STAT 375 Homework 4 Solutions

Problem (1)

The naive mean field energy is given by

$$\mathbb{F}_{MF}(b) = \mathbb{E}_b \log \psi_{\text{tot}} + H(b)$$

where the belief b factorizes into a product of beliefs over the individual vertices. For the homogenous Ising model over the torus, we get:

$$\mathbb{F}_{MF}(b) = \sum_{i \in V_l} \mathbb{E}_{b_i}(\theta_v x_i) + \sum_{(i,j) \in E_l} \mathbb{E}_{b_i b_j}(\theta_e x_i x_j) + \sum_{i \in V_l} H(b_i)$$

With the restriction that the belief is independent of the vertex the above expression simplifies to:

$$\mathbb{F}_{MF}(b_v) = \theta_v l^2 \mathbb{E}_{b_v}(x) + 4\theta_e l^2 \left(\mathbb{E}_{b_v}(x)\right)^2 + l^2 h(b_v)$$

where $h(b_v) := -b_v(+1)\log b_v(+1) - b_v(-1)\log b_v(-1)$ is the binary entropy function. The mean field free energy obtained is plotted below:

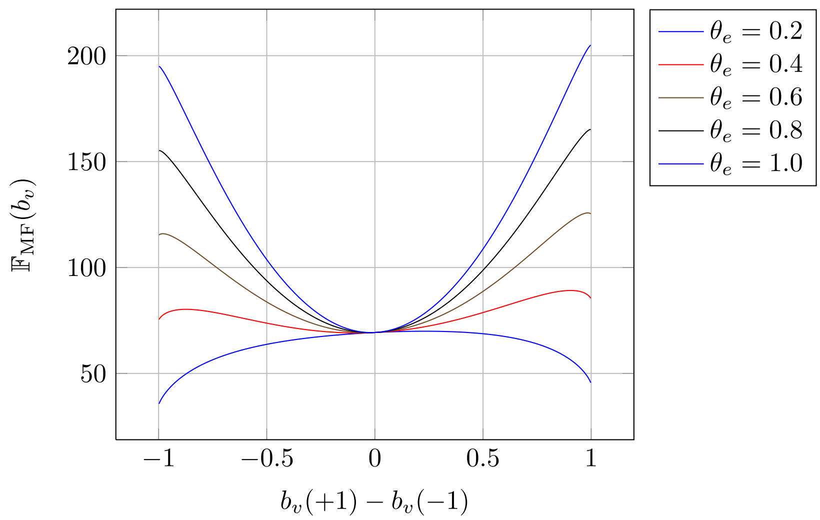

Figure 1: The naive mean field energy as a function of $b_v(+1) - b_v(-1)$

Problem (2)

The explicit expression for the Bethe free energy is given by:

$$\mathbb{F}(b) = \sum_{(i,j) \in E_l} \mathbb{E}_{b_{ij}}(\theta_e x_i x_j) + \sum_{i \in V_l} \mathbb{E}_{b_i}(\theta_v x_i) + \sum_{i \in V_l} (1 - deg(i)) H(b_i) + \sum_{(i,j) \in E_l} H(b_{ij})$$

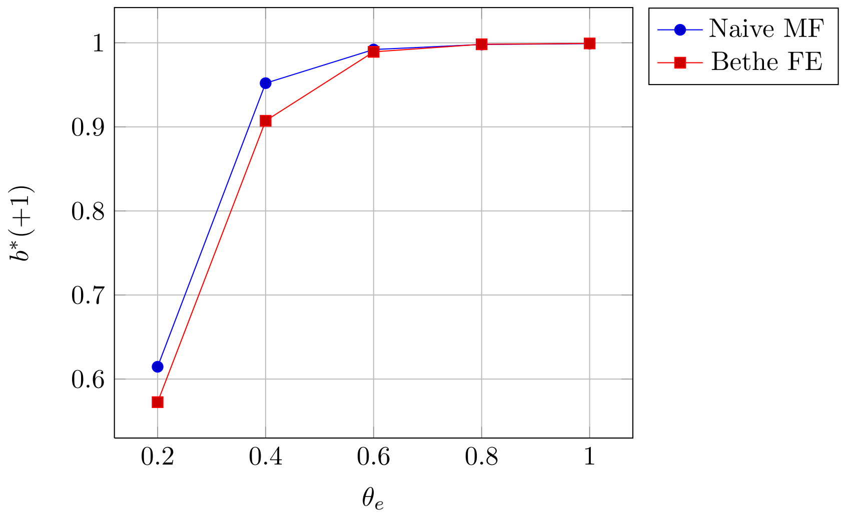

Figure 2: $b_v^*(+1)$ versus θ_e for the naive mean field and Bethe free energy approximations

If the belief is independent of the vertex, the above expression simplifies to:

$$\mathbb{F}(b) = 2l^2\theta_e \mathbb{E}_{b_e}(x_1x_2) + l^2\theta_v \mathbb{E}_{b_v}(x_1) - 3l^2H(b_v) + 2l^2H(b_e)$$

where $H(\cdot)$ denotes the entropy of the argument distribution. Reparametrizing the problem with $b_v = \begin{bmatrix} p_1 \\ 1-p_1 \end{bmatrix}$ and $b_e = \begin{bmatrix} p_2 & p_1 - p_2 \\ p_1 - p_2 & 1 - 2p_1 + p_2 \end{bmatrix}$ yields the following expression for the free energy:

$$\begin{aligned} \mathbb{F}(b) = {} & 2l^2\theta_e(1 - 4p_1 + 4p_2) + l^2\theta_v(2p_1 - 1) + -3l^2(g(p_1) + g(1-p_1)) \\ & + 2l^2\left(g(p_2) + 2g(p_1 - p_2) + g(1 - 2p_1 + p_2)\right) \end{aligned}$$

where $g(x) := -x\log(x)$. The above function can be maximized to obtain the optimal free energy.

For proving the existence of multiple stationary points, we use the log-likelihood representation of the BP update equations. Let $l_e = \frac{1}{2}\log\left(\frac{\nu(+1)}{\nu(-1)}\right)$ denote the message (in the log-likelihood domain) along an edge. By symmetry, the messages are the same for all edges. Let l_v denote marginal b_v in log-likelihood domain. Then, at a stationary point we have:

$$\begin{aligned} l_e &= \theta_v + 3\,\mathrm{arctanh}(\tanh(\theta_e)\tanh(l_e)) \\ l_v &= \theta_v + 4\,\mathrm{arctanh}(\tanh(\theta_e)\tanh(l_e)) \end{aligned}$$

In particular, we can write the first equation as:

$$\tanh\left(\frac{l_e - \theta_v}{3}\right) = \tanh(\theta_e)\tanh(l)$$

We plot the left and right hand sides of the above equation with $\theta_e = 1$ in the plot in figure 4, demonstrating the existence of two stationary points.

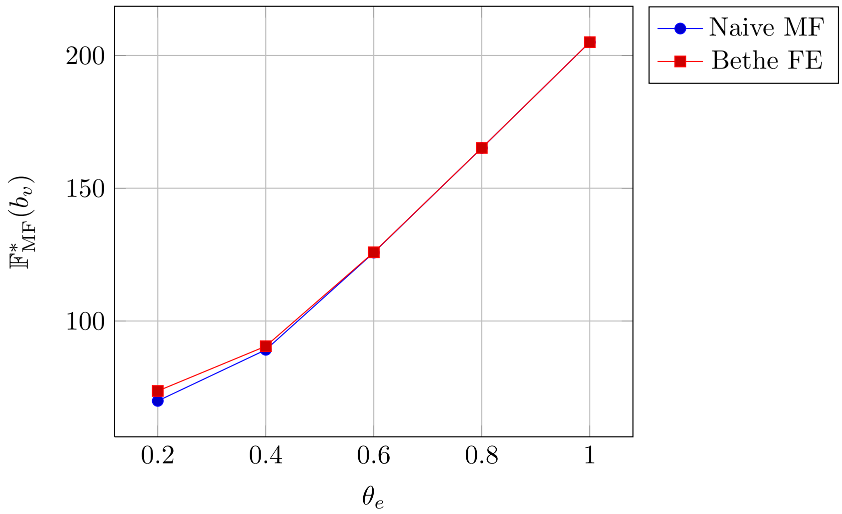

Figure 3: Optimal free energy versus θ_e for the naive mean field and Bethe free energy approximations

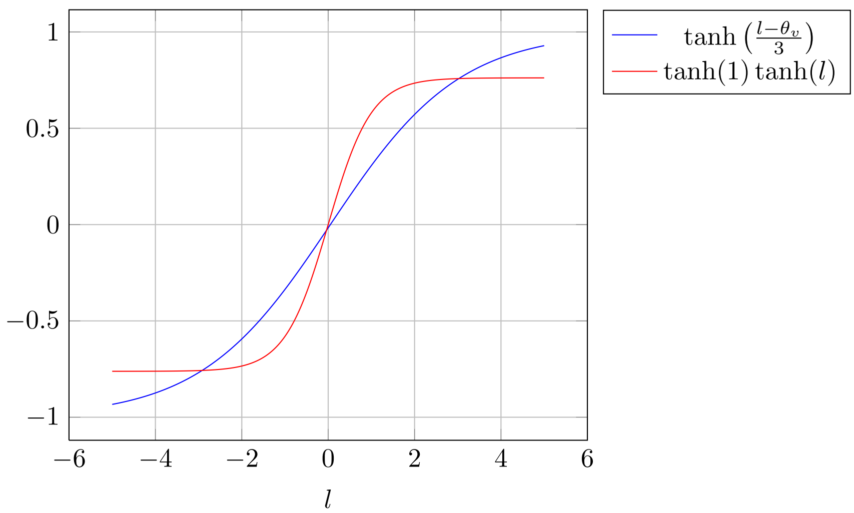

Figure 4: The points of intersection represent fixed points for the BP iteration