Stat 375 Inference in Graphical Models

Homework 2

Due - 4/16/2012

Please return this homework in class or to Packard 272.

As in the first homework, we will consider the uniform measure over the *independent sets* of a graph $G = (V, E)$. Explicitly, this is the the measure over $x \in \{0, 1\}^V$ defined by

$$\mu_G(x) = \frac{1}{Z(G)} \prod_{(i,j)\in E} \mathbb{I}\big((x_i, x_j) \neq (1, 1)\big) . \tag{1}$$

Specifically, we consider $G = T_{k,\ell}$ to be the rooted regular tree with ℓ generations and branching factor k. Hence the root has k descendants and each other node has one ancestor and k descendants (with the exception of the leaves). The number of vertices is $(k^{\ell+1} - 1)/(k - 1)$, and $T_{k,\ell=0}$ is the graph consisting only of the root.

Denote by ø the root of $T_{k,\ell}$.

(1) Denote by $Z_\ell = Z(T_{k,\ell})$ the number of independent sets of $G = T_{k,\ell}$. Let $Z_\ell(0)$ be the number of independent sets $T_{k,\ell}$ such that $x_ø = 0$, and by $Z_\ell(1)$ the number of independent sets such that $x_ø = 1$.

Of course $Z_0(0) = Z_0(1) = 1$. Derive a recursion expressing $(Z_{\ell+1}(0), Z_{\ell+1}(1))$ as a function of $(Z_\ell(0), Z_\ell(1))$.

(2) Using the above, derive a recursion for the probability that the root belongs to a uniformly random independent set. Explicitly, derive a recursion for

$$p_\ell = \mu_{T_{k,\ell}}\big(\{x_ø = 1\}\big) . \tag{2}$$

(3) Program this recursion and plot p_ℓ as a function of $\ell \in \{0, \dots, 50\}$ for four values of k, e.g. $k \in \{1, 2, 3, 10\}$. Comment on the qualitatative behavior of these plots. Can you prove any of your observations? [Answering to the last question is optional.]