Stat 310B/Math 230B Theory of Probability

Practice Final

Andrea Montanari *March 13, 2024*

This practice final is longer than the actual final!
Solutions should be complete and concisely written. Please, use a separate booklet for each problem.

You have 3 hours but you are not required to solve all the problems!!!

Just solve those that you can solve within the time limit.
For any clarification on the text, one of the TA's will be available in his office, and Andrea by phone/email.
You can one of the recommended textbooks and your notes. You cannot use computers, and in particular you cannot use the web. You can cite theorems (propositions, corollaries, lemmas, etc.) from Amir Dembo's lecture notes by number, and exercises you have done as homework by number as well. Any other non-elementary statement must be proved!

Problem 1 [30 points]

I am placing bets on a game of chance. I start with an integer number of m dollars, and each time n I can bet an integer number of dollars D_n (as long as my current capital is at least D_n). With probability 1/2 (independently of the past) I win, and I increase my capital by D_n dollars, and with probability 1/2 I loose, and I decrease it by D_n dollars.

Let X_n the capital I have at time n. Answer the following questions providing proofs of your statements.

(*a*) Let $X_\infty = \lim_{n\to\infty} X_n$. Does this quantity exist? What is the betting strategy that maximizes the success probability $\mathbb{P}(X_\infty \ge m+1)$? Compute the optimal success probability.

(*b*) How does your answer to question (*a*) changes if I want to maximize $\mathbb{P}(X_\infty \ge k)$ for a general integer $k \ge 0$? Is the strategy that maximizes $\mathbb{P}(X_\infty \ge k)$

(*c*) Fix $k > m$. Consider an arbitrary strategy that maximizes $\mathbb{P}(X_\infty \ge k)$. Does any such strategy maximizes $\mathbb{P}(X_n \ge k)$ for all n?

Problem 2 [20 points]

Let $G = (V, E, w)$ be a finite weighted network with vertex set V, edge set[1] $E \subseteq \binom{V}{2}$ and strictly positive weights $w_{xy} = w_{yx} > 0$ for each edge $(x, y) \in E$, and consider the simple random walk on this network, i.e. the Markov chain with state space V and transition probabilities

$$p(x, y) = \frac{w_{xy}}{\sum_{z\in\partial x} w_{xz}}\, , \tag{1}$$

where we denote by $\partial x := \{z : (x, z) \in E\}$ the neighborhood of x. We assume that the network is connected, the chain is aperiodic, and that weights are normalized so that $2\sum_{(x,y)} w_{xy} = 1$. The network has a distinguished vertex $v \in V$.

We know the graph G and observe a single trajectory of the Markov chain $(X_k)_{0\le k\le n}$ initialized at $X_0 = v$, and want to estimate the weights. To this end let, $\widehat{W}_{xy}$, $(x, y) \in E$ be the normalized counts

$$\widehat{W}_{xy}(n) := \frac{1}{2n}\sum_{k=1}^{n} \big[\mathbf{1}(X_{k-1} = x, X_k = y) + \mathbf{1}(X_{k-1} = y, X_k = x)\big]\, . \tag{2}$$

[1] $\binom{V}{2}$ is the set of subsets of size 2 of V.

(a) Show that, for any $u, z \in V$, $\lim_{n\to\infty} \mathbb{P}_v(X_n = z) = \pi(z)$ for $\pi(z) = \sum_{y\in V} w_{zy}$ unique invariant measure.

(b) Show that, for any $u, v \in V$, there exists a coupling of $X^u \sim \mathbb{P}_u$, $X^v \sim \mathbb{P}_v$ such that

$$\mathbb{P}(X_n^u \neq X_n^v) \leq C_1 e^{-c_2 n}\,,$$

for some constants $C_1, c_2 > 0$.

(c) Let $Z_n = \sum_{k=1}^n \mathbf{1}(X_{k-1} = x, X_k = y)$, and $\mathcal{F}_k$ the canonical filtration of the Markov chain. Deduce that there exists a constant C_3 such that

$$\big|\mathbb{E}[Z_n|\mathcal{F}_k] - \mathbb{E}[Z_n|\mathcal{F}_{k-1}]\big| \leq C_3\,. \tag{3}$$

(d) Use Azuma-Hoeffding to show that

$$\mathbb{P}_v\Big(|Z_n - \mathbb{E}[Z_n] \geq t\Big) \leq 2\,e^{-t^2/C_4 n}\,. \tag{4}$$

(e) Prove that, almost surely

$$\lim_{n\to\infty} \widehat{W}_{xy}(n) = w_{x,y}\,. \tag{5}$$

(f) Prove that there exists a constant $C > 0$ such that:

$$\lim_{n_0\to\infty} \mathbb{P}\Big(\exists n \geq n_0 \text{ s.t. } |\widehat{W}_{xy}(n) - w_{x,y}| \geq C\sqrt{n\log n}\Big) = 0\,. \tag{6}$$

Problem 3 [60 points]

Recall that given a graph $G = (V, E)$ and two disjoint sets of vertices $A, B \subseteq V$, such that $V_0 = V \setminus (A \cup B)$ is finite, a current flow from A to B is a map $i : \vec{E} \to \mathbb{R}$ such that[2]

1. $i(x, y) = -i(y, x)$.
2. For any $x \in V_0$, $\sum_{y\in\partial x} i(x, y) = 0$.
3. There exists a voltage function $h : V \to \mathbb{R}$ such that, for all $[x, y] \in \vec{E}$, $i(x, y) = h(x) - h(y)$, and $h|_A = 1$, $h|_B = 0$.

Recall also that the effective conductance $C_{A,B}$ is the total current across a cut separating A and B, the effective resistance is $R_{A,B} = 1/C_{A,B}$. If $A = \{v\}$ is a single vertex, we write $C_{v,B}$ and $R_{v,B}$ instead of $C_{\{v\},B}$ and $R_{\{v\},B}$.

(a) Prove that effective resistances add if networks are connected in series, and effective conductances add if they are connected in series.

[We talked about this in class.]

(b) Let G_d be the regular infinite tree with degree $d \geq 3$, rooted at an arbitrary vertex ρ, and denote by B_n the set vertices at distance at least n from root. Compute the effective resistance R_{ρ,B_n}.

(c) Use the above to prove that the simple random walk (SRW) on G_d is transient.

[2] We denote by $\partial x := \{z : (x, z) \in E\}$ the neighborhood of x and $\vec{E} \subseteq V \times V$ the set of directed edges in G, containing two directed edges $[x, y]$, $[y, x]$ for any (undirected) $(x, y) \in E$.

(d) Given a sequence of integers $\boldsymbol{d} = (d_n : n \geq 0)$, with $d_n \geq 2$ for all n, let $T_{\boldsymbol{d}}$ be the infinite tree rooted at ρ such that all vertices at distance n from the root have degree d_n. Prove that, if there exists n_0 such that $d_n \geq 3$ for all $n \geq n_0$, then the SRW on $T_{\boldsymbol{d}}$ is transient.

(e) Prove that, if there exists n_0 such that $d_n = 2$ for all $n \geq n_0$, then the SRW on $T_{\boldsymbol{d}}$ is recurrent.

(f) Can you come up with examples of sequences $\boldsymbol{d}$, with $d_n \geq 2$ for all n, such that the condition at point (e) does not hold and yet the SRW on $T_{\boldsymbol{d}}$ is recurrent?

Problem 4

[Just for fun!]

Let Ω be the space of Borel probability measures ω on $\mathbb{R}$, $\mathcal{F}$ be the σ algebra generated by the events $E_{A,t} := \{\omega : \omega(A) \leq t\}$ for $A \in \mathcal{B}$, $t \in \mathbb{R}$.

Let μ_0 be a Borel probability measure on $\mathbb{R}$ and consider the following sequence of random probability measures ξ_n, $n \geq 1$.

- $\xi_1 = \delta_{Z_1}$ for $Z_1 \sim \mu_0$.
- For each $n \geq 1$, draw $Z_{n+1} \sim \mu_0$ independently of the past, $X_{n+1} \sim (n\mu_0 + \delta_{Z_{n+1}})/(n+1)$, and set

$$\xi_{n+1} = \frac{1}{n+1}(n\mu_0 + \delta_{X_{n+1}})\,. \tag{7}$$

Define a suitable filtration $\mathcal{F}_n$, and show that the process $(\xi_n)_{n\geq 0}$ is an inhomogeneous Markov chain with respect to that filtration.