**Stat 316 Stochastic Processes on Graphs**

# A single full calculation

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The objective of this lecture is to spell out a single calculation from the course in all of its painful details. The choice of which calculation was made on the basis of its level of complication (moderate) rather than on it intrinsic interest.

We shall deal with the ferromagnetic Ising model on random regular graphs, which we treated a couple of weeks ago. Recall that this is a model over variables $x_i \in \{+1, -1\}$, $i \in \{1, \dots, N\}$ with distribution

$$\mu(x) = \frac{1}{Z_G} \exp\left\{\beta \sum_{(i,j)\in E} x_i x_j\right\} . \tag{1}$$

Here $E$ is the edge set of a random graph $G = (V = [N], E)$ that we shall take to be a random graph with degree $k$. More precisely $G$ is generated as follows (*configuration model*). Associate to each vertex $i \in V$ $k$ 'half edges,' sample a uniformly random pairing over $kN$ objects, and pair all the half-edges accordingly. Two distinct pairings are considered as distinct graphs.

We will be interested in the *restricted partition function* defined by constraining the usual sum to configurations of vanishing magnetization:

$$Z_G^* \equiv \sum_{x:\,\sum_i x_i = 0} \exp\left\{\beta \sum_{(i,j)\in E} x_i x_j\right\} . \tag{2}$$

A few lectures ago, we made the following claim, that we will now prove.

**Lemma 1.** *Assume $N$ to be even. Then the expectation of the restricted partition function $Z_G^*$ is, to the leading exponential order,*

$$\mathbb{E}\{Z_G^*\} \doteq 2^N (\cosh\beta)^{kN/2} . \tag{3}$$

**Proof** Throughout the proof $M = Nk/2$ will denote the number of edges in $G$, and $\mathbb{G}_{N,k}$ the graph ensemble. Finally, for a set $\mathcal{S}$, we shall denote by $|\mathcal{S}|$ its cardinality.

Let $\Delta_G(x)$ denote the number of edges $(i,j) \in E$ such that $x_i \neq x_j$, and $Z_G^*(\Delta)$ be the number of configurations $x$ such that $\Delta_G(x) = \Delta$. Then we clearly have $\sum_{(i,j)\in E} x_i x_j = M - 2\Delta_G(x)$. As a consequence

$$Z_G^* = e^{\beta M} \sum_{\Delta=0}^{M} Z_G^*(\Delta)\, e^{-2\beta\Delta} . \tag{4}$$

By linearity of expectation, and since the graph distribution is invariant under vertices permutations, we have

$$\mathbb{E}\, Z_G^*(\Delta) = \sum_{\sum_i x_i = 0} \mathbb{P}\{\Delta_G(x) = \Delta\} = \binom{N}{N/2} \mathbb{P}\{\Delta_G(x_*) = \Delta\} = \tag{5}$$

$$= \binom{N}{N/2} \frac{|\{G \in \mathbb{G}_{N,k} \text{ st } \Delta_G(x_*) = \Delta\}|}{|\mathbb{G}_{N,k}|} . \tag{6}$$

Here $x_*$ denote the configuration consisting in $N/2$ $+1$'s followed by $N/2$ $-1$'s.

The number of graph in the ensembles is just the number of pairings of $Nk$ objects

$$|\mathbb{G}_{N,k}| = \mathfrak{P}(Nk) \equiv \frac{(Nk)!}{(Nk/2)!2^{Nk/2}}\,. \tag{7}$$

On the other hand it is not too hard to compute the number of such pairings for which the number edges with unequal end-points is $\Delta$:

$$|\{G \in \mathbb{G}_{N,k} \text{ st } \Delta_G(x_*) = \Delta\}| = \binom{M}{\Delta}^2 \Delta!\, \mathfrak{P}(M-\Delta)^2 \tag{8}$$

Putting everything together we get

$$\begin{aligned}
\mathbb{E}\{Z_G^*\} &= e^{\beta M}\binom{N}{N/2}\frac{1}{\mathfrak{P}(2M)}\sum_{\Delta=0}^{M}\binom{M}{\Delta}^2 \Delta!\, \mathfrak{P}(M-\Delta)^2 e^{-2\beta\Delta} &(9)\\
&\doteq e^{\beta M}\binom{N}{N/2}\frac{1}{\mathfrak{P}(2M)}\sup_{\Delta\in\{0,\dots,M\}}\binom{M}{\Delta}^2 \Delta!\, \mathfrak{P}(M-\Delta)^2 e^{-2\beta\Delta}\,. &(10)
\end{aligned}$$

Let us recall the exponential behaviors (for $q \in [0,1]$)

$$\binom{N}{Nq} \doteq e^{NH(q)}\,, \qquad \mathfrak{P}(N) \doteq \left(\frac{N}{e}\right)^{N/2}\,, \tag{11}$$

where $H(x) \equiv -x\log x - (1-x)\log(1-x)$ is the binary entropy function.

Substituting in the expression for $\mathbb{E}\{Z_G^*\}$ we get

$$\begin{aligned}
\mathbb{E}\{Z_G^*\} &\doteq e^{\beta M}2^N\left(\frac{2M}{e}\right)^{-M}\sup_{\delta\in[0,1]}\left\{e^{2MH(\delta)}\left(\frac{M\delta}{e}\right)^{M\delta}\left(\frac{M(1-\delta)}{e}\right)^{M(1-\delta)}e^{-2M\beta\delta}\right\} &(12)\\
&= 2^N\, e^{\beta M}\, 2^{-M}\, \exp\left\{M\sup_{\delta\in[0,1]}[H(\delta)-2\beta\delta]\right\} &(13)
\end{aligned}$$

The sup is achieved when $H'(\delta) = 2\beta$, which implies $\delta = \delta_*(\beta) \equiv (1+e^{2\beta})^{-1}$. At this point we have $H(\delta_*) - 2\beta\delta_* = \log(1+e^{-2\beta})$, which yields

$$\mathbb{E}\{Z_G^*\} \doteq 2^N\, e^{\beta M}\, 2^{-M}\, \exp\left\{M\log(1+e^{-2\beta})\right\} = 2^N(\cosh\beta)^M\,. \tag{14}$$

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