Stat 375 Inference in Graphical Models

Homework 1

Due - 04/09/2012

Please return this homework in class or to Packard 272.

Given a graph $G = (V, E)$, an *independent set* of G is a subset $S \subseteq V$ of the vertices such that if $i, j \in S$ then $(i, j) \notin E$. We let $\mathrm{IS}(G)$ denote the set of independent sets of G, and $Z(G) = |\mathrm{IS}(G)|$ denote its size. We consider the uniform measure

$$\mu_{\mathrm{IS},G}(S) = \frac{1}{Z(G)} \mathbb{I}\big(S \in \mathrm{IS}(G)\big) . \tag{1}$$

(1) The set $S \subseteq V$ can be encoded by a binary vector $x \in \{0, 1\}^V$ letting $x_i = 1$ if and only if $i \in S$. Denote by $\mu_G(x)$ the probability distribution induced on this vector when $S \sim \mu_{\mathrm{IS},G}$. Show that $\mu_G(x)$ is a pairwise graphical model on G.

(2) Let L_n be the line graph with n vertices, i.e. the graph with vertex set $V(L_n) = \{1, 2, 3, \dots, n\}$ and edge set $E(L_n) = \{(1, 2), (2, 3), \dots, (n-1, n)\}$. Derive a formula for $Z(L_n)$.
[Hint: Write a recursion over n, and solve it by matrix representation.]

(3) With the above definitions, derive a formula for $\mu_{L_n}(x_i = 1)$, $i \in \{1, \dots, n\}$. Plot $\mu_{L_n}(x_i = 1)$ versus i for $n = 11$. Describe the main features of this plot. Can you explain them intuitively?
[Hint: Use the same recursion as in point (2).]

(4) The same measure $\mu_{L_n}(x)$ can be described as a Bayesian network. Using the results in point (3), write the conditional probability distributions for such a network.