Stat 375 Inference in Graphical Models

Homework 4

Due - 5/7/2012

Please return this homework in class or in the bin in Packard second floor, or by e-mail to Yiqun Liu (liuyiqun1124@gmail.com).

This week we shall consider $G_\ell = (V_\ell, E_\ell)$, an $\ell \times \ell$ two-dimensional torus. This has vertex set $V_\ell = [\ell] \times [\ell]$ and, for any two vertices $i, j \in V_\ell$, $i = (i_1, i_2)$, $j = (j_1, j_2)$, $i_1, i_2, j_1, j_2 \in [\ell]$, we let $(i,j) \in E_\ell$ if and only if either $i_1 = j_1$ and $(i_2 - j_2) \in \{+1, -1\}$ *modulo* ℓ, or $i_2 = j_2$ and $(i_1 - j_1) \in \{+1, -1\}$ *modulo* ℓ.

We consider the *homogeneous* Ising model over $x \in \{+1, -1\}^{V_\ell}$

$$\mu(x) = \frac{1}{Z_G} \exp\Big\{\theta_{\rm e} \sum_{(i,j)\in E_\ell} x_i x_j + \theta_{\rm v} \sum_{i \in V_\ell} x_i\Big\}, \tag{1}$$

where $\theta_{\rm e}, \theta_{\rm v}$ are parameters.
[It is rare to encounter such a symmetric model in applications. On the other hand, such toy examples are very useful for developing intuition.]

In the following, fix $\ell = 10$, $\theta_{\rm v} = 0.05$.

(1) Consider the *naive mean field approximation*, and write the naive mean field free energy $\mathbb{F}_{\rm MF}(b)$.

Assume then the further restriction $b_i(x_i) = b_{\rm v}(x_i)$ for all $i \in V_\ell$ (i.e. the belief is independent of the vertex). Write an expression $\mathbb{F}_{\rm MF}(b_{\rm v})$ as a function of $b_{\rm v}$. Plot the free energy $\mathbb{F}_{\rm MF}(b_{\rm v})$ as a function of $[b_{\rm v}(+1) - b_{\rm v}(-1)]$ for $\theta_{\rm e} \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$.

Maximize $\mathbb{F}_{\rm MF}(b_{\rm v})$ with respect to $b_{\rm v}$ and plot the optimal value $b^*_{\rm v}(+1)$ and $\mathbb{F}_{\rm MF}(b^*_{\rm v})$ as a function of $\theta_{\rm e}$.

(2) Repeat the same exercise for the *Bethe free energy*: Write explicitly the Bethe free energy $\mathbb{F}(b)$.

Assume the further restriction $b_i(x_i) = b_{\rm v}(x_i)$ for all $i \in V_\ell$, $b_{ij}(x_i, x_j) = b_{\rm e}$ (i.e. the belief is independent of the vertex). Write an expression $\mathbb{F}(b_{\rm v}, b_{\rm e})$ as a function of $b_{\rm v}, b_{\rm e}$. Consider $\theta_{\rm e} = 1.0$, and show that $\mathbb{F}(b_{\rm v}, b_{\rm e})$ has more than one stationary point.

Try to[1] maximize $\mathbb{F}(b_{\rm v}, b_{\rm e})$ with respect to $b_{\rm v}, b_{\rm e}$ and plot the optimal value $b^*_{\rm v}(+1)$ and the free energy $\mathbb{F}(b^*_{\rm v}, b^*_{\rm e})$ as a function of $\theta_{\rm e}$.

(3), Optional. Construct a region-based approximation to the free energy that improves over Bethe's free energy. Assume that the belief are 'symmetric' over the graph, and optimize the free energy. Plot the resulting approximation for $\mu(x_i = +1)$ and compare it with the one obtained at points **(1)**, **(2)** above.