mention two papers of Kurz [1996] and Kurz and Beltratti [1996].

Common to both papers is the econometric implications of the "mistake measure" of an agent which is the difference between his probability belief and the true equilibrium probability. The econometric implications focus on the orthogonality conditions of the theory. To explain these return to equation (5) above

Copyright © 2001 All Rights Reserved

which we write in the explicit form $\eta_t(I_t, \varepsilon_{t+1}) = \xi_t(I_t) + \varepsilon_{t+1}$. When $Q = \Pi$ the usual orthogonality theorem of conditional expectations imply that $\xi_t \equiv 0$. Under rational beliefs typically $\xi_t \neq 0$ and this means that excess utility returns are present in the market and are predictable in the sense that there are information variables in I_t which are correlated with stock returns. Kurz [1996] postulates that the period 1947–1992 in the U.S. consists of three distinct intervals representing three regimes. The predictions of the theory, which he tests for the period at hand, are that for each regime $\xi_t \neq 0$ and that ξ_t are significantly different across regimes. He then argues that the bulk of the fluctuations in stock prices represent endogenous uncertainty.

Kurz and Beltratti [1996] show that the presence of mistake functions lead to biased estimates of structural parameters based on equations like (5). Utilizing data on the asset composition of a sample of mutual funds they estimate the mistake functions of the funds and find the risk aversion coefficients of most fund managers to be between 2 and 4. They argue that the debate on the "equity premium" is flawed by the exclusion of endogenous asset price uncertainty. By replacing rational expectations with rational belief they extend the uncertainty allowed and argue that in equilibrium the demanded premium would be higher than could be accounted for by models in which the only perceived uncertainty is the exogenous uncertainty of future dividends. To make their case Kurz and Beltratti [1996] construct a two-agent economy and compare the results of a simulation of the RBE of this economy with the simulation results of Mehra and Prescott [1985]. The parameters of the real part of the economy are the same as the Mehra and Prescott [1985] specifications and in accord with the empirical evidence. They show that the calculations for the unique rational expectations equilibrium entail the usual unsatisfactory results of the equity premium puzzle whereas for the specified RBE the moments closely correspond to the historical record.

Turning to issues of public policy it can be seen that in a market with endogenous uncertainty public policy and collective action can have an important effect. Consider, for example, a collective action which aims to restrict price fluctuations in a specified market to a target range. Under any theory where the state of belief has no effect on prices, such a policy implies that with probability 1 the public sector will end up needing to directly intervene (by either buying or selling) in order to ensure the success of the policy. Under rational beliefs there exists a target range of prices and quantities and a credible public stabilizing action which will be effective *but the policy will never require any actual market intervention.* Furthermore, the larger is the endogenous component of economic fluctuations the larger is the range of effectiveness of public policy. Finally, if a market is dominated by irrational traders, one cannot make any prediction as to the effect of public policy.

Without structural knowledge by agents, individuals and firms who hold rational beliefs make allocation decisions based on their differing expectations. In such an economy prices and quantities change not only in response to changes in fundamental states but also in response to changes in the state of belief. A credible public policy will alter the beliefs of agents and therefore can guide the economy onto an alternative RBE in which the level of fluctuations of prices and quantities is *permanently* altered.

Copyright © 2001 All Rights Reserved

The methodological objections to the use of diversity of beliefs are weak relative to the issue of public policy. What matters is the fact that if the theory of rational beliefs correctly describes human behavior and if endogenous uncertainty is present in equilibrium then public policy has an impact in an RBE! Moreover the theory leads to specific predictions of the range in which public policy can be effective as well as to the manner in which public policy should be executed. We hasten to add that this does not mean that public policy under rational beliefs is a simple matter since in the postulated environment there are important difficulties in identifying criteria for formulating *desirable* public policies and in the practical execution of such policies when identified. The fact that the theory of rational belief predicts the effectiveness of public policy suggests that testing this prediction may be an important way in which both the validity of the theory as well as its usefulness can be judged.

References

Arrow, K. J.. Le role des valeurs Boursieres pour la repartition la meillure des risques. Econometrie CNRS **40**, 41–47 (1953)

Arrow, K. J.: Debreu, G.: Existence of an equilibrium for a competitive economy. Econometrica **22**, 265–290 (1954)

Aumann, R. J.: Correlated equilibrium as an expression of Bayesian Rationality. Econometrica **55**, 1–18 (1987)

Bewley, T. F.: Existence of equilibria in economies with infinitely many commodities. Journal of Econ. Theory **4**, 514–540 (1972)

Brennscheidt, G.: Predictive behaviour: An experimental study. Lecture Notes in Economics and Mathematical Systems No. 403, Berlin. Heidelberg, New York: Springer Verlag 1993

Cabrales, A., Hoshi, T.: Heterogeneous beliefs, wealth accumulation, and asset price dynamics. Discussion paper 93-11, Department of Economics. University of California at San Diego 1993

Chuang, C. S.: Ergodic properties of conditional forecast functions of stable systems. Economic Theory **8**, 521–530 (1996)

De Long, J. B, Shleifer, A., Summers, L. H., Waldman. R. J.: Noise trader risk in financial markets Journal of Political Economy **98**, 703–738 (1990)

Detemple, J., Murthy, S.: Intertemporal asset pricing with heterogeneous beliefs. Journal of Economic Theory **62**, 294–320 (1994)

Frankel, J. A., Froot, K. A.: Using survey data to test standard propositions regarding exchange rate expectations. American Economic Review **77**, 133–153 (1987)

Frankel, J. A, Froot, K A.: Chartists, fundamentalists and the demand for dollars. In: Courakis, A. S., Taylor, M P. (eds) Private behaviour and government policy in interdependent economies. New-York Oxford University Press, 73–126 (1990)

Frankel, J. A.. Rose, A. K., A survey of empirical research on nominal exchange rates. In: Grossman, G. M., Rogoff, K. (eds) Chapter 33 of the Handbook of International Economics, vol. III. Amsterdam: North Holland, 1689–1729 1995

Froot, K. A., Frankel, J. A · Forward discount bias: is it an exchange risk premium? Quarterly Journal of Economies **104**, 139–161 (1989)

Harris, M., Raviv, A.: Differences of opinion make a horse race. Review of Financial Studies **6**, 473–506 (1993)

Henrotte, P.: Construction of a state space for interrelated securities with an application to temporary equilibrium theory. Economic Theory **8**, 423–459 (1996)

Kandel, E., Pearson, N. D.: Differential interpretation of public signals and trade in speculative markets. Journal of Political Economy **4**, 831–872 (1995)

Kurz, M.: The Kesten-Stigum Model and the treatment of uncertainty in equilibrium theory. In· Balch, M. S., McFadden, D. L., Wu, S. Y. (eds) Essays on economic behavior under uncertainty. Amsterdam: North Holland, 389–399 1974

Copyright © 2001 All Rights Reserved

Kurz, M.: On the structure and diversity of rational beliefs. Economic Theory **4**, 877–900 (1994a)
Kurz, M.: On rational belief equilibria. Economic Theory **4**, 859–876 (1994b)
Kurz, M : Asset prices with rational beliefs. Draft, Stanford University 1996 (to appear in a forthcoming monograph in the Springer series *Studies in Economic Theory* entitled *Endogenous Economic Fluctuations: Studies in the Theory of Rational Beliefs*, 1997)
Kurz, M., Beltratti, A : The equity premium is no puzzle. Draft, Stanford University 1996 (to appear in a forthcoming monograph in the Springer series *Studies in Economic Theory* entitled *Endogeneous Economic Fluctuations: Studies in the Theory of Rational Beliefs*, 1997)
Kurz, M , Schneider, M : Coordination and correlation in markov rational belief equilibria Economic Theory **8**, 489–520 (1996)
Kurz, M., Wu, H. M : Endogenous uncertainty in a general equilibrium model with price contingent contracts. Economic Theory **8**, 461–488 (1996)
Mehra. R , Prescott, E. C.: The equity premium: a puzzle. Journal of Monetary Economics **15**, 145–162 (1985)
Morris, S. The common prior assumption in economic Theory. Economics and Philosophy **11**, 227–253 (1995)
Morris, S.: Trade with heterogeneous prior beliefs and asymmetric information Econometrica **62**, 1327–1347 (1994)
Nielsen, C. K.: Weakly rational beliefs, structural independence and rational belief structures. A dissertation submitted to the Department of Economics. Stanford University 1994
Nielsen, C K : Rational belief structures and rational belief equilibria. Economic Theory **8**, 399–422 (1996a)
Nielsen, C. K : On some topological properties of stable measures. Economic Theory **8**, 531–553 (1996b)
Radner, R : Existence of equilibrium of plans, prices and price expectations in a sequence of market. Econometrica **40**, 289–303 (1972)
Radner, R.: Rational expectations equilibrium generic existence and the information revealed by prices. Econometrica **47**, 655–678 (1979)
Savage, L. J.: The foundations of statistics. New York: Wiley 1954
Svensson, L E. O.: Efficiency and speculation in a model with price-contingent contracts. Econometrica **49**, 131–151 (1981)
Takagi, S.: Exchange rate expectations: A survey of survey studies. International Monetary Fund Staff Papers **38** 156–183 (1991)
Taylor, M.: The economics of exchange rates. Journal of Economic Literature **33**, 13–47 (1995)

Copyright © 2001 All Rights Reserved