STATS 375: Homework 2 Solutions

Problem (1)

As before, we assume that an empty set is an independent set, by definition. An independent set not containing the root $\varnothing$ is formed by choosing an independent set from each subtree rooted at one of the children of $\varnothing$. Also, an independent set containing the root $\varnothing$ cannot have any of the children of $\varnothing$ and thus is formed of $\varnothing$ in addition to independent sets not containing the root in the subtrees of the children of $\varnothing$. This yields the following recursion equations:

$$\begin{aligned}
Z_{l+1}(0) &= (Z_l(0) + Z_l(1))^k \\
Z_{l+1}(1) &= Z_l(0)^k \\
\text{with } Z_0(0) &= Z_0(1) = 1
\end{aligned}$$

Problem (2)

We have the following immediately:

$$\begin{aligned}
p_{l+1} &= \frac{Z_{l+1}(1)}{Z_{l+1}(0) + Z_{l+1}(1)} \\
&= \frac{Z_l(0)^k}{(Z_l(0) + Z_l(1))^k + Z_l(0)^k} \\
&= \frac{1}{1 + (1 - p_l)^{-k}}
\end{aligned}$$

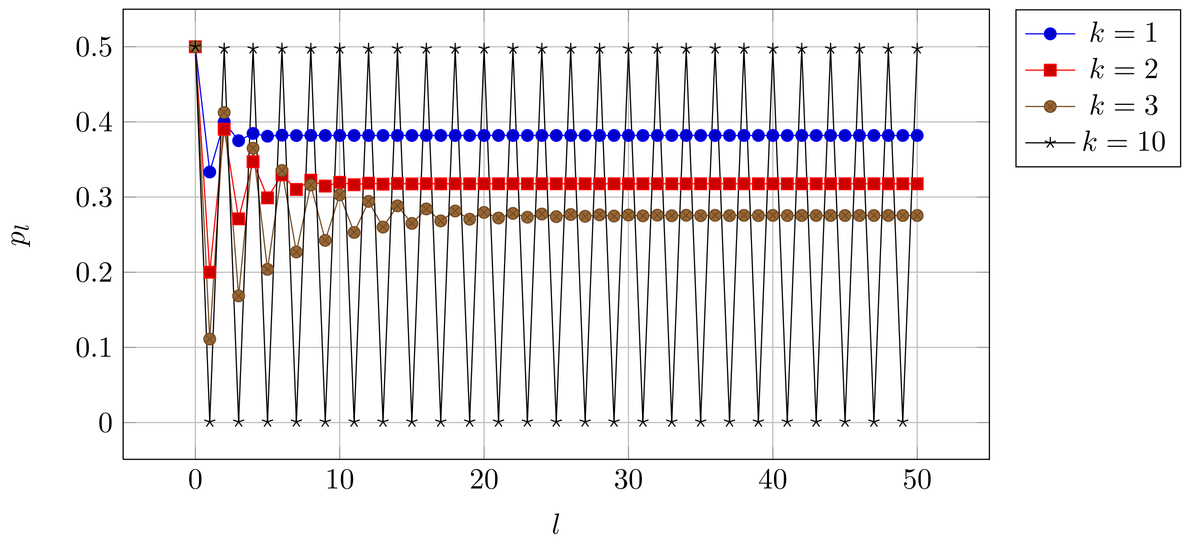
Problem(3)

The following code plots p_l for the relevant values of k and l:

```
k_vals = [1 2 3 10];
iters = length(k_vals);
l_vals = 1:50;

p = zeros(iters, 1+length(l_vals));
p(:, 1) = 0.5*ones(iters, 1); %initialization

spec = {'b' 'g' 'r' 'k'} ;

figure(1)
```

```
hold on
for i = 1:iters
    k = k_vals(i);
    for l = 1:length(l_vals)
        p(i, l+1) = (1-p(i, l))^k/(1+(1-p(i, l))^k);
    end
    plot([0 l_vals], p(i,:), spec{i});
end
hold off
legend('1', '2', '3', '10');
```

The plot is as follows:



The recursion converges to a fixed point for $k = 1, 2, 3$ but fails to (or appears to fail to) converge for $k = 10$.

Proof of convergence for $k = 1, 2, 3$

Let $f_k : [0, 1] \to [0, 1]$ be the mapping (as in the recursion) parametrized by k:

$$f_k(x) = \frac{(1-x)^k}{1+(1-x)^k}$$

We use Banach's fixed point theorem to prove convergence.

Theorem 1 (Banach)**.** *Let X be a complete metric space and $f : X \to X$ be a contraction mapping. Then f has a unique fixed point x^*. Also the sequence $\{x_i\}_{i=1}^{\infty}$ generated by $x_i = f(x_{i-1})$ converges to x^*.*

Definition 1. *Let X be a metric space and $d(\cdot, \cdot)$ be the associated metric. $f : X \to X$ is a contraction mapping with parameter β on X if $\exists\, 0 \le \beta < 1$ such that:*

$$\forall x_1, x_2 \in X : \quad d(f(x_1), f(x_2)) \le \beta d(x_1, x_2)$$

In our case, the space is the interval $[0, 1]$ with the associated metric being the absolute value of the difference: $d(x_1, x_2) = |x_1 - x_2|$.

We know that a fixed point exists as the mappings f_k are continuous, decreasing and map onto $[0, 0.5]$ for all k. However, this does not guarantee that the recursion converges. To use the fixed point theorem, we prove that for $k = 1, 2, 3$, f_k are contraction mappings. For this we use the following lemma:

Lemma 1. *Let $f : [a, b] \to [a, b]$ be a differentiable function such that $|f'(x)|$ is bounded uniformly by $\beta < 1$ in its domain. Then f is a contraction mapping with parameter β, the distance metric being the absolute value of the difference.*

Proof. Consider $a \le x_1 < x_2 \le b$. By the intermediate value theorem, $\exists c \in [x_1, x_2]$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Thus $|f(x_2) - f(x_1)| = |f'(c)||(x_2 - x_1)| \le \beta|(x_2 - x_1)|$. □

A little calculus shows that the maximum value of $|f'_k(x)|$ occurs at $x = 1 - \left(\frac{k-1}{k+1}\right)^{\frac{1}{k}}$, whereby we get:

$$|f'_k(x)| \le \frac{(k+1)^2}{4k} \left(\frac{k-1}{k+1}\right)^{\frac{k-1}{k}}$$

For $k = 2, 3$ this yields that f_k is indeed a contraction map by Lemma 1. Thus, by the fixed point theorem, the recursion converges to its unique fixed point. For $k = 1$, we cannot use this directly as the maximum is at $x = 0$ and $f'_1(0) = -1$. However this can be remedied by restricting the domain of f_1 to $[\epsilon, 1]$ for some small $\epsilon > 0$ whereupon it becomes a contraction map on the restricted domain since $|f'_1| \le \frac{1}{(1+\epsilon)^2}$.

Non-convergence for $k > 4$

One argument for the non-convergence of the recursion for larger k is the following condition: there must exist a neighborhood around the fixed point x^* in which $|f'_k(x)| < 1$ holds. This is because the linearization of f_k around its fixed point must be a stable linear system, i.e. have eigenvalues within the unit circle. For values of $k > 4$, in particular for the value $k = 10$, this condition fails to hold.