Stat 316 Stochastic Processes on Graphs

Ferromagnetic Ising models

Amir Dembo *Lecture 3-4 - 10/5/2007*

1 Introduction

By ferromagnetic Ising models we mean the study of the large-N behavior of the measures

$$\mu_{G,\beta,h}(x) = \frac{1}{Z_G(\beta,h)} \exp\left\{\widehat{H}(x)\right\} = \frac{1}{Z_G(\beta,h)} \exp\left\{\beta \sum_{(i,j)\in E} x_i x_j + h \sum_{i\in V} x_i\right\}, \tag{1}$$

of the spin variables $x_i \in \mathcal{X} = \{-1,1\}$, $i \in V$, given a (possibly random) finite graph $G = (V,E)$ of $|V| = N$ vertices, and its dependence on the magnetic field $h \geq 0$ and inverse temperature $\beta > 0$ parameters.

The Curie-Weiss model is perhaps the simplest special case of (1), a mean field model corresponding to the complete graph $G = K_N$ (where there is an edge between each pair of vertices $i \neq j \in V = [N]$, and consequently, the inverse temperature is scaled down by the connectivity factor N). In this context we already derived the simple mean field equation $m = \tanh(h + \beta m)$ for the average magnetization $m \approx \mathbb{E}\, X_i$ under the measure μ at a typical vertex of G, whose analysis reveals the existence of a phase transition when $h = 0$ and $\beta > \beta_c = 1$ and the lack thereof when either $h > 0$ or $h = 0$ and $\beta < \beta_c$ (in which case the average magnetization concentrates near a non-random solution $m_*(\beta,h) \geq 0$ of the mean field equation).

Our goal here is to derive similar mean field equations for other families of graphs. Specifically, we focus on the mean field model corresponding to random k-regular graph G, where $k \geq 2$ is a fixed parameter and E is uniformly chosen at random subject to the constraints $|\{j : (i,j) \in E\}| = k$ for all $i \in V = [N]$. [1] We show that the mean field equation in this case is

$$\overline{m} = f_{\beta,h,k-1}(\overline{m}) := \tanh(h + (k-1)\tanh^{-1}(\tanh(\beta)\overline{m}))\,, \tag{2}$$

with a similar picture as for the Curie-Weiss model, albeit now with $\beta_c = \tanh^{-1}(1/(k-1))$, and where the average magnetization at a uniformly chosen vertex of G is given for $N \to \infty$ by

$$m = f_{\beta,h,k}(\overline{m}) = \tanh(h + k\tanh^{-1}(\tanh(\beta)\overline{m}))\,, \tag{3}$$

which simplifies to $m = \overline{m}(1+\tanh(\beta))/(1+\overline{m}^2\tanh(\beta))$ upon combining (2) with the identity $\tanh^{-1}(r) = \frac{1}{2}\log((1+r)/(1-r))$.

Unlike the Curie-Weiss model, here we can not directly and explicitly compute the relevant probabilities under the measure μ, so instead we resort to the more robust approach of *local weak convergence* (as detailed in Section 3), whereby using *Griffiths inequalities* for ferromagnetic Ising measures, we locally approximate the given model by Ising models on a suitable (possibly random) tree and boundary conditions. To this end, we first turn to the general analysis of ferromagnetic Ising measures on finite trees, from which we draw the mean field equations (2) and (3). In doing so, we pay attention in particular to the effect of boundary conditions on the magnetization at a vertex well inside the tree. Building on our analysis of the mean field equations, we shall also address the issue of phase transition in a future lecture.

2 Ferromagnetic Ising measures on trees

Let $\mu_T(x;b)$ denote the ferromagnetic Ising measure for $G = T$ a finite tree, as given by (1), subject to fixed boundary conditions $x_u = b_u$ for $u \in T_*$ a possibly empty subset of vertices of T (the parameters $h \geq 0$ and

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[1] To simplify our presentation we shall slightly deviate from this ensemble of graphs, by allowing self edges and multiple edges in G.

$\beta > 0$ are fixed so $\mu_T(x; b)$ stands hereafter for $\mu_{G,\beta,h}(x|x_u = b_u \text{ for } u \in T_*))$. Utilizing the tree structure we get an efficient recursive evaluation of $m_v = m_v(b) = \mu_T(x_v = 1; b) - \mu_T(x_v = -1; b)$ for all $v \in T$.

Lemma 1. *Let $\{u_1, \ldots, u_K\}$ be the vertices of a sub-tree $T(w)$ of T that are adjacent to w in $T(w)$ (so $K = K(w)$ is the degree of w in $T(w)$), and for $i = 1, \ldots, K$ let $T(u_i)$ denote the sub-tree of $T(w)$ containing u_i after the removal of the edge (w, u_i). With $\overline{m}_u = \mu_{T(u)}(x_u = 1; b) - \mu_{T(u)}(x_u = -1; b)$ and in particular, $\overline{m}_u = b_u$ for $u \in T_*$, we have the recursion*

$$z_w = h + \sum_{i=1}^{K(w)} F_\beta(z_{u_i}), \qquad F_\beta(z) = \tanh^{-1}(\tanh(\beta)\tanh(z)), \tag{4}$$

for $z_w = \tanh^{-1}(\overline{m}_w)$, which when initiated with $T(v) = T$ leads to $m_v = \tanh(z_v)$ after at most N applications.

Proof Let $\widehat{H}_w(x) = \beta \sum_{(i,j)\in T(w)} x_j x_j + h \sum_{i\in T(w)} x_i$ denote the contribution to $\widehat{H}(x)$ from the vertices and edges within $T(w)$. Since $T(w)$ is a tree, for $K = K(w)$, the vertices of $T(w)$, apart of w, are precisely the union of the disjoint sets of vertices in $T(u_i)$ for $i = 1, \ldots, K$, and the edges of $T(w)$ are merely (w, u_i) and the disjoint collections of edges within $T(u_i)$, for $i = 1, \ldots, K$. Consequently, with x^i denoting the projection of x on the vertices of $T(u_i)$, we have that

$$\widehat{H}_w(x) = \sum_{i=1}^{K} \widehat{H}_{u_i}(x^i) + h x_w + \beta x_w \sum_{i=1}^{K} x_{u_i}.$$

This implies that $S_w(\xi) = \mu_{T(w)}(x_w = \xi; b)$ for $\xi \in \{-1, 1\}$ is such that

$$S_w(\xi) = \frac{1}{Z_w} \sum_{x: x_w = \xi} \exp(\widehat{H}_w(x)) = \frac{e^{h\xi}}{Z_w} \prod_{i=1}^{K} \sum_{x^i} e^{\beta\xi x_{u_i}} \exp(\widehat{H}_{u_i}(x^i)) = \widehat{Z}_w e^{h\xi} \prod_{i=1}^{K} \left[e^{\beta\xi} S_{u_i}(1) + e^{-\beta\xi} S_{u_i}(-1)\right],$$

where $\widehat{Z}_w$ is independent of ξ. Recall that $S_w(\xi) = (1 + \xi\,\overline{m}_w)/2$, so considering $S_w(1)/S_w(-1)$ we find that

$$\frac{1 + \overline{m}_w}{1 - \overline{m}_w} = \frac{e^h \prod_{i=1}^K \left[e^{\beta}(1 + \overline{m}_{u_i}) + e^{-\beta}(1 - \overline{m}_{u_i})\right]}{e^{-h} \prod_{i=1}^K \left[e^{-\beta}(1 + \overline{m}_{u_i}) + e^{\beta}(1 - \overline{m}_{u_i})\right]}.$$

With z_w being half the logarithm of the left side of this identity, upon verifying that

$$\frac{1}{2} \log\left[\frac{e^{\beta}(1+m) + e^{-\beta}(1-m)}{e^{-\beta}(1+m) + e^{\beta}(1-m)}\right] = \tanh^{-1}(\tanh(\beta) m),$$

we arrive at (4). □

The monotonicity of $b \mapsto m_v(b)$ is a direct consequence of Lemma 1, coupled with the analysis of $F_\beta(z)$ for $z \in [-\infty, \infty]$.

Corollary 2. *For any $v \in T$ a tree, $h \geq 0$, $\beta > 0$ and $T_* \subseteq T$, if the boundary conditions $\hat{b}$ and b are such that $\hat{b}_u \geq b_u$ at each $u \in T_*$, then $m_v(\hat{b}) \geq m_v(b)$. Hence, for any boundary condition b on T_**

$$m_v(-) \leq m_v(b) \leq m_v(+), \qquad z_v(-) \leq z_v(b) \leq z_v(+), \tag{5}$$

with $m_v(\pm)$ and $z_v(\pm)$ corresponding to the extreme, that is the all plus (and the all minus), boundary conditions.

Proof With m_v a monotone increasing function of z_v it suffices to prove the monotonicity of $b \mapsto z_w(b)$ at each w in our recursive evaluation of z_v. This clearly holds for $w \in T_*$, whereby $z_w = \tanh^{-1}(b_w)$. More generally, from (4) we have that $\partial z_w / \partial z_{u_i} = F'_\beta(z_{u_i})$ and the stated monotonicity follows since

$$F'_\beta(z) = \frac{\tanh(\beta)}{1 + (1 - \tanh^2(\beta)) \sinh^2(z)} \geq 0 \,, \tag{6}$$

for any $\beta \geq 0$ and all z. □

We next consider the root v of the Galton-Watson tree T associated with the first $t+1$ generations of a branching process having offspring distribution K of finite mean and one ancestor at the 0-th generation. Here T_* consists of the vertices in T of distance $t+1$ from the root. In this case, setting $\beta_c = \tanh^{-1}(1/\mathbf{E}K)$ we next show that if $\beta < \beta_c$ then as $t \to \infty$ the effect of boundary conditions on z_v, hence on m_v is negligible.

Lemma 3. *Let v be the root of a Galton-Watson tree T of depth $t+1$ and offspring distribution K. Then, for boundary conditions on the set of vertices of distance $t+1$ from v,*

$$e(t+1) = \mathbf{E}[\sup_{b,\hat{b}} |z_v(b) - z_v(\hat{b})|] \leq e(1)(\tanh(\beta)\mathbf{E}K)^t$$

where $e(1) = 2\beta\mathbf{E}K < \infty$. In particular, if $\tanh(\beta)\mathbf{E}K < 1$, then the effect of boundary values on the magnetization at the root decays exponentially in t.

Proof From (5) we have that $\sup_{b,\hat{b}} |z_v(b) - z_v(\hat{b})| = |z_v(+) - z_v(-)|$. We also have from (6) that $|F'_\beta(z)| \leq \tanh(\beta)$ and consequently, it follows from the recursion (4) that for any $w \in T$

$$|z_w(+) - z_w(-)| \leq \sum_{i=1}^{K(w)} |F_\beta(z_{u_i}(+)) - F_\beta(z_{u_i}(-))| \leq \tanh(\beta) \sum_{i=1}^{K(w)} |z_{u_i}(+) - z_{u_i}(-)| \,. \tag{7}$$

Starting the recursion at v which is the root of the Galton-Watson tree we see that $T(w)$ is the sub-tree of all descendants of w up to generation $t+1$. The disjoint random sub-trees $T(u_i)$ of descendants of each offspring of w are thus identically distributed, independent of each other and of the number $K(w)$ of offspring of w. As $\mathbf{E}|z_u(+) - z_u(-)|$ depends only on the distance s between u and T_*, we denote it by $e(s)$, and considering the expectation of (7) arrive at $e(s+1) \leq (\tanh(\beta)\mathbf{E}K)e(s)$ for $s = 1, \ldots, t$. To complete the proof note that for $b \in \{-1, 0, 1\}$,

$$F_\beta(b \times \infty) = \tanh^{-1}(b \tanh(\beta)) = b\beta \,, \tag{8}$$

so considering (4) with $u_i \in T_*$ we deduce that $e(1) = 2\beta\mathbf{E}K$ as stated. □

As we next see, the claimed mean field equation (2) for the random k-regular graph, is just the fixed point equation associated with our recursion, when specialized to the trees one finds inside a random k-regular graph and to constant boundary conditions.

Corollary 4. *Suppose the tree T is also the closed ball of radius $t+1$ and center v for a k-regular graph G, with T_* the set of vertices of distance $t+1$ from v in G. That is, each of the $N = (k(k-1)^t - 2)/(k-2)$ vertices of $T \setminus T_*$ is of degree k. Suppose further that the boundary values are constant, i.e. $b_u = b$ for some $b \in \{-1, 0, +1\}$ and all $u \in T_*$. Then, $m_v = m_v(t; b) = f_{\beta,h,k}(\overline{m}(t; b))$ where $\overline{m}(s; b) = f_{\beta,h,k-1}(\overline{m}(s-1; b))$ for $s = 2, \ldots, t$, $\overline{m}(1; b) = \tanh(h + (k-1)b\beta)$ and $f_{\beta,h,k}(\cdot)$ is as in (3).*

Proof We apply the recursion (4), where our degree condition translates to $K(v) = k$ and thereafter having $K(w) = k - 1$. Further, here the shape of the tree $T(w)$ depends only on the distance s of w from the boundary T_* of T, and having constant boundary conditions implies the same for $\overline{m}_w$ which we thus denote by $\overline{m}(s; b) = \tanh(z(s; b))$. With $z(0; b) = b \times \infty$ (and $0 \times \infty = 0$), we have from (4) and (8) that $\overline{m}(1; b) = \tanh(h + (k - 1)b\beta)$. Similarly, it then follows from (4) that $z(s) = h + (k - 1)F_\beta(z(s - 1))$ for $s = 2, \ldots, t$ while $z_v = z(t + 1) = h + kF_\beta(z(t))$, which $m = \tanh(z)$ maps to the stated recursion for $\overline{m}(s; b)$ and $m_v(t; b)$. □

We conclude this section with the analysis of equation (2).

Lemma 5. *The equation (2) has a unique positive solution* $\overline{m}_*(\beta, h)$ *when* $h > 0$ *and a unique negative solution* $\overline{m}_*(\beta, h) = -\overline{m}_*(\beta, -h)$ *when* $h < 0$. *If* $h = 0$ *it has a unique solution,* $\overline{m}_*(\beta, 0) = 0$*, for* $\beta \leq \beta_c = \tanh^{-1}(1/(k - 1))$*, and three solutions,* 0 *and* $\pm\overline{m}_*(\beta, 0)$*, with* $\overline{m}_*(\beta, 0) > 0$*, for each* $\beta > \beta_c$*. Further,* $\overline{m}_*(\beta, h) \downarrow \overline{m}_*(\beta, 0)$ *as* $h \downarrow 0$.

Proof Let $g(z) = z - (k - 1)F_\beta(z)$ for $F_\beta(z)$ of (4). From (6) we see that $F'_\beta(0) = \tanh(\beta)$ and $F''_\beta(z) < 0$ for all $z > 0$. Consequently, the smooth function $g(z)$ is strictly convex on $(0, \infty)$, and with $F_\beta(\cdot)$ an odd function, so is $g(\cdot)$. We thus deduce the existence of a unique solution $z(\beta, h)$ of $g(z) = h$ for any $h > 0$, and a unique negative solution $z(\beta, h) = -z(\beta, -h)$ of $g(z) = h$ when $h < 0$. Turning to $h = 0$, with $g(\cdot)$ a smooth odd function that is strictly convex on $(0, \infty)$, the equation $g(z) = 0$ has a unique solution $z(\beta, 0) = 0$ if $g'(0) = 1 - (k - 1)\tanh(\beta) \geq 0$, that is, for $\beta \leq \beta_c$, and three solutions, 0 and $\pm z(\beta, 0)$ for some $z(\beta, 0) > 0$, otherwise. Finally, for such a function $g(\cdot)$ we have that $z(\beta, h) \downarrow z(\beta, 0)$ as $h \downarrow 0$. We complete the proof by observing that solutions of (2) are of the form $\overline{m} = \tanh(z)$ with $g(z) = h$, while $\tanh(\cdot)$ is a monotone increasing odd function. □

Exercise 1: Derive the analog of (4) for the probability vector $S_w(\cdot)$ in the simplex over $\mathcal{X}$, first when $\mathcal{X} = \{-1, 1\}$ and then for an arbitrary finite set $\mathcal{X}$.

Open problem: Most of what we do for random k-regular graphs can be adapted for other ensembles of random graph. Of particular interest is the Erdös-Rényi random graph $\mathcal{G}(c/n, n)$, where each pair (i, j) is chosen to be in E with probability c/n independently of all other pairs. This ensemble reduces to a Galton-Watson tree with a Poisson(c) offspring distribution. Check that for $h > 0$ the mean field equation replacing (2) in this case corresponds to finding positive random variable Z that satisfies the equality in distribution

$$Z \overset{d}{=} h + \sum_{i=1}^{K} F_\beta(Z_i)\,,$$

where Z_i are i.i.d. copies of Z which are independent of K. The uniqueness of such Z when $\tanh(\beta)\mathbf{E}K \geq 1$ and $h > 0$ (that is, the analog of Lemma 5), is an open problem.

3 Griffiths inequalities and local weak convergence

Griffiths inequalities allow us to compare certain marginals of ferromagnetic Ising measures for one graph G and non-negative parameters β, h with certain other choices for G, β and h. To this end, we consider the *extended* ferromagnetic Ising measure

$$\mu_J(x) = \frac{1}{Z(J)} \exp\{\widehat{H}_J(x)\} = \frac{1}{Z(J)} \exp\Big\{\sum_{R \subseteq V} J_R x_R\Big\}\,, \tag{9}$$

for a finite set V and parameters $J_R \geq 0$, where hereafter $x_R = \prod_{u \in R} x_u$ and $x = (x_u, u \in V)$ for spin variables $x_u \in \mathcal{X} = \{-1, 1\}$. We note in passing that the Ising measure $\mu_{G,\beta,h}$ of (1) is merely μ_J in case $J_{\{i\}} = h$ for all $i \in V$, $J_{\{i,j\}} = \beta$ for all $(i, j) \in E$ and $J_R = 0$ for all other subsets of V.

In this context Griffiths inequalities are [2]

Proposition 6 (Griffiths inequalities)**.** *For $A, B \subseteq V$ and any $J = (J_R, R \subseteq V)$ with $J_R \geq 0$,*

$$\mathbf{E}_J[x_A] = \frac{1}{Z(J)} \sum_x x_A \exp\left\{\widehat{H}_J(x)\right\} \geq 0\,, \tag{10}$$

$$\frac{d}{dJ_B}\mathbf{E}_J[x_A] = \mathrm{Cov}_J(x_A, x_B) \geq 0\,. \tag{11}$$

Proof Fixing $A \subseteq V$ we start with (10), where for V finite,

$$\begin{aligned} \sum_x x_A \exp\left\{\widehat{H}_J(x)\right\} &= \sum_x x_A \sum_{n=0}^{\infty} \frac{1}{n!}\widehat{H}_J(x)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_x x_A (\sum_R J_R x_R)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{R_1,\dots,R_n} \prod_{\ell=1}^{n} J_{R_\ell} \sum_x x_A \prod_{\ell=1}^{n} x_{R_\ell}\,. \end{aligned}$$

Since $x_u^2 = 1$ for all u we have that $x_A \prod_{\ell=1}^n x_{R_\ell} = x_C$ for $C = \{u \in V : u$ in an odd number of sets among $A, R_1, \dots, R_n\}$. Further, with $\sum_x x_u = 0$ it follows that $\sum_x x_C = 0$ if C is non-empty (and $\sum_x x_C^2 = 2^{|V|} > 0$ for $C = \emptyset$). Thus $\sum_x x_A \prod_{\ell=1}^n x_{R_\ell} \geq 0$ for all $A, R_1, \dots, R_n$, and with $J_R \geq 0$ for all R, we have established (10). Turning to deal with (11) we fix $A, B \subseteq V$ and check that

$$\frac{d}{dJ_B}\mathbf{E}_J[x_A] = \frac{d}{dJ_B} \frac{\sum_x x_A \exp\{\sum_R J_R x_R\}}{\sum_x \exp\{\sum_R J_R x_R\}} = Z(J)^{-2} \sum_{x,y} (x_A x_B - x_A y_B) \exp\{\sum_R J_R(x_R + y_R)\}$$

which is precisely the covariance of x_A and x_B under $\mu_J(x)$. We shall use (10) to verify that this quantity is non-negative. To this end, let $z_u = x_u y_u \in \mathcal{X}$ noting that $y_R = x_R z_R$ for any $R \subseteq V$ (as $x_R^2 = 1$), and as before $x_A x_B = x_C$ for the symmetric difference C between A and B. Consequently,

$$\sum_{x,y} (x_A x_B - x_A y_B) \exp\{\sum_R J_R(x_R + y_R)\} = \sum_z (1 - z_B) \sum_x x_C \exp\{\sum_R J_R(z) x_R\}$$

where $J_R(z) = J_R(1 + z_R) \geq 0$. From (10) we thus have that $\sum_x x_C \exp\{\sum_R J_R(z) x_R\} \geq 0$ for each $z \in \mathcal{X}^V$, and with $1 - z_B \geq 0$ we complete the proof of (11). ☐

Fixing $\beta > 0$ and $h \geq 0$, for any finite graph G let $m_v(G) = \mu_{G,\beta,h}(x_v = 1) - \mu_{G,\beta,h}(x_v = -1)$ denote the magnetization at $v \in V$ induced by the corresponding (ferromagnetic) Ising measure. For $S \subseteq V$ we similarly define $m_v(S; b)$ as the magnetization at v induced by the same Ising measure subject to fixed boundary conditions $x_u = b_u$ for $u \notin S$. Of particular interest to us are $m_v(S; +)$ and $m_v(S; f)$ corresponding to $b_u = 1$, respectively $b_u = 0$, for all $u \notin S$. The latter are called *free boundary conditions* since subject to $b_u = 0$, $u \notin S$, the restriction of the Ising measure $\mu_{G,\beta,h}$ to $(x_u, u \in S)$ coincides with the Ising measure $\mu_{G|_S,\beta,h}$ for the restriction $G\,|_S$ of G to S (i.e. with S as its vertices and $\{(i,j) \in E : i \in S, j \in S\}$ as its edges). We then get by Griffiths inequalities the following comparison results

Lemma 7. *If $v \in S \subseteq V$ then $m_v(S; f) \leq m_v(G) \leq m_v(S; +)$. Further, $S \mapsto m_v(S; f)$ is monotone non-decreasing and $S \mapsto m_v(S; +)$ is monotone non-increasing, both with respect to set inclusion (among sets S that contain v).*

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[2] Our source for both statement and proof is [Lig85, Theorem IV.1.21], see also [Gin70] for more general results in this direction.

Proof From Griffiths inequalities we know that $J \mapsto \mathbf{E}_J[x_v]$ is monotone non-decreasing (where $J \geq \hat{J}$ if and only if $J_R \geq \hat{J}_R$ for all $R \subseteq V$).

Recall further that $m_v(G) = \mathbf{E}_{J^0}[x_v]$ where $J^0_{\{i\}} = h$, $J^0_{\{i,j\}} = \beta$ when $(i,j) \in E$ and all other values of J^0 are zero. Considering

$$J^{\eta,S}_R = J^0_R + \eta \mathbf{1}_{R \subseteq S^c, |R|=1} ,$$

with $\eta \mapsto J^{\eta,S}$ non-decreasing, so is $\eta \mapsto \mathbf{E}_{J^{\eta,S}}[x_v]$. In addition, $\mu_{J^{\eta,S}}(x_i = -1) \leq c e^{-2\eta}$ when $i \notin S$, hence as $\eta \uparrow \infty$ the measure $\mu_{J^{\eta,S}}$ converges to μ_J subject to the fixed boundary conditions $x_u = 1$ for $u \notin S$. Consequently, $m_v(G) \leq \mathbf{E}_{J^{\eta,S}}[x_v] \uparrow m_v(S;+)$.

Similarly, let $J^S_R = J^0_R \mathbf{1}_{R \subseteq S}$ noting that under μ_{J^S} the random vector $(x_u, u \in S)$ is distributed according to the Ising measure $\mu_{G|_S,\beta,h}$. With $v \in S$ we thus deduce that $m_v(S;f) = \mathbf{E}_{J^S}[x_v] \leq \mathbf{E}_{J^0}[x_v] = m_v(G)$.

Finally, the stated monotonicity of $S \mapsto m_v(S;f)$ and $S \mapsto m_v(S;+)$ are in view of Griffiths inequalities the direct consequence of the monotonicity (with respect to set inclusions) of $S \mapsto J^S$ and $S \mapsto J^{\eta,S}$, respectively. □

Applying this lemma, we next find that the magnetization at the root under the Ising measure on a k-regular tree of large depth is for $h > 0$ or $\beta \leq \beta_c$ and both free and plus boundary conditions, near the unique $m_*(\beta,h)$ specified by our mean field equations (2) and (3).

Corollary 8. *Let $m_v(t;b)$ denote the magnetization at the root v of k-regular tree T of depth $t+1$ under the Ising measure with constant boundary values $b_u = b \in \{-1,0,1\}$ at the leaves of T, as in Corollary 4. Then, for $h > 0$ or $\beta \leq \beta_c = \tanh^{-1}(1/(k-1))$ both $m_v(t;0) \uparrow m_*(\beta,h)$ and $m_v(t;1) \downarrow m_*(\beta,h)$ as $t \uparrow \infty$, where $m_*(\beta,h) = f_{\beta,h,k}(\overline{m}_*(\beta,h))$ for $\overline{m}_*(\beta,h)$ of Lemma 5 (and $f_{\beta,h,k}(\cdot)$ as in (3)).*

Proof Fixing an offspring w of v in the k-regular tree, let $T(w)$ and $\overline{m}(t;b)$ denote the corresponding sub-tree (of depth t), and associated magnetization at w, respectively. Recall Corollary 4 that $m_v(t;b) = f_{\beta,h,k}(\overline{m}(t;b))$ with $f_{\beta,h,k}(\cdot)$ non-decreasing and continuous. It thus suffices to show that both $\overline{m}(t;0) \uparrow \overline{m}_*(\beta,h)$ and $\overline{m}(t;1) \downarrow \overline{m}_*(\beta,h)$ as $t \uparrow \infty$ (when $h > 0$ or $\beta \leq \beta_c$).

To this end, embedding the sub-tree $T(w)$ for a k-regular tree of depth t in the sub-tree $T(w)$ for a k-regular tree of larger depth, the monotonicity in t (and hence convergence) of the latter two sequences is a direct consequence of Lemma 7. Further, by the recursion of Corollary 4 the limit of such a sequence must be a solution of (2). We also saw there that $0 \leq \overline{m}(t;0) < \overline{m}(t;1) \leq 1$ for $t = 1$, and with $f_{\beta,h,k-1}$ mapping $[0,1]$ to itself, the same applies for all t, hence for the limit $t \uparrow \infty$. We conclude the proof upon noting that when either $h > 0$ or $\beta \leq \beta_c$ we have from Lemma 5 the uniqueness of a non-negative solution $\overline{m}_*(\beta,h)$ of (2). □

Suppose kN is even. A k-regular random graph over $V = [N]$ is constructed by assigning k labeled half-edges to each vertex $i \in V$ and creating the set of edges E by a uniform random matching of the kN half-edges (as mentioned before, this is the same as choosing a k-regular graph uniformly at random, apart from possibly having self edges and multiple edges).

In view of Lemma 7 and Corollary 8 we complete the derivation of the mean field equations by showing that

Lemma 9. *Fix $t < \infty$ and a positive integer I. The probability $Q(N,t)$ that the closed ball of radius t and center I in a k-regular graph G over $[N]$ is a tree, converges to one as $N \to \infty$*

Remark This is of course also the probability of such event when choosing $I \in \{1,\ldots,N\}$ uniformly.

Proof Let $B_N(t)$ denote the subgraph induced by vertices of distance at most t from I in G. Ordering the vertices of the tree T according to their distance from the root, it is easy to verify that $Q(N,t) = \mathbb{P}(B_N(t) = T) = \prod_{v=0}^{|T|} q(v)$ where $q(v) \geq 1 - C/N$ for some $C = C(k,t)$, any N and $v \leq |T|$. □

References

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