Stat 350 - Probabilistic Concepts in Statistical Physics and Information Theory

This course is oriented towards research in applied probability with active participation from *all students attending it*. Students are to get involved in a (small) research project by:

1. Forming a small team (of 2-3 students).
2. Reviewing some recent literature on a research problem chosen in a list proposed by us, and presenting an overview of this material in class. The lectures of February 4, 6 and 11 are assigned to this presentations that are of interest to the whole class and need to be pedagogical.
3. Thinking (!) independently to the problem. By 'think' we mean try a few approaches to solving, or make progress on the problem. Some of these will be suggested by us, and some (hopefully) will come from you.

   Teams conclusions will be presented at the end of the course. You are encouraged to report both negative ('we tried this and did not work') and positive ones ('we solved the problem').

Here is a list of three possible topics. More details are available from us:

Non-ultrametric overlap structures. The *Sherrington-Kirkpatrick model* is the Boltzmann distribution

$$\mu(x) = \frac{1}{Z_N(\beta,h)} \exp\Big\{\frac{\beta}{\sqrt{N}}\sum_{(i,j)} J_{ij}\sigma_i\sigma_j + h\sum_i \sigma_i\Big\}\,. \tag{1}$$

where the J_{ij} are iid normal random variables and $Z_N(\beta,h)$ is a normalization constant. Of interest is the free entropy density

$$\phi(\beta,h) \equiv \lim_{N\to\infty}\frac{1}{N}\log Z_N(\beta,h)\,. \tag{2}$$

In [1], Aizenman and coworkers proved a variational principle for this quantity. More precisely the free entropy density is the infimum of a given functional over the space of 'random overlap structures' (ROSt). If the ROSt is chosen within the so-called 'ultrametric' family, this variational principle reduces to the celebrated Parisi formula.

It is an open problem to find a non-ultrametric ROSt on which the functional can be effectively evaluated, and understand its significance.

[1] M. Aizenman, R. Sims, and S. Starr, "Extended variational principle for the Sherrington-Kirkpatrick spin-glass model," Phys. Rev. B 68, 214403 (2003)
[2] M. Aizenman, R. Sims, and S. Starr, "Mean-Field Spin Glass models from the Cavity–ROSt Perspective," `arXiv:math-ph/0607060v1`

Spin glasses with correlated couplings. Consider again the spin glass model (1). Mathematically rigorous studies have been confined until now to the case of iid couplings J_{ij}. On the other hand, physicists have considered various cases of correlated couplings. We propose to apply mathematical techniques to examples of this type. The objectives would be:

1. Compute the free entropy density. In particular, investigate the possibility of proving interpolation formulae a la Guerra.

2. Prove some form of mean field (TAP) equations.

Specific models (in order of increasing difficulty) could be: $\{J_{ij}\}$ form a gaussian process; $J = \{J_{ij}\}$ is a uniformly (Haar) distributed random orthogonal matrix; $J = ODO^T$ for D diagonal given and O a random orthogonal matrix.

[3] M. Talagrand, " Spin Glasses, a Challenge to Mathematicians," Springer, 2003
[4] R. Cherrier, D. S. Dean, A. Lefèvre, "The role of the interaction matrix in mean-field spin glasses" Phys. Rev. E 67, 046112 (2003) `arXiv:cond-mat/0211695`

Ruelle probability cascades and Competing particle systems. Ruelle probability cascades are a special class of point processes on the real line, and whose distribution has a special ultrametric structure. They play a crucial role in spin glass theory. It has been shown [5,6] that Ruelle cascade are characterized uniquely (apart from some technical condictions) by the property of being quasi-stationary with respect to a class of perturbations. Such perturbations consist of shifting each particle by a function of a gaussian random variable. The covariance among the shifts depends on the particles 'identities'.

We propose to investigate the case of more general perturbations, and the related point processes.

[5] A. Ruzmaikina and M. Aizenman, "Characterization of invariant measures at the leading edge for competing particle systems," Ann. Probab. 33 (2005) 82-113
[6] L.-P. Arguin and M. Aizenman, "On the Structure of Quasi-Stationary Competing Particle Systems," `arXiv:0709.2901v1`

The Hopfield model. The Hopfield model is a simple model of associative memory, introduced by John Hopfield in 1982. It stores patterns $\{\xi^1, \dots, \xi^p\}$, where $\xi^k = (\xi^k_1, \dots, \xi^k_N) \in \{+1, -1\}^N$ is a vector, in the matrix

$$J_{ij} = \frac{1}{\sqrt{p}} \sum_{\mu=1}^{p} \xi^k_i \xi^k_j \,. \tag{3}$$

The idea is that patterns can be retrieved by looking for local minima of the energy function $E(x) = -x^T Jx$, where $x \in \{+1, -1\}^N$. For instance if $p = 1$ $E(x) = -(\xi^T x)^2$ has its unique minima at $x = \pm\xi$.

In studying the properties of this energy function, it is useful to consider the Boltzmann measure (1), with the J_{ij} defined as above in terms of patterns. The case of iid patterns, whereby each entry ξ^k_i is an independent Bernoulli(1/2) random variables has been intensively studied.

We propose students to check whether the techniques that were successful for the SK model (interpolation, ROSt, ...) do succeed in the present case.

[7] J. J. Hopfield, "Neural networks and physical systems with emergent collective computational properties," Proc. Natl. Acad. Sci. USA, 79 (1982) 2554
[8] M. Talagrand, "Exponential inequalities and convergence of moments in the replica-symmetric regime of the Hopfield model," Ann. Probab. 28 (2000), 1393-1469