Stat 310B/Math 230B Theory of Probability

Midterm

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Due on 2/7/2014

Solutions should be complete and concisely written. Please, use a separate booklet for each problem.

You have 3 hours but you are not required to solve all the problems!!!

Just solve those that you can solve within the time limit.

For any clarification on the text, one of the TA's will be outside the room, and Andrea in Packard 272.

You can consult Amir Dembo's lecture notes and your own notes. You cannot use other textbooks, you cannot use computers, and in particular you cannot use the web. You can cite theorems (propositions, corollaries, lemmas, etc.) from the lecture notes by number, and exercises you have done as homework by number as well. Any other non-elementary statement must be proved!

Problem 1 (30 points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, denote by $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ the vector space of square integrable random variables. Given $X_1, \dots, X_n \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{W}(\{X_i\}_{i\in[n]})$ be the vector space of linear combinations of these random variables:

$$\mathcal{W}(\{X_i\}_{i\in[n]}) = \Big\{ X = \sum_{i=1}^{n} c_i X_i : \;\; c_i \in \mathbb{R} \Big\} . \tag{1}$$

(*a*) Prove that orthogonal projection from $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathcal{W}(\{X_i\}_{i\in[n]})$ is well defined. Namely, given $Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, there exist $\widehat{Y} \in \mathcal{W}(\{X_i\}_{i\in[n]})$, such that

$$\mathbb{E}\{(\widehat{Y} - Y)^2\} = \inf \Big\{ \mathbb{E}\{(Z - Y)^2\} : \;\; Z \in \mathcal{W}(\{X_i\}_{i\in[n]}) \Big\} . \tag{2}$$

Further such a projection is unique, in the sense that given two random variables $\widehat{Y}$, $\widehat{Y}'$ satisfying the above conditions, $\widehat{Y} = \widehat{Y}'$ almost surely.

Finally, prove that $\mathbb{E}\{(\widehat{Y} - Y)W\} = 0$, for every $W \in \mathcal{W}(\{X_i\}_{i\in[n]})$.

(*b*) Prove that, if $(X_1, \dots, X_n, Y)$ are jointly Gaussian, with zero mean, then

$$\widehat{Y} = \mathbb{E}\{Y | X_1, X_2, \dots, X_n\} . \tag{3}$$

(*c*) Again, assuming that $(X_1, \dots, X_n, Y)$ are jointly Gaussian define

$$\sigma^2 = \mathbb{E}\{(Y - \widehat{Y})^2\} , \tag{4}$$

and assume $\sigma^2 > 0$. Further let γ_{μ,σ^2} be Gaussian the probability distribution with mean μ and variance σ^2. Namely (with $\mathrm{d}x$ the Lebesgue measure on $\mathbb{R}$)

$$\gamma_{\mu,\sigma^2}(\mathrm{d}x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, \mathrm{d}x . \tag{5}$$

Prove that the regular conditional probability distribution of Y given $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ is given by

$$\widehat{\mathbb{P}}_{Y|\mathcal{F}_n}(\,\cdot\,|\omega) = \gamma_{\widehat{Y}(\omega),\sigma^2}\,, \tag{6}$$

with $\widehat{Y} = \sum_{i=1}^n c_i X_i$ the orthogonal projection of Y onto $\mathcal{W}(\{X_i\}_{i\in[n]})$.

Problem 2 (20 points)

Let X, Y be two independent and identically distributed random variables, with common density function f with respect to Lebesgue measure. Define $M = \max(X, Y)$.

What is the regular conditional probability distribution of X given M? Prove your answer.

Problem 3 (30 points)

Fix $\epsilon \in (0,1)$, and let $f : [0,1] \to \mathbb{R}_{\ge 0}$ be a Borel function with $f(x) \le \min(x, 1-x)$, for all $x \in [0,1]$.

For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\{\xi_k\}_{k\ge1}$ be a sequence of i.i.d. random variables, with $\mathbb{P}(\xi_k = +1) = \mathbb{P}(\xi_k = -1) = 1/2$. Let $\mathcal{F}_n = \sigma(\xi_1, \xi_2, \dots, \xi_n)$ be the filtration generated by the ξ_i's, and define, recursively,

$$X_n = X_{n-1} + \xi_n\, f(X_{n-1})\,, \qquad X_0 = \epsilon\,. \tag{7}$$

(*a*) Show that $(X_n, \mathcal{F}_n)$ is a martingale, and prove that $X_n \to X_\infty$, both almost surely and in L^1, for some random variable X_∞ taking values in $[0,1]$.

(*b*) Let $S \equiv \{x \in [0,1] : f(x) = 0\}$. Prove that $\mathbb{P}(X_\infty \in S) = 1$.

(*c*) Assume $f(x) > 0$ for all $x \in (0,1)$ (and clearly $f(0) = f(1) = 0$). Compute the law of X_∞.

Problem 4 (20 points)

Let $(X_n, \mathcal{F}_n)$ be a martingale with $\sup_n \mathbb{E}\{X_n^2\} < \infty$, $\{a_k\}_{k\ge1}$ a deterministic sequence, and define (with $Y_0 = 0$)

$$Y_n = \sum_{k=1}^n a_k (X_k - X_{k-1})\,. \tag{8}$$

Prove that, if $\sum_k a_k^2 < \infty$, then Y_n converges almost surely and in L^2.