Chapter 2: A Model with Differential Risk

2A.1 Some comments on the model used in the text

The model specified in the text allows individual firms to optimize by setting their own pricing and allocation strategies based on their own productivities since individual production functions were defined by

$$Y_{jt} = \zeta_t \Psi_{jt} (K_{jt})^{\alpha} (A_t L_{jt})^{(1-\alpha)}.$$

The productivity of growth of the individual firm is defined relative to the aggregate exogenous process. The full reasoning behind this model is developed in Chapter 7 where the productivity growth of individual firm is impacted by its R&D program. However, in the aggregation stage of Chapter 2 the variability of $\Psi_t = (\Psi_{1t}, \Psi_{2t}, \ldots, \Psi_{Mt})$ was not studied and is expressed by the assumption that $\Psi_{jt} = 1$. This fact leaves the aggregate undiversifiable randomness of $(\zeta_t, \mathcal{P}_t)$ as the only source of risk taken by all. Consequently, the rental rate R_{jt} and profits Π_{jt} are equally risky, and in equilibrium both risks are associated with $(\zeta_t, \mathcal{P}_t)$. If the effect of individual Ψ_{jt} is included in the model and these are not independently distributed, then the riskiness of firm strategic behavior would be reflected in the expression (2.9)

$$\varphi_t = \left[\sum_{j=1}^{M} \vartheta_{jt}^{\theta_t} \Psi_{jt}^{\theta_t - 1}\right]^{\frac{1}{\theta_t - 1}}$$

and this is the added risk being considered here.

As to the dynamic assumptions, I continue to specify the standard form of capital accumulation and technological change:

$$K_{t+1} = (1-\delta)K_t + I_t$$

$$\log(\zeta_{t+1}) = \lambda_\zeta \log(\zeta_t) + \varepsilon_{t+1}^{\zeta} \quad , \quad \varepsilon_{t+1}^{\zeta} \sim N(0, \sigma_\zeta^2)\,, \text{ i.i.d.}$$

The dynamics of aggregate market power is formulated in terms of the markup

$$\mathcal{P}_t = \frac{\theta_t}{\theta_t - 1},$$

and they reflect the theory developed in Chapter 1 that views technology and policy as the two forces that determine the dynamics of $\mathcal{P}_t$. The dynamics of market power, which is centrally important for the theory, is approximated by a Markov process

$$\mathcal{P}_{t+1} = g \mathcal{P}_t^{\lambda_{\mathcal{P}}} [\zeta_{t+1}^{\mu_\zeta} \zeta_t^{(\mu_\zeta)^2} \zeta_{t-1}^{(\mu_\zeta)^3} \zeta_{t-2}^{(\mu_\zeta)^4} \zeta_{t-3}^{(\mu_\zeta)^5}] e^{\varepsilon_{t+1}^{\mathcal{P}}} \;,\; \varepsilon_{t+1}^{\mathcal{P}} \sim N(0, \sigma_{\mathcal{P}}^2) \quad , \varepsilon_{t+1}^{\mathcal{P}} \text{ i.i.d., independent of } \varepsilon_{t+1}^{\zeta}.$$

2A.2 Differential risk

As motivated in Chapter 7, the term Ψ_{jt}, which is the relative productivity effect of firm j, is an unconditional mean 1 random variable, ζ_t is a common Markov productivity process with unconditional mean of 1 and $A_t = g_0 g^t$ is a common deterministic productivity factor growing at a constant rate g. Date t profit function is

$$\Pi_t^j = P_{jt} Y_{jt} - P_t W_t L_{jt} - P_t R_t K_{jt},$$

where factor costs (W_t, R_t) are in units of consumption or "real" terms.

In the context at hand, the unique feature is Ψ_{jt} that measures the changes in the market condition of firm j due to all factors associated with its strategic behavior: unexpected change in innovations of firm j, effects of a price war, acquisition of technology or firms etc. In a complete model this would be the outcome of a strategic game among the M firms in the economy where the strategies of the firms are those noted. This would require an integration of standard Walrasian competitive conditions with a dynamic strategic game, a task which is avoided here as being far away from the central topic of the present study. As an alternative, assume the outcome of this process is an exogenous stochastic process of the vector

$$\Psi_t = (\Psi_{1t}, \Psi_{2t}, \ldots, \Psi_{Mt}) \quad , \quad \log\Psi_{t+1} = \lambda_\Psi \log\Psi_t + \varepsilon^\Psi_{t+1} \quad , \quad E_{t-1}\Psi_{jt} = \Psi^{\lambda_\Psi}_{j,t-1}$$

where ε^Ψ_t is an i.i.d. random vector with a variance-covariance matrix to be specified. It will introduce the additional risk of profits as distinct from capital income.

2A.2.1 Two stage equilibration at each date

The risk taken by capital and labor in contracting to work for the firm is lower than the risk taken by profits. This difference is expressed by capital and labor being hired after the aggregate risk expressed by $(\zeta_t, \mathcal{P}_t)$ is resolved, but before the firm specific risk expressed by Ψ_{jt} is resolved. I assume $(\zeta_t, \mathcal{P}_t)$ is revealed at the *start of date t* and capital and labor are hired at that time since they do not assume the firm's own technological risk expressed by Ψ_{jt} and do not benefit from its outcome. As noted, the risk of Ψ_{jt} is associated with the firm's pricing and innovation and I assume that Ψ_{jt} becomes known at a later stage in *date t*. There is no real model time difference between the *beginning and second stageof date t; ,* the difference is only in the timing at which information is revealed and acted upon within each period. The model must then distinguish between activities at the beginning and those at the second stage of date t, but this difference does not entail a difference in time discounting. Therefore I will define a rolling equilibrium that is developed in two stages within the same time unit, with the following general scheme:

Stage 1: At start of date t $(\zeta_t, \mathcal{P}_t)$ is revealed, K_t is known from the end of t-1, firms form a *date t* expectation $(E_{j,t-1}\Psi_{jt})$ and equilibrium $(\hat{W}_t, \hat{R}_t, \hat{L}_t, \hat{K}_{t+1}, \hat{I}_t, \hat{C}_t, \hat{Y}_t, \hat{\Pi}_t)$ is selected as in the text. It leads to interim factor prices $(\hat{W}_t, \hat{R}_t)$ at which labor is actually contracted. Interim forecasts for $(\hat{C}_{t+1}, \hat{L}_{t+1}, \hat{R}_{t+1}, \hat{\Pi}_{t+1})$ are made (but are revised when full information is available). It solves for factor prices, for *actual* capital and labor income $(\hat{R}_t K_t, \hat{W}_t \hat{L}_t)$ and for *planned* profits output and consumption $(\hat{\Pi}_t, \hat{C}_t, \hat{Y}_t)$.

Stage 2: $(K_t, \hat{L}_t)$ *are now fixed*, and then all Ψ_{jt} are revealed. A new equilibrium is reached in which the values of (C_t, Y_t) and of profits $\Pi_t = Y_t - \hat{W}_t \hat{L}_t - \hat{R}_t K_t$ are updated. It is now the real time for households to choose an optimal K_{t+1}, which will depend on a forecast of

$(C_{t+1},\hat{L}_{t+1},\hat{R}_{t+1},\Pi_{t+1})$. The final date t solution is $(\hat{W}_t,\hat{R}_t,\hat{L}_t,K_{t+1},C_t,Y_t,\Pi_t)$ with income distribution of $(\hat{W}_t\hat{L}_t\,,\hat{R}_tK_t\,,\Pi_t)$.

The key to the distinction made here is the fact that capital income $\hat{R}_tK_t$ is less risky than profits and is known before profits are determined as a risky residual.

To write down the equilibrium conditions recall from the text that

$$P_t=[\sum_{j=1}^{M}\vartheta_{jt}(\frac{P_{jt}}{\vartheta_{jt}})^{1-\theta_t}]^{\frac{1}{1-\theta_t}}=[\sum_{j=1}^{M}\vartheta_{jt}^{\theta_t}P_{jt}^{1-\theta_t}]^{\frac{1}{1-\theta_t}}\ \text{ with }\ 1=[\sum_{j=1}^{M}\vartheta_{jt}^{\theta_t}(\frac{P_{jt}}{P_t})^{1-\theta_t}]^{\frac{1}{1-\theta_t}}$$

In stage 1 the firms choose their own prices and quantity of labor and capital employed to maximize profits at each date:

$$\mathrm{Max}_{P_{jt},N_{jt},K_{jt}}\Big[P_{jt}\hat{Y}_{jt}-P_tW_tL_{jt}-P_tR_tK_{jt}+\lambda_{jt}[\zeta_t(E_{t-1}\Psi_{jt})K_{jt}^{\alpha}(A_tL_{jt})^{1-\alpha}-\hat{Y}_{jt}]\Big]\ ,\ \hat{Y}_{jt}=(\frac{P_{jt}}{\vartheta_{jt}P_t})^{-\theta_t}\hat{Y}_t$$

The first order conditions are then

$$(\theta-1)(\frac{P_{jt}}{\vartheta_{jt}P_t})^{-\theta_t}\hat{Y}_t=\lambda_{jt}\theta_t(\frac{P_{jt}}{\vartheta_{jt}P_t})^{-\theta_t}\hat{Y}_tP_{jt}^{-1}\ \Rightarrow\ \lambda_{jt}=P_{jt}\frac{(\theta_t-1)}{\theta_t}\ \text{ all j}$$

$$W_t=\frac{P_{jt}}{P_t}\frac{(\theta_t-1)}{\theta_t}(1-\alpha)(E_{t-1}\Psi_{jt})[\zeta_tA_t^{1-\alpha}](K_{jt})^{\alpha}(L_{jt})^{-\alpha}=\frac{P_{jt}}{P_t}\frac{(\theta_t-1)}{\theta_t}\frac{\partial\hat{Y}_{jt}}{\partial L_{jt}}.$$

$$R_t=\frac{P_{jt}}{P_t}\frac{(\theta_t-1)}{\theta_t}\alpha(E_{t-1}\Psi_{jt})[\zeta_tA_t^{1-\alpha}](K_{jt})^{\alpha-1}(L_{jt})^{1-\alpha}=\frac{P_{jt}}{P_t}\frac{(\theta_t-1)}{\theta_t}\frac{\partial\hat{Y}_{jt}}{\partial K_{jt}}.$$

I now proceed to the aggregation. To that end, I start with the following proposition, taking advantage of the symmetry of the producers:

Proposition A2.1: In equilibrium

(i) Relative prices satisfy $\frac{P_{jt}}{P_t}(E_{t-1}\Psi_{jt})=\hat{\varphi}_t$ for all j where

(A2.1a) $$\hat{\varphi}_t=[\sum_{j=1}^{M}\vartheta_{jt}^{\theta_t}(E_{t-1}\Psi_{jt})^{\theta_t-1}]^{\frac{1}{\theta_t-1}}=[\sum_{j=1}^{M}(E_{t-1}\Psi_{jt})^{\theta_t-1}]^{\frac{1}{\theta_t-1}}\qquad\text{if}\quad\vartheta_{jt}=1$$

(ii) Equilibrium quantities act in accordance with an aggregate production function and

(A2.1b) $$Y_t=\hat{\varphi}_t\zeta_t(K_t)^{\alpha}(A_tL_t)^{(1-\alpha)}.$$

(A2.1c) $$(\frac{P_{jt}}{P_t}(E_{t-1}\Psi_{jt}))\zeta_t(K_{jt})^{\alpha}(A_tL_{jt})^{(1-\alpha)}=\hat{\varphi}_t\zeta_t(K_{jt})^{\alpha}(A_tL_{jt})^{(1-\alpha)}.$$

The proposition shows that, adjusting for the factor Ψ_{jt}, the relative price of an intermediate good j to the price of the consumption good is the same for all intermediate goods. In addition, (A2.1a) also shows that any change in the quality of intermediate good j shows up as a change in the relative price of intermediate goods to consumption good. I will then disregard the effect of change in quality and assume $\vartheta_{jt} = 1$. It then follows that after aggregation

$$\hat{W}_t = \frac{(\theta_t - 1)}{\theta_t}(1-\alpha)\hat{\varphi}_t \zeta_t A_t (K_t)^{\alpha} (A_t \hat{L}_t)^{-\alpha}$$

$$\hat{R}_t = \frac{(\theta_t - 1)}{\theta_t}\alpha\hat{\varphi}_t \zeta_t (K_t)^{\alpha-1} (A_t \hat{L}_t)^{1-\alpha}$$

$$\hat{Y}_t = \hat{\varphi}_t \zeta_t K_t^{\alpha} (A_t \hat{L}_t)^{1-\alpha}$$

These are then the three equations for the firm in equilibrium.

2A.3 Optimal household behavior

I have assumed that capital is owned by households and they rent it to the firms. Their saving decision is based on the expected rental rate of capital to be realized in the next period, when the demand for capital by firms is formulated. This means that households make decisions in the two stages as follows:

In stage 1: At the start of each period, the level of capital invested has already been decided at t-1 and therefore it is known, consequently the household makes only its labor decision given the wage rate which is specified in equilibrium.
In stage 2: When both the rental rate and total profits are specified, the household makes its final revised consumption and its capital investment decisions. Thus, households make capital and consumption decisions when all information is available and therefore all demand functions can be deduced from the same optimization. One could require the consumers to make their final consumption decision in stage 1, before knowing their profits, but this raises the possibility of default, which is a complication avoided at this point.

I study the dynamic optimization of a representative household, which takes the form of

$$\mathrm{Max}_{(C,I,L)} \sum_{\tau=0}^{\infty} \beta^{\tau} [\frac{1}{1-\sigma} C_{\tau}^{1-\sigma} (1 - \frac{\mathcal{H}}{1+\eta} L_{\tau}^{1+\eta})],$$

although in the simulations I also explore the effect of the alternative utility function

$$\mathrm{Max}_{(C,I,L)} \sum_{\tau=0}^{\infty} \beta^{\tau} [\log C_{\tau} - \frac{\mathcal{H}}{1+\eta} L_{\tau}^{1+\eta}],$$

subject to the budget constraint

$$C_t + K_{t+1} = W_t L_t + K_t(R_t + (1-\delta)) + + \Pi_t \quad , \quad R_t = r_t + \delta .$$

Note that in this model we ignore the stock price and quantity demanded. R the rental rate, is the sum of interest plus depreciation. The parameter $\mathcal{H}$ pins down the steady state value of L at the value of 0.3 reflecting the estimate that the fraction of time of a year spent at work is 30%.

2A.4 A two stage equilibrium

Stage 1: at the start of date t

Since I have assumed the joint Markov Process

$$\Psi_t = (\Psi_{1t}, \Psi_{2t}, \ldots, \Psi_{Mt}) \quad , \quad \log\Psi_{t+1} = \lambda_\Psi \log\Psi_t + \varepsilon^\Psi_{t+1} \quad , \quad E_{t-1}\Psi_{jt} = \Psi^{\lambda_\Psi}_{j,t-1},$$

it follows that for any sequence of random vectors $(\zeta_t, \mathcal{P}_t, \Psi_t)$ the following values are known:

$$\hat{\varphi}_t = [\textstyle\sum_{j=1}^{M} (E_{t-1}\Psi_{jt})^{\theta_t - 1}]^{\frac{1}{\theta_t - 1}}.$$

Therefore, in stage 1 we have that aggregate output is

$$\hat{Y}_t = \hat{\varphi}_t \zeta_t (\hat{K}_t)^\alpha (A_t \hat{L}_t)^{(1-\alpha)}.$$

The equilibrium conditions of stage 1 are then

$$(\hat{C}_t)^{-\sigma}(1 - \frac{\mathcal{H}}{1+\eta}\hat{L}_t^{1+\eta}) = \beta E_t[(\hat{C}_{t+1})^{-\sigma}(1 - \frac{\mathcal{H}}{1+\eta}\hat{L}_{t+1}^{1+\eta})(\hat{R}_{t+1} + (1-\delta))]$$

$$\frac{\mathcal{H}}{1-\sigma}\hat{L}_t^{\eta} = (1 - \frac{\mathcal{H}}{1+\eta}\hat{L}_t^{1+\eta})\frac{\hat{W}_t}{\hat{C}_t}$$

$$\hat{W}_t = \frac{\hat{\varphi}_t}{\mathcal{P}_t}(1-\alpha)\zeta_t g^{\,t}(K_t)^\alpha (g^{\,t}\hat{L}_t)^{-\alpha}$$

$$\hat{R}_t = \frac{\hat{\varphi}_t}{\mathcal{P}_t}\alpha\zeta_t (K_t)^{\alpha-1}(g^{\,t}\hat{L}_t)^{1-\alpha}$$

$$\hat{C}_t + (\hat{K}_{t+1} - K_t(1-\delta)) = \hat{Y}_t = \hat{W}_t\hat{L}_t + \hat{K}_t\hat{R}_t + \hat{\Pi}_t$$

$$\hat{Y}_t = \hat{\varphi}_t\zeta_t(K_t)^\alpha(A_t\hat{L}_t)^{(1-\alpha)}.$$

This is the same as the equilibrium in the text with one minor difference. In the text it is assumed that $\hat{\varphi}_t$=1 for all t, while here $\hat{\varphi}_t$ in the definition of productivity changes over time. The stochastic process by which it changes is predictable and presents no difficulty for solution with perturbation methods.

Stage 2: At the end of date t, after firm specific productivities $\Psi_t = (\Psi_{1t}, \ldots, \Psi_{Mt})$ *are revealed.*

A new value is selected for

$$\varphi_t = [\textstyle\sum_{j=1}^{M} (\Psi_{jt})^{\theta_t - 1}]^{\frac{1}{\theta_t - 1}}$$

and a revised equilibrium is attained. In this new equilibrium the variables $(\hat{R}_t, \hat{W}_t, \hat{L}_t, K_t)$ are fixed and the problem is to determine profits, output, consumption and future capital sock holding. In that equilibrium agents must revise their forecasts of $(C_{t+1}, \hat{R}_{t+1}, \hat{W}_{t+1}, \hat{L}_{t+1},)$ which are different from earlier forecasts because they have more information. I will explain how the revision is made but will not create a new notation for these revised forecasts.

The four variables to be determined are now $(C_t, Y_t, K_{t+1}, \Pi_t)$ and the four equations for their determination are:

$$(C_t)^{-\sigma}(1-\frac{\mathcal{H}}{1+\eta}\hat{L}_t^{1+\eta}) = \beta E_t[(C_{t+1})^{-\sigma}(1-\frac{\mathcal{H}}{1+\eta}\hat{L}_{t+1}^{1+\eta})(\hat{R}_{t+1}+(1-\delta))]$$

$$Y_t = \hat{W}_t\hat{L}_t + K_t\hat{R}_t + \Pi_t$$

$$C_t + (K_{t+1} - K_t(1-\delta)) = Y_t$$

$$Y_t = \varphi_t\zeta_t K_t^{\alpha}(A_t\hat{L}_t)^{1-\alpha}$$

The final solution is $(\hat{W}_t, \hat{R}_t, \hat{L}_t, K_{t+1}, I_t, C_t, Y_t, \Pi_t)$ remove P^s here

To study these conditions, use the normalization

$$Y_t = \varphi_t\zeta_t(K_t)^{\alpha}(A_tL_t)^{(1-\alpha)} = \varphi_t\zeta_t A_t(\frac{K_t}{A_t})^{\alpha}(L_t)^{1-\alpha} = \varphi_t\zeta_t A_t(k_t)^{\alpha}(L_t)^{1-\alpha} \quad , \quad k_t = (\frac{K_t}{A_t}),$$

and define now the variables scaled by the growth rate

$$c_t = (\frac{C_t}{A_t}) \ , \ w_t = (\frac{W_t}{A_t}) \ , \ y_t = (\frac{Y_t}{A_t}) \ , \ g = (\frac{A_{t+1}}{A_t}) \ , \ y_t = \varphi_t\zeta_t k_t^{\alpha}L_t^{1-\alpha}.$$

The stage 2 four equilibrium conditions are then

$$c_t + g k_{t+1} = \varphi_t\zeta_t k_t^{\alpha}\hat{L}_t^{1-\alpha} + k_t(1-\delta)$$

$$(1-\frac{\mathcal{H}}{1+\eta}\hat{L}_t^{1+\eta}) = \beta E_t[(g\frac{c_{t+1}}{c_t})^{-\sigma}(1-\frac{\mathcal{H}}{1+\eta}\hat{L}_{t+1}^{1+\eta})(\hat{R}_{t+1}+(1-\delta))]$$

$$y_t = \hat{w}_t\hat{L}_t + k_t\hat{R}_t + \pi_t$$

$$y_t = \varphi_t\zeta_t k_t^{\alpha}(\hat{L}_t)^{1-\alpha}$$

Note that solution of stage 2 equilibrium requires forecasting $(c_{t+1}, \hat{R}_{t+1}, \hat{L}_{t+1})$ given all information in stage 2. This appears in conflict or in addition to the forecasts of stage 1, which are made by using the state variable $\hat{\varphi}_{t+1}$. But this is only a conflict of notation.

To clarify this issue, we review the way forecasting is done in stage 1. Forecasting in a dynamic equilibrium model is made by the use of the equilibrium map itself and this is the way forecasts of $(\hat{c}_{t+1}, \hat{L}_{t+1})$ are made. They are approximating functions of the forecasted four state variables $(\hat{k}_{t+1}, E_t[\hat{\varphi}_{t+1}], \zeta_{t+1}, \mathcal{P}_{t+1})$, but $\hat{R}_{t+1}$ is forecasted with the explicit equilibrium condition

$$\hat{R}_{t+1} = \frac{E_t[\hat{\varphi}_{t+1}]}{\mathcal{P}_{t+1}}\alpha\zeta_{t+1}k_{t+1}^{\alpha-1}\hat{L}_{t+1}^{1-\alpha}.$$

To understand the forecasts made in the second stage, it is important to keep in mind that since

$$\hat{\varphi}_t = [\textstyle\sum_{j=1}^{M} (\Psi_{j,t-1}^{\lambda_\Psi})^{\theta_t - 1}]^{\frac{1}{\theta_t - 1}}$$

there is a difference between the forecasts made in stage 1 and the one in stage 2. In stage 1

(A2.2a) $$E_t[\hat{\varphi}_{t+1}] = E_t[(\textstyle\sum_{j=1}^{M} (\Psi_{jt}^{\lambda_\Psi})^{\theta_{t+1}-1})^{\frac{1}{\theta_{t+1}-1}} | \Psi_{j,t-1}, k_t, \zeta_t, \mathcal{P}_t]$$

but in stage 2 the forecast is different and without creating new notation we observe that it is now

(A2.2b) $$E_t[\hat{\varphi}*_{t+1}] = E_t[(\sum_{j=1}^{M} (\Psi_{jt}^{\lambda_\Psi})^{\theta_{t+1}-1})^{\frac{1}{\theta_{t+1}-1}} | k_{t+1}, \zeta_t, \mathcal{P}_t].$$

Consequently, in stage 2 we repeat the procedure used in stage 1 but this time we use the forecasts $(k_{t+1}, E_t[\hat{\varphi}*_{t+1}], \zeta_{t+1}, \mathcal{P}_{t+1})$. Given these forecasts, the procedure now is as follows:

(i) forecast of $\hat{c}_{t+1}$ is updated to be a forecast of c_{t+1} simply by using the equilibrium map

(ii) forecast of $\hat{L}_{t+1}$ is updated by using the equilibrium map of stage 1 and the new forecasts

(iii) forecast of $\hat{R}_{t+1}$ is updated, as in stage 1, by using the exact definition.