STAT375: Homework 1 Solutions

Problem (1)

We define the following functions for all $(i, j) \in E$:

$$\psi_{ij}(x_i, x_j) = \begin{cases} 0 & \text{if } x_i = x_j = 1 \\ 1 & \text{otherwise} \end{cases}$$

We then have:

$$\mu_G(x) = \frac{1}{Z(G)} \prod_{(i,j)\in E} \psi_{ij}(x_i, x_j)$$

since the product of the ψ_{ij}'s yields the indicator function $\mathbb{I}(S \in \mathrm{IS}(G))$ for the subset S encoded by x. Thus $\mu_G(x)$ is a pairwise graphical model.

Problem (2)

We assume throughout that the empty set is, by definition, an independent set. This is merely for convenience of representation. Now $Z(L_n)$ is the number of independent sets in the graph L_n. Let $Z(L_n) = A_n + B_n$ where A_n (B_n) denotes the number of independent sets in L_n containing (excluding) the vertex n. We can then write the following recurrences for A_n and B_n:

$$\begin{aligned} A_n &= B_{n-1} \\ B_n &= A_{n-1} + B_{n-1} \end{aligned}$$

The first recurrence follows from the fact that if $S \subseteq [n]$ containing n is an independent set of L_n, then $S \backslash \{n\}$ is an independent set of L_{n-1}. The second, similarly, is because an independent set of L_n not containing vertex n is basically an independent set of L_{n-1}.

Defining $X_n = [A_n \; B_n]^T$, we can write the recurrence relation as:

$$\begin{aligned} X_n &= PX_{n-1} \\ \text{where } P &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\ \text{and } X_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

This yields:

$$X_n = P^{n-1} X_1$$

As $Z(L_n) = [1\ 1]^T X_n$, diagonalizing P yields the following closed form solution:

$$Z(L_n) = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^{n-1} + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^{n-1}$$
$$\text{where } c_1 = 1 + \frac{2}{\sqrt{5}},\ c_2 = 1 - \frac{2}{\sqrt{5}}$$

Another solution is to write a second order recurrence relation for $Z(L_n)$ (using similar arguments as above):

$$Z(L_n) = Z(L_{n-1}) + Z(L_{n-2})$$
$$Z(L_0) = 1,\ Z(L_1) = 2$$

Problem (3)

For $i \in \{1, n\}$, i.e. i being an end vertex, the number of independent sets containing i is simply $Z(L_{n-2})$. If i is an intermediate vertex, then an independent set containing i is formed by choosing an independent set from $[i-2]$ and an independent set from $[n] \backslash [i+1]$. Thus we obtain the marginal as:

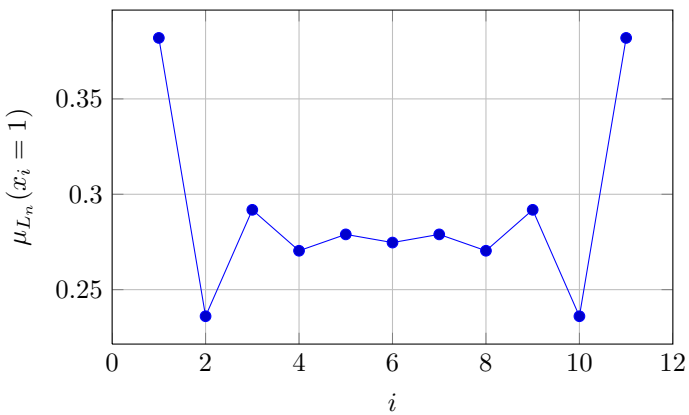
$$\mu_{L_n}(x_i = 1) = \begin{cases} \frac{Z(L_{n-2})}{Z(L_n)} & \text{if } i \in \{1, n\} \\ \frac{Z(L_{i-2}) Z(L_{n-i-1})}{Z(L_n)} & \text{otherwise} \end{cases}$$

The following MATLAB code produces the required values and plots:

```
n = 11;
n_range = 0:n;
c1 = 1+2/sqrt(5);
c2 = 1-2/sqrt(5);
r1 = (1+sqrt(5))/2;
r2 = (1-sqrt(5))/2;

%z(1) ... z(12) contains Z_0 to Z_11
z = c1*r1.^(n_range-1) + c2*r2.^(n_range-1);

%compute marginals mu
mu = zeros(1, n);
mu(1) = z(end-2)/z(end);
mu(n) = mu(1);

for i = 2:(n-1)
```

```
    mu(i) = z(i-1)*z(n-i)/z(n+1);
end

plot(1:n, mu)
```

The plot is as follows:



The exponent in the numerator is constant for $i = 2, \ldots n-1$, hence we see a relatively flat marginal curve in this region. The marginal increases at either end, since the end vertices impose fewer restrictions on the inclusion of other vertices in the independent set.

Problem (4)

By the law of conditional probability, we have:

$$\mu_{L_n}(x) = \mu_{L_n}(x_1)\mu_{L_n}(x_2|x_1)\mu_{L_n}(x_3|x_2x_1)\cdots\mu_{L_n}(x_n|x_1\cdots x_{n-1})$$

Since the inclusion of vertex i is dependent only on its neighbors, we have $\mu_{L_n}(x_i|x_1\cdots x_{i-1}) = \mu_{L_n}(x_i|x_{i-1})$. This is equivalent to creating a Bayesian network by directing all the edges in L_n towards the larger index, i.e. letting the parent $\pi(k)$ of a vertex k be $k-1$, $k = 2, \ldots n$. Using similar arguments as before, we have that:

$$\mu_{L_n}(x_i = 1|x_{i-1}) = \begin{cases} 0 & \text{if } x_{i-1} = 1 \\ \frac{Z(L_{n-i-1})}{Z(L_{n-i+1})} & \text{otherwise} \end{cases}$$