Stat 310B/Math 230B Theory of Probability

Homework 4 Solutions

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Exercise [5.2.10]

It is easy to check that $\{S_n^2 - s_n^2\}$ is a martingale (but you should do it). Let $A = \{\max_{k=1}^n |S_k| > x\}$ and $\tau = \inf\{k : |S_k| > x\} \wedge n$. Then

$$(x + K)^2 \mathbf{P}(A) \geq \mathbf{E}(S_\tau^2 I_A) \geq \mathbf{E}((S_\tau^2 - s_\tau^2) I_A) = \mathbf{E}((S_n^2 - s_n^2) I_A),$$

since τ is bounded by n implies $\mathbf{E}(S_\tau^2 - s_\tau^2) = \mathbf{E}(S_n^2 - s_n^2)$ (see Corollary 5.1.33), and $\tau = n$ on A^c. Now add $\mathbf{E}(S_n^2 I_{A^c})$ to both sides to get

$$(x + K)^2 \mathbf{P}(A) + \mathbf{E}(S_n^2 I_{A^c}) \geq \mathbf{E}(S_n^2) - s_n^2 \mathbf{P}(A).$$

But $\mathbf{E}(S_n^2 I_{A^c}) \leq (x + K)^2 \mathbf{P}(A^c)$ so

$$(x + K)^2 = (x + K)^2 \mathbf{P}(A) + (x + K)^2 \mathbf{P}(A^c) \geq \mathbf{E}(S_n^2) - s_n^2 \mathbf{P}(A) = s_n^2 \mathbf{P}(A^c).$$

Exercise [5.2.14]

1. Note that

$$\log M_n = S_n = \sum_{i=1}^n X_i$$

for the i.i.d. variables $X_i = \log Y_i$. Let $X_1^{(m)} = \max(X_1, -m)$. Since

$$\exp(X_1^{(m)}) = \max(Y_1, e^{-m}) \leq Y_1 + e^{-m}$$

and $\mathbb{E}Y_1 = 1$, it follows from Jensen's inequality that

$$\mathbb{E}X_1^{(m)} \leq \log \mathbb{E}[\exp(X_1^{(m)})] \leq \log(1 + e^{-m})\,.$$

Since $(X_1)_+ = X_1^{(0)}$ we thus deduce that $\mathbb{E}[(X_1)_+]$ is finite, which suffices for the strong law of large numbers to apply (c.f. Theorem **??**). That is, $n^{-1} \log M_n$ converges almost sure to

$$\mu = \mathbb{E}X_1 \leq \lim_{m\to\infty} \mathbb{E}X_1^{(m)} \leq 0\,.$$

Further, if $\mu = 0$ then yet another application of Jensen's inequality results with

$$1 = \exp(\mu/2) \leq \mathbb{E}\exp(X_1/2) = \mathbb{E}\sqrt{Y_1} \leq \sqrt{\mathbb{E}Y_1} = 1$$

i.e. with $\mathrm{Var}(\sqrt{Y_1}) = 0$, which is ruled out by our assumption that Y_1 is a non-constant random variable.

2. Since a.s. $n^{-1} \log M_n \to \mu < 0$, we have that with probability one $n^{-1} \log M_n \leq \mu/2$ for all n large enough. That is, $M_n \leq \exp(\mu n/2) \to 0$ as $n \to \infty$. For $\{M_n\}$ uniformly integrable this would imply that $M_n \to 0$ in L^1 and in particular that $\mathbb{E}M_n \to 0$ as $n \to \infty$. However, clearly $\mathbb{E}M_n = 1$ for all n, so necessarily $\{M_n\}$ is not U.I.

3. If Doob's L^p maximal inequality applies for $p = 1$, then for any non-negative martingale X_n

$$\mathbb{E}[\max_{k \le n} X_k] \le q\mathbb{E}X_n = q\mathbb{E}X_0 < \infty \,.$$

Taking $n \to \infty$ it then follows that $\sup_k X_k$ is integrable, hence that $\{X_n\}$ is uniformly integrable. As we have seen in part (b) a counter example to the latter statement, we conclude that Doob's L^p maximal inequality can not extend as is to $p = 1$.

Exercise [5.3.9]

For part (a) let $X_n = -1/n$ (non-random), so $X_n^2 = 1/n^2$. Then, obviously $\{X_n\}$ is a submartingale whereas $\{X_n^2\}$ is a super-martingale. No contradiction with Proposition **??** since $\Phi(x) = x^2$ is a decreasing function on the support $(-\infty, 0)$ of the sequence $\{X_n\}$. For part (b) consider $S_n = \sum_{k=1}^n \xi_k$ and independent $\{\xi_k\}$ such that $\xi_k = k^2 - 1$ with probability $1/k^2$ and otherwise $\xi_k = -1$. Clearly, $\mathbf{E}\xi_k = 0$ for all k so $\{S_n\}$ is a martingale. However, by Borel-Cantelli II we have that $\mathbf{P}(\xi_k = -1, e.v.) = 1$ hence $S_n \to -\infty$ almost surely. Theorem **??** does not apply for this example as $\mathbb{E}(S_n)_-$ is unbounded. Indeed, take ℓ finite and large enough so $\sum_{k>\ell} k^{-2} \le 1/2$ and set $b = \sum_{k=1}^{\ell} k^2$. Then, with probability at least half, $\xi_k = -1$ for all $k > \ell$ hence $S_n \le b - n$. Consequently, $\mathbb{E}[(S_n)_-] \ge \frac{1}{2}(n-b)_+ \to \infty$ as $n \to \infty$.

Exercise [5.3.10]

Consider the adapted processes $Q_n = \prod_{i=1}^n (1 + Y_i) \ge Q_0 = 1$ and $W_n = (1 + X_n)/Q_{n-1}$. Since $\sum_i Y_i < \infty$ a.s. and Y_i are non-negative, Q_n converges a.s. to a finite limit, say $Q_\infty = \sup_n Q_n$. Therefore, if W_n converges a.s. to a finite limit then so does $X_n = W_n Q_{n-1} - 1$. Note that W_n is integrable and

$$\mathbf{E}(W_{n+1}|\mathcal{F}_n) \le (1 + Y_n)(1 + X_n)/Q_n = W_n \;,$$

implying that W_n is a super-martingale. Further, X_n is non-negative, hence so is W_n which by Doob's convergence theorem then converges a.s. to a finite limit.

Exercise [5.2.11]

The same line of reasoning applies in all three parts of this exercise. Namely, for a certain convex non-negative function $\Phi(\cdot)$ the relevant inequality trivially holds when $\mathbb{E}\Phi(Y_n)$ is infinite. In part (a) assuming $\mathbb{E}\Phi(Y_n) < \infty$ for the non-decreasing $\Phi(y) = (y)_+^p$, we have from the L^p-maximal inequalities that $\mathbb{E}\Phi(Y_k)$ is finite for all $k \le n$, so $\{X_k = \Phi(Y_k), k \le n\}$ is a sub-MG (by Proposition **??**). In parts (b) and (c) we start with a martingale $\{Y_k\}$ so again $X_k = \Phi(Y_k)$ is a sub-MG although the relevant functions $\Phi(y) = |y|^p$ and $\Phi(y) = (y + c)^2$, for $c \ge 0$, respectively, are non-decreasing only for $y \ge 0$. In all three cases if $Y_k \ge y > 0$ then $X_k \ge x = \Phi(y)$, so we bound $\mathbf{P}(\max_k Y_k \ge y)$ by $\mathbf{P}(\max_k X_k \ge x)$ which in turn is further bounded via Doob's inequality. This procedure results with the stated bounds of (a) and (b), whereas in part (c) it provides the bound

$$\mathbf{P}(\max_{k=0}^{n} Y_k \ge y) \le (y + c)^{-2}\mathbb{E}(Y_n + c)^2 \,.$$

However, here $\mathbb{E}Y_n = \mathbb{E}Y_0 = 0$, so the preceding bound simplifies to $(y + c)^{-2}(\mathbb{E}Y_n^2 + c^2)$ and setting $c = \mathbb{E}Y_n^2/y$ yields after a bit of algebra the stated bound of part (c).

Exercise [5.2.19]

1. Since $|W_n|$ and $|Y_n|$ are both bounded by $|X_n^1| + |X_n^2|$, the integrability of W_n and Y_n follows from that of X_n^1 and X_n^2. It is also easy to see that for an $\mathcal{F}_n$-stopping time τ and $\mathcal{F}_n$-adapted $\{X_n^1\}$, $\{X_n^2\}$ the processes $\{W_n\}$ and $\{Y_n\}$ are also $\mathcal{F}_n$-adapted. Our assumption that $X_\tau^1 \ge X_\tau^2$ implies that

$$W_n \le W_n + (X_\tau^1 - X_\tau^2)I_{\{\tau=n\}} = Y_n \,.$$

Further, τ is an $\mathcal{F}_n$-stopping time, so the event $\{\tau < n\} = \{\tau \leq n-1\}$ and its complement $\{\tau \geq n\} = \{\tau > n-1\}$ are both in $\mathcal{F}_{n-1}$. Hence, taking out the known $I_{\{\tau<n\}}$ and $I_{\{\tau\geq n\}}$ we deduce from the sup-MG property of X_n^1 and X_n^2 that

$$\begin{aligned}\mathbb{E}[W_n|\mathcal{F}_{n-1}] &\leq \mathbb{E}[Y_n|\mathcal{F}_{n-1}] \\ &= I_{\{\tau\geq n\}}\mathbb{E}[X_n^1|\mathcal{F}_{n-1}] + I_{\{\tau<n\}}\mathbb{E}[X_n^2|\mathcal{F}_{n-1}] \\ &\leq X_{n-1}^1 I_{\{\tau>n-1\}} + X_{n-1}^2 I_{\{\tau\leq n-1\}} = W_{n-1} \leq Y_{n-1}\,.\end{aligned}$$

That is, both $\{W_n\}$ and $\{Y_n\}$ are sup-MGs for $\mathcal{F}_n$.

2. Fixing a positive integer n, consider the partition of Ω to the disjoint events $\{A_\ell, B_\ell, \ell \geq 0\}$ where $A_\ell = \{\omega : \tau_\ell(\omega) < n \leq \theta_\ell(\omega)\}$ and $B_\ell = \{\omega : \theta_\ell(\omega) < n \leq \tau_{\ell+1}(\omega)\}$ for $\ell = 0, 1, \ldots$. As τ_ℓ, $\ell \geq 0$ and θ_ℓ, $\ell \geq 0$ are stopping time for the filtration $\{\mathcal{F}_n\}$ it is easy to check that each of these events is in $\mathcal{F}_{n-1}$. We claim that if the event A_ℓ occurs, then $Z_n - Z_{n-1} \leq 0$. Indeed, for $\omega \in A_\ell$ either $Z_n = Z_{n-1} = a^{-\ell}b^\ell$ or in case $n = \theta_\ell$, by definition $X_n \leq a$ and $Z_n - Z_{n-1} = a^{-\ell}b^\ell(X_n/a - 1) \leq 0$. We further claim that if the event B_ℓ occurs, then $Z_n - Z_{n-1} \leq a^{-\ell-1}b^\ell(X_n - X_{n-1})$. Indeed, for $\omega \in B_\ell$ the preceding inequality holds with equality except when $n = \tau_{\ell+1}$ in which case it follows from the fact that $X_{\tau_{\ell+1}} \geq b$. Thus, decomposing $Z_n - Z_{n-1}$ according to this partition of Ω, we deduce that $Z_n - Z_{n-1} \leq V_n(X_n - X_{n-1})$ with $V_n = \sum_{\ell=0}^\infty a^{-\ell-1}b^\ell I_{B_\ell}$ non-negative and measurable on $\mathcal{F}_{n-1}$. Further, as $\theta_\ell \geq 2\ell$, the disjoint events B_ℓ are empty for $\ell \geq n/2$, hence $V_n \leq a^{-1}(b/a)^{n/2}$ is bounded. Taking out the known V_n and recalling that $(X_n, \mathcal{F}_n)$ is a sup-MG, we see that $V_n(X_n - X_{n-1})$ is integrable with
$$\mathbb{E}[V_n(X_n - X_{n-1})|\mathcal{F}_{n-1}] = V_n\mathbb{E}[X_n - X_{n-1}|\mathcal{F}_{n-1}] \leq 0\,.$$
Consequently, as $Z_0 \leq 1$, the non-negative
$$Z_n \leq Z_0 + \sum_{k=1}^n V_k(X_k - X_{k-1})$$
are integrable, with
$$\mathbb{E}[Z_n - Z_{n-1}|\mathcal{F}_{n-1}] \leq V_n\mathbb{E}[X_n - X_{n-1}|\mathcal{F}_{n-1}] \leq 0\,,$$
establishing that $(Z_n, \mathcal{F}_n)$ is a sup-MG, as claimed.

3. We have that $\theta_0 = 0$ if and only if $X_0 \leq a$ and consequently, $Z_0 = \min(1, X_0/a)$. Recall that $Z_n \geq 0$ and $Z_{\tau_\ell} = a^{-\ell}b^\ell$ when τ_ℓ is finite, that is, when $U_\infty(a,b) \geq \ell$. Further from part (b) we have that $\mathbb{E}Z_0 \geq \mathbb{E}[Z_{n\wedge\tau_\ell}]$ for any $n, \ell \geq 1$. Thus, taking $n \to \infty$ and applying Fatou's lemma we see that
$$\mathbb{E}[\min(X_0/a, 1)] \geq \mathbb{E}[\liminf_{n\to\infty} Z_{n\wedge\tau_\ell}] \geq \mathbb{E}[Z_{\tau_\ell} I_{\{\tau_\ell<\infty\}}] = \left(\frac{b}{a}\right)^\ell \mathbf{P}(U_\infty(a,b) \geq \ell).$$