Stat 310B/Math 230B Theory of Probability

Homework 5 Solutions

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Exercise [5.3.20]

1. We claim that

$$\mathbf{E}[h|\mathcal{F}_n] = 2^n \sum_{i=1}^{2^n} \Big[\int_{A_{i,n}} h(u)du \Big] I_{A_{i,n}}(t)\,. \tag{1}$$

Indeed, integrability and $\mathcal{F}_n$-measurability of the RHS of (1) are obvious. Further, denoting the RHS of (1) by f_n, clearly $\mathbf{E}[f_n I_{A_{i,n}}] = \mathbf{E}[h I_{A_{i,n}}]$ for each value of i and n (as $\mathbf{E}(I_{A_{i,n}}) = \mathbf{P}(A_{i,n}) = 2^{-n}$), which suffices since $\{A_{i,n} : i = 1, \ldots, 2^n\}$ is a π-system.

2. Each function $X_n(t)$ is piecewise constant on the intervals generating $\mathcal{F}_n$, having the value $h_{i,n}$ on the interval $A_{i,n}$. Thus, $X_n(t)$ is $\mathcal{F}_n$ measurable and further integrable with

$$c_n = \mathbf{E}|X_n| = 2^{-n} \sum_{i=1}^{2^n} |h_{i,n}|$$

finite. It is easy to check that $h_{i,n} = (h_{2i-1,n+1} + h_{2i,n+1})/2$, namely, the constant value of $X_n(t)$ on each interval $A_{i,n}$ is the average of the values that $X_{n+1}(t)$ take on the two adjacent intervals $A_{2i-1,n+1}$ and $A_{2i,n+1}$ into which $A_{i,n}$ split. By part (a)

$$\mathbf{E}[X_{n+1}|\mathcal{F}_n] = \sum_{i=1}^{2^n} g_{i,n} I_{A_{i,n}}(t)\,,$$

where $g_{i,n}$ is the expected value of $X_{n+1}(t)$ with respect to the uniform measure on $A_{i,n}$. Since $A_{i,n}$ is the disjoint union of the intervals $A_{2i-1,n+1}$ and $A_{2i,n+1}$ of same length on each of which $X_{n+1}(t)$ is constant, it follows that $g_{i,n} = h_{i,n}$ and consequently, that X_n is a martingale.

3. Let $c_n = \mathbf{E}|X_n|$ and $c = \sup_n c_n$. If $c = \infty$ then there exist $n_k \geq k$ such that $c_{n_k} \geq 2^k$. Taking for each k the collection of intervals A_{i,n_k} with the $m = m^{(k)} = 2^{n_k - k}$ largest values of $|h_{i,n_k}|$ then yields some $s_1^{(k)} < t_1^{(k)} \leq s_2^{(k)} < t_2^{(k)} \cdots t_m^{(k)}$ for which

$$\sum_{\ell=1}^{m} |t_\ell^{(k)} - s_\ell^{(k)}| \leq 2^{-k} \underset{k\to\infty}{\longrightarrow} 0 \quad \text{while} \quad \sum_{\ell=1}^{m} |x(t_\ell^{(k)}) - x(s_\ell^{(k)})| \geq 1$$

contradicting the absolute continuity of $x(\cdot)$. Therefore, $c < \infty$ and hence for all $\rho > 0$ by Markov's inequality

$$2^{-n} \sum_{j=1}^{2^n} I_{|h_{j,n}| > \rho} = \mathbf{P}(|X_n| > \rho) \leq \mathbf{E}|X_n|/\rho \leq c/\rho \tag{2}$$

Further, note that

$$\mathbf{E}[|X_n| I_{|X_n|>\rho}] = \sum_{\{j:|h_{j,n}|>\rho\}} 2^{-n} |h_{j,n}|\,. \tag{3}$$

Let $\epsilon > 0$ be fixed and $\delta = \delta(\epsilon, x) > 0$ be determined as in the definition of absolute continuity. Taking $\rho = c/\delta$ observe that (2) and (3) imply by the absolute continuity of $x(\cdot)$ that

$$\mathbf{E}[|X_n| I_{|X_n|>\rho}] \leq \epsilon \quad \text{for all} \quad n,$$

hence $\{X_n\}$ is U.I.

4. Assuming hereafter that $x(\cdot)$ is absolutely continuous, hence $\{X_n\}$ is a U.I. martingale, by Corollary **??**,

$$X_n = \mathbf{E}[h|\mathcal{F}_n] \quad \text{for some} \quad h = X_\infty \in L^1.$$

Hence, by (1) we see that $x(i2^{-n}) - x((i-1)2^{-n}) = \int_{(i-1)2^{-n}}^{i2^{-n}} h(u)du$ for $n = 0, 1, \cdots$ and $i = 1, \cdots, 2^n$. By linearity of the integral we thus have

$$x(j2^{-n}) - x(i2^{-n}) = \int_{i2^{-n}}^{j2^{-n}} h(u)du \quad \text{for all} \quad j \geq i, \quad \text{and all} \quad n \quad \text{values.} \tag{4}$$

Consider now arbitrary $1 > t \geq s \geq 0$ and let $j_m \geq i_m, n_m$ be such that $j_m 2^{-n_m} \to t$ and $i_m 2^{-n_m} \to s$ as $m \to \infty$. By continuity of $x(\cdot)$, the LHS of (4) for this sequence converges to $x(t) - x(s)$, while the RHS of (4) is $\int_0^1 h(u) I_{[i_m 2^{-n_m}, j_m 2^{-n_m}]}(u)du$, with the integrand converging a.e. to $h(u)I_{[s,t]}(u)$ and dominated by the integrable function $|h|$. Thus, by dominated convergence these integrals converge to $\int_s^t h(u)du$.

5. Consider now

$$\Delta^{-1}[x(s + \Delta) - x(s)] - h(s) = \Delta^{-1} \int_s^{s+\Delta} [h(u) - h(s)]du.$$

So that

$$\lim_{\Delta \to 0} |\Delta^{-1}(x(s + \Delta) - x(s)) - h(s)| \leq \lim_{\Delta \to 0} \Delta^{-1} \int_s^{s+\Delta} |h(u) - h(s)|du = 0 \text{ a.e. } [0, 1).$$

Hence $h(t) = \frac{dx}{dt}$ a.e. $[0, 1)$ as claimed.

Exercise [5.3.40]

Set $\lambda > 0$ and let $\psi = e^\lambda - \lambda - 1$.

1. Obviously, N_n is measurable on $\mathcal{F}_n$. By our assumptions about the L^2 martingale $(M_n, \mathcal{F}_n)$, part (a) of Exercise **??** applies for the law of $Y = \lambda(M_{n+1} - M_n)$ conditional on $\mathcal{F}_n$, taking there $\kappa = \lambda$ and

$$\lambda^{-2}\mathbb{E}[Y^2|\mathcal{F}_n] = \mathbb{E}[(M_{n+1} - M_n)^2|\mathcal{F}_n] = \langle M \rangle_{n+1} - \langle M \rangle_n \,.$$

With $\langle M \rangle_{n+1} \in m\mathcal{F}_n$, we consequently have that

$$\mathbb{E}[N_{n+1}|\mathcal{F}_n] = N_n \exp(-\psi(\langle M \rangle_{n+1} - \langle M \rangle_n))\mathbb{E}[e^Y|\mathcal{F}_n] \leq N_n \,,$$

implying that $(N_n, \mathcal{F}_n)$ is a non-negative sup-MG.

2. Since $\lambda > 0$, for any finite $n \geq 0$, the event $A_n := \{M_{n\wedge\tau} > u, \langle M \rangle_{n\wedge\tau} \leq r\}$ implies that $N_{n\wedge\tau} \geq a$ for $a = \exp(\lambda u - \psi r)$. Fixing an $\mathcal{F}_n$-stopping time τ, since $\{N_{n\wedge\tau}\}$ is a sup-MG, it follows by Markov's inequality that for any $n \geq 0$,

$$\mathbf{P}(A_n) \leq \mathbf{P}(N_{n\wedge\tau} \geq a) \leq a^{-1}\mathbb{E}N_{n\wedge\tau} \leq a^{-1}\mathbb{E}N_0 = a^{-1} \,.$$

Thus, by Fatou's lemma also $\mathbf{P}(\liminf A_n) \leq a^{-1}$. Since $\langle M \rangle_{n\wedge\tau} \uparrow \langle M \rangle_\tau$, the event $\{M_\tau > u, \langle M \rangle_\tau \leq r\}$ is contained in $\liminf A_n$, yielding the bound

$$\mathbf{P}(M_\tau > u, \langle M \rangle_\tau \leq r) \leq a^{-1} = \exp(-\lambda u + r\psi) \,.$$

To complete the proof, consider the preceding for $u_\ell \uparrow u$.

3. Recall that the L^2 bounded martingale S_n of Example **??** converge a.s. (and in L^2) to a finite limit S_∞. As the martingale S_n has independent increments, we deduce that $\langle S\rangle_n = \mathbb{E}(S_n^2)$ is a non-random sequence which converges to the finite constant $\langle S\rangle_\infty = \sum_k \mathbb{E}\xi_k^2$. By part (a) and our assumption that $|\xi_k| \le 1$ we further have that $N_n = \exp(\lambda S_n - \psi\langle S\rangle_n)$ is a non-negative sup-MG for any $\lambda > 0$. Thus, by Doob's convergence theorem $N_n \to N_\infty$ almost surely and $\mathbb{E}N_\infty \le \mathbb{E}N_0 = 1$. Since necessarily $N_\infty = \exp(\lambda S_\infty - \psi\langle S\rangle_\infty)$, it follows that $\mathbb{E}[\exp(\lambda S_\infty)] \le \exp(\psi\langle S\rangle_\infty)$ is finite. This conclusion extends to all $\lambda \in \mathbb{R}$ since the same argument applies also for the martingale $-S_n$.

Exercise [5.4.12]

This is the strictly positive product martingale M_n of Example **??**, for the positive i.i.d. variables $Y_k = e^{\lambda\xi_k}/M(\lambda)$ of mean one.

1. For $p = 1 - q \ge 1/2$ we know from parts (c) and (d) of Exercise **??** that τ_b is finite a.s. Further, by definition $S_{n\wedge\tau_b} \le b$ for all n and as $M(\lambda) = pe^{\lambda} + qe^{-\lambda} \ge 1$ whenever $\lambda \ge 0$, in this case

$$M_{n\wedge\tau_b} = \exp(\lambda S_{n\wedge\tau_b} - (n\wedge\tau_b)\log M(\lambda)) \le \exp(\lambda b)\,.$$

Thus, $\{M_{n\wedge\tau_b}\}$ is a uniformly bounded, hence U.I. martingale. With $S_{\tau_b} = b$, it then follows from Doob's optional stopping theorem that

$$1 = \mathbf{E}M_0 = \mathbf{E}M_{\tau_b} = e^{\lambda b}\mathbf{E}[M(\lambda)^{-\tau_b}]\,.$$

2. Setting $0 < s < 1$ there exists for $p \ge 1/2$ a unique $\lambda > 0$ such that $M(\lambda) = pe^{\lambda} + qe^{-\lambda} = 1/s$. Indeed, solving $qsx^2 - x + ps = 0$ for $x = e^{-\lambda}$ in $(0,1)$, we find from part (a) that $\mathbf{E}[s^{\tau_b}] = x^b$ and

$$\mathbf{E}[s^{\tau_1}] = x = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs}\,,$$

when $q \in (0, 1/2]$, whereas $x = s$ in the trivial case $q = 0$.

3. Since $\tau_{a,b} = \min(\tau_b, \tau_{-a})$ is finite a.s. and $\tau_{-a} \ge a$ we have that

$$\mathbf{P}(\tau_b < a) \le \mathbf{P}(\tau_b < \tau_{-a}) \le \mathbf{P}(\tau_b < \infty)\,.$$

Consequently, $\mathbf{P}(\tau_b < \tau_{-a}) \to \mathbf{P}(\tau_b < \infty)$ as $a \to \infty$. To complete the proof recall that from Corollary **??** we have that

$$\mathbf{P}(\tau_b < \tau_{-a}) = 1 - r = \frac{1 - e^{-\lambda_* a}}{e^{\lambda_* b} - e^{-\lambda_* a}} \to e^{-\lambda_* b}$$

when $a \to \infty$ (as $\lambda_* > 0$).

4. Clearly, $\{\tau_b < \infty\}$ if and only if $\{Z \ge b + 1\}$. Hence, for any positive integer b,

$$\mathbf{P}(Z = b) = \mathbf{P}(Z \ge b) - \mathbf{P}(Z \ge b + 1) = (1 - e^{-\lambda_*})e^{-\lambda_*(b-1)}\,.$$

Exercise [5.4.14]

Recall that ξ_k are i.i.d. random variables taking values in $\{A, B, \ldots, Z\}$ such that $\mathbf{P}(\xi_k = x) = 1/26$ if $x \in \{A, B, \ldots, Z\}$ and 0 otherwise.
(a). The amount of money the gamblers have collectively earned by time n is $M_n = \sum_{i=1}^n Z_{i,n}$, where $Z_{i,n}$ is the amount of money that the gambler who enters the game just before time step i *earns* by time step n. For example, if the letter the monkey types at time i is P then $Z_{i,i} = \$25$, since this gambler paid \$1 to bet and received \$26 upon successfully betting on P.

Clearly, M_n is measurable on $\mathcal{F}_n^{\boldsymbol{\xi}} = \sigma(\xi_k, k \leq n)$ and since each $Z_{i,n}$ is bounded, so is M_n. To show that M_n is a MG with respect to $\mathcal{F}_n^{\boldsymbol{\xi}}$ observe that

$$\mathbf{E}(M_{n+1}|\mathcal{F}_n^{\boldsymbol{\xi}}) = \sum_{i=1}^{n} \mathbf{E}(Z_{i,n+1}|\mathcal{F}_n^{\boldsymbol{\xi}}) + \mathbf{E}(Z_{n+1,n+1}|\mathcal{F}_n^{\boldsymbol{\xi}}) .$$

The gambler who enters the game at time $n+1$ is independent of all the events through time n, hence

$$\mathbf{E}(Z_{n+1,n+1}|\mathcal{F}_n^{\boldsymbol{\xi}}) = \mathbf{E}(Z_{n+1,n+1}) = 25\frac{1}{26} + (-1)\frac{25}{26} = 0 .$$

The earnings of a gambler who entered the game at time $i \leq n$ and has either lost by the end of time step n, or left the game upon wining all 11 rounds, are unchanged from time n to $n+1$. Any other gambler who entered the game at time $i \leq n$ bets all he has, that is $1 + Z_{i,n}$, on the outcome of Monkey's typing at step $n+1$. With probability $1/26$ he will have as a result $26(Z_{i,n}+1)$ and otherwise, he loses everything and departs. Since he initially paid \$1 for his first gamble, his total earnings after time step $n+1$ are $Z_{i,n+1} = 26(Z_{i,n}+1) - 1$ with probability $1/26$ and -1 with probability $25/26$. Thus,

$$\mathbf{E}(Z_{i,n+1}|\mathcal{F}_n^{\boldsymbol{\xi}}) = \frac{1}{26}\mathbf{E}[26(Z_{i,n}+1) - 1|\mathcal{F}_n^{\boldsymbol{\xi}}] - \frac{25}{26} = Z_{i,n} .$$

Consequently, $\mathbf{E}(M_{n+1}|\mathcal{F}_n^{\boldsymbol{\xi}}) = M_n$ with $M_0 = 0$.
(b). Let $\xi_i^j = (\xi_i, \xi_{i+1}, \ldots, \xi_j)$ for $j > i$ noting that the monkey first types PROBABILITY at the stopping time
$\widehat{\tau} = \inf\{n : \xi_{n-10}^n = (P, R, O, B, A, B, I, L, I, T, Y)\}$ with respect to the filtration $\{\mathcal{F}_n^{\boldsymbol{\xi}}\}$. Setting $a = 26^{11}$, at time $\widehat{\tau}$ exactly one player have won and all the others have lost, from which it follows that $M_{\widehat{\tau}} = a - \widehat{\tau}$. As each gambler quits after he wins the entire word, we have that always $|M_{n+1} - M_n| \leq a$, so $\{M_n\}$ is a MG of bounded increments. Since $\mathbf{P}(\widehat{\tau} \leq n + r \,|\, \mathcal{F}_n^{\boldsymbol{\xi}}) \geq \varepsilon$ for $\varepsilon = 26^{-11}$ and $r = 11$ we further know that $\mathbf{E}\widehat{\tau}$ is finite (see part (c) of Exercise **??**). Consequently, $\{M_{n\wedge\widehat{\tau}}\}$ is U.I. (see part (a) of Proposition **??**), and by Doob's optional stopping theorem, $\mathbf{E}M_{\widehat{\tau}} = \mathbf{E}M_0 = 0$. Equivalently, $\mathbf{E}\widehat{\tau} = a$. Since the word ABRACADABRA has partial repeats (ABRA and A), the answer for the stopping time τ corresponding to the first time that the monkey produces ABRACADABRA, is different. Indeed, following the preceding analysis in case of gambling scheme for ABRACADABRA, at time τ we have that three gamblers won, betting on ABRACADABRA, ABRA, and A (with all others losing), leading to $M_\tau = a' - \tau$ for $a' = 26^{11} + 26^4 + 26$, and consequently, to $\mathbf{E}\tau = a' > a$.
(c). Let $T_0 = 0$ and $T_k = \inf\{n \geq 11 : L_n = k\}$, $k \geq 1$, denote the k-th time the monkey typed PROBABILITY. These are almost surely finite $\{\mathcal{F}_n^{\boldsymbol{\xi}}\}$-stopping times and their increments $\tau_k = T_k - T_{k-1}$ are independent (iteratively apply part (a) of Exercise **??** to get the independence of $T_k \in m\mathcal{F}_{T_k}^{\boldsymbol{\xi}}$ and $\tau_{k+1} \in m\sigma(\xi_{T_k+r}, r \geq 1)$). Further, the word PROBABILITY has no partial repeats, so $\tau_k \geq 11$ for all k. That is, at each time T_k the monkey has to start afresh and type by chance this word, from which we deduce that τ_k are identically distributed as $\widehat{\tau}$ of part (b). We are thus in the *renewal theory* setting of Example **??** with $L_n = \sup\{k \geq 0 : T_k \leq n\}$ and from Exercise **??** we know that $n^{-1}L_n \stackrel{a.s.}{=} 1/\mathbf{E}\widehat{\tau}$. Further, recall part (b) and Exercise **??**(b) that $\mathbf{P}(\widehat{\tau} > kr) \leq (1-\varepsilon)^k$ for some r finite and $\varepsilon > 0$, hence by part (a) of Lemma **??**,

$$\mathbf{E}\widehat{\tau}^2 = 2\sum_{y=1}^{\infty} y\mathbf{P}(\widehat{\tau} \geq y) \leq 2r^2 \sum_{k=0}^{\infty} (k+1)\mathbf{P}(\widehat{\tau} > kr)$$

is finite. We thus deduce from the renewal theory CLT that $(L_n - n/a)/\sqrt{vn} \xrightarrow{\text{d}} G$ for the finite, positive constant $v = a^{-3}\text{Var}(\widehat{\tau})$ (c.f. part (b) of Exercise 3.2.9, applied to $Y_k = \tau_k/a$).

Exercise [5.5.19]

Since $Y_n \to Y_{-\infty}$ a.s. when $n \to -\infty$, clearly $|Y_{-\infty}| \leq \sup_n |Y_n| = Z$ is integrable by our assumption that Z is integrable. Further, by (cJensen) and monotonicity of the C.E. we have that for any $n \leq r$ finite,

$$|\mathbb{E}[Y_n|\mathcal{F}_n] - \mathbb{E}[Y_{-\infty}|\mathcal{F}_n]| \leq \mathbb{E}[|Y_n - Y_{-\infty}| \,|\, \mathcal{F}_n] \leq \mathbb{E}[W_r|\mathcal{F}_n]\,,$$

where $W_r := \sup_{n \leq r} |Y_n - Y_{-\infty}|$ is bounded by $2Z$ hence integrable. By Lévy's downward theorem $\mathbb{E}[W_r|\mathcal{F}_n] \to \mathbb{E}[W_r|\mathcal{F}_{-\infty}]$ as $n \to -\infty$ both a.s. and in L^1. Thus, considering the limit $n \to -\infty$ of the preceding inequality and of the corresponding one between the expectations of both sides, we deduce that for any $r \leq 0$,

$$\limsup_{n \to -\infty} |\mathbb{E}[Y_n|\mathcal{F}_n] - \mathbb{E}[Y_{-\infty}|\mathcal{F}_n]| \leq \mathbb{E}[W_r|\mathcal{F}_{-\infty}]\,,$$

$$\limsup_{n \to -\infty} \|\mathbb{E}[Y_n|\mathcal{F}_n] - \mathbb{E}[Y_{-\infty}|\mathcal{F}_n]\|_1 \leq \mathbb{E}W_r\,.$$

The sequence $\{W_r\}$ is dominated by the integrable $2Z$ and by its construction, $W_r \downarrow 0$ as $r \downarrow -\infty$. Hence, by (DOM) and (cDOM), both $\mathbb{E}W_r \downarrow 0$ and $\mathbb{E}[W_r|\mathcal{F}_{-\infty}] \downarrow 0$ for $r \downarrow -\infty$. We have thus shown that $\mathbb{E}[Y_n|\mathcal{F}_n] - \mathbb{E}[Y_{-\infty}|\mathcal{F}_n] \to 0$ both a.s. and in L^1 when $n \to -\infty$. We are now done, since from Lévy's downward theorem, when $n \to -\infty$ also $\mathbb{E}[Y_{-\infty}|\mathcal{F}_n] \to \mathbb{E}[Y_{-\infty}|\mathcal{F}_{-\infty}]$ (a.s. and in L^1).

Exercise [5.4.10]

1. Since τ is a stopping time for $\mathcal{F}_n^{\boldsymbol{\xi}}$, we know that $I_{k \leq \tau} = 1 - I_{\tau \leq k-1}$ is measurable on $\mathcal{F}_{k-1}^{\boldsymbol{\xi}}$, and hence independent of ξ_k. Consequently, with ξ_k identically distributed,

$$\mathbf{E}\xi_k I_{k \leq \tau} = \mathbf{E}\xi_k \mathbf{P}(k \leq \tau) = \mathbf{E}\xi_1 \mathbf{P}(\tau \geq k)\,.$$

   The representation $S_\tau = \sum_{k=1}^\infty \xi_k I_{k \leq \tau}$ applies when $\tau < \infty$ a.s. (hence when $\mathbf{E}\tau < \infty$ as assumed). Thus, by Fubini's theorem with respect to the product of the probability measure $\mathbf{P}$ and the counting measure on $k \in \{1, 2, \ldots\}$, we find that

$$\mathbf{E}S_\tau = \mathbf{E}[\sum_{k=1}^\infty \xi_k I_{k \leq \tau}] = \sum_{k=1}^\infty \mathbf{E}[\xi_k I_{k \leq \tau}] = \mathbf{E}[\xi_1] \sum_{k=1}^\infty \mathbf{P}(\tau \geq k) = \mathbf{E}[\xi_1]\mathbf{E}[\tau]\,,$$

   where the integrability condition for Fubini's theorem is merely that

$$\sum_{k=1}^\infty \mathbf{E}[|\xi_k| I_{k \leq \tau}] = \sum_{k=1}^\infty \mathbf{E}[|\xi_1|]\mathbf{P}(\tau \geq k) = \mathbf{E}[|\xi_1|]\mathbf{E}[\tau]$$

   is finite. As the latter follows from the assumed finiteness of $\mathbf{E}\tau$, we are done.

2. Without loss of generality assume that $\mathbf{E}\xi_1 = 0$, for otherwise, we can always work with $\{\xi_i - \mathbf{E}\xi_i\}$ which are i.i.d. and have the same variance as $\{\xi_i\}$. Setting $v := \mathrm{Var}(\xi_1)$, recall that $X_n = S_n^2 - vn$ is a martingale with $X_0 = 0$. Since $\mathbf{E}X_{n \wedge \tau} = \mathbf{E}X_0 = 0$ and $\tau < \infty$ a.s., we have by monotone convergence that as $n \to \infty$

$$\mathbf{E}S_{n \wedge \tau}^2 = v\mathbf{E}[n \wedge \tau] \uparrow v\mathbf{E}\tau < \infty\,.$$

   This shows that the martingale $\{S_{n \wedge \tau}\}$ is L^2-bounded and by Doob's L^2-martingale convergence theorem, $S_{n \wedge \tau} \to S_\tau$ in L^2, resulting with

$$\mathbf{E}S_\tau^2 = \lim_{n \to \infty} \mathbf{E}S_{n \wedge \tau}^2 = v\mathbf{E}\tau\,.$$

3. When establishing Wald's identity in part (a) we used the condition $\mathbf{E}\tau < \infty$ only for justifying the representation $S_\tau = \sum_{k=1}^\infty \xi_k I_{k\le\tau}$ and for establishing Fubini's theorem integrability condition when interchanging the order of summation (over k) and expectation (with respect to $\mathbf{P}$). For a non-negative sequence ξ_k we have a non-negative integrand, in which case Fubini's theorem requires no integrability assumption (under the convention that $0 \times \infty = 0$), and the representation for S_τ is then valid even when $\tau(\omega) = \infty$.

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1. For $p = 1 - q \ge 1/2$ we know from parts (c) and (d) of Exercise **??** that τ_b is finite a.s. Further, by definition $S_{n\wedge\tau_b} \le b$ for all n and as $M(\lambda) = pe^\lambda + qe^{-\lambda} \ge 1$ whenever $\lambda \ge 0$, in this case

$$M_{n\wedge\tau_b} = \exp(\lambda S_{n\wedge\tau_b} - (n\wedge\tau_b)\log M(\lambda)) \le \exp(\lambda b)\,.$$

Thus, $\{M_{n\wedge\tau_b}\}$ is a uniformly bounded, hence U.I. martingale. With $S_{\tau_b} = b$, it then follows from Doob's optional stopping theorem that

$$1 = \mathbf{E}M_0 = \mathbf{E}M_{\tau_b} = e^{\lambda b}\mathbf{E}[M(\lambda)^{-\tau_b}]\,.$$

2. Setting $0 < s < 1$ there exists for $p \ge 1/2$ a unique $\lambda > 0$ such that $M(\lambda) = pe^\lambda + qe^{-\lambda} = 1/s$. Indeed, solving $qsx^2 - x + ps = 0$ for $x = e^{-\lambda}$ in $(0,1)$, we find from part (a) that $\mathbf{E}[s^{\tau_b}] = x^b$ and

$$\mathbf{E}[s^{\tau_1}] = x = \frac{1-\sqrt{1-4pqs^2}}{2qs}\,,$$

when $q \in (0, 1/2]$, whereas $x = s$ in the trivial case $q = 0$.

3. Since $\tau_{a,b} = \min(\tau_b, \tau_{-a})$ is finite a.s. and $\tau_{-a} \ge a$ we have that

$$\mathbf{P}(\tau_b < a) \le \mathbf{P}(\tau_b < \tau_{-a}) \le \mathbf{P}(\tau_b < \infty)\,.$$

Consequently, $\mathbf{P}(\tau_b < \tau_{-a}) \to \mathbf{P}(\tau_b < \infty)$ as $a \to \infty$. To complete the proof recall that from Corollary **??** we have that

$$\mathbf{P}(\tau_b < \tau_{-a}) = 1 - r = \frac{1 - e^{-\lambda_* a}}{e^{\lambda_* b} - e^{-\lambda_* a}} \to e^{-\lambda_* b}$$

when $a \to \infty$ (as $\lambda_* > 0$).

4. Clearly, $\{\tau_b < \infty\}$ if and only if $\{Z \ge b+1\}$. Hence, for any positive integer b,

$$\mathbf{P}(Z = b) = \mathbf{P}(Z \ge b) - \mathbf{P}(Z \ge b+1) = (1 - e^{-\lambda_*})e^{-\lambda_*(b-1)}\,.$$