Stat 310B/Math 230B Theory of Probability

Homework 3 Solutions

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Exercise [5.1.8]

(a). Recall Proposition 5.1.5 that for all $k \geq 1$ a.s. $\mathbf{E}[D_k|\mathcal{F}_{k-1}] = 0$. Obviously this a.s. zero random variable has zero expectation, which amounts to D_k having zero mean. Further, by the tower property, a.s. $\mathbf{E}[D_k|\mathcal{F}_n] = 0$ for all $k > n \geq 0$.

Clearly, the differences $D_n = X_n - X_{n-1}$ of a square-integrable process $\{X_n\}$ are also square-integrable, from which we deduce by the Cauchy-Schwarz inequality that $D_k D_m$ is integrable for any $k, m \geq 0$. In case $k > m \geq n$ we thus get by the tower property and taking out the known D_m, that

$$\mathbf{E}[D_k D_m|\mathcal{F}_n] = \mathbf{E}[D_m \mathbf{E}(D_k|\mathcal{F}_m)|\mathcal{F}_n] = 0\,.$$

Considering the expectation of the latter identity, we conclude by the trivial tower property (4.2.1) that $\mathbf{E}D_k D_m = 0$, i.e. D_n are uncorrelated.

(b). By linearity of the C.E., for any $\ell \geq n \geq 0$,

$$\mathbf{E}[X_\ell - X_n|\mathcal{F}_n] = \sum_{k=n+1}^{\ell} \mathbf{E}[D_k|\mathcal{F}_n] = 0\,.$$

The same applies for $\{Y_n\}$, hence upon taking out the known $X_n, Y_n \in m\mathcal{F}_n$, by linearity of the C.E. also

$$\Delta_{\ell,n} := \mathbf{E}[Y_n(X_\ell - X_n) + X_n(Y_\ell - Y_n)|\mathcal{F}_n] = 0\,.$$

Clearly, for any $\ell \geq n \geq 0$,

$$\mathbf{E}[X_\ell Y_\ell|\mathcal{F}_n] - X_n Y_n - \Delta_{\ell,n} = \mathbf{E}[(X_\ell - X_n)(Y_\ell - Y_n)|\mathcal{F}_n]\,,$$

from which we thus get the first claimed identity. In particular, considering $\ell = k$ and $n = k-1$ we have that for any $k \geq 1$,

$$\mathbf{E}[X_k Y_k|\mathcal{F}_{k-1}] - \mathbf{E}[X_{k-1}Y_{k-1}|\mathcal{F}_{k-1}] = \mathbf{E}[(X_k - X_{k-1})(Y_k - Y_{k-1})|\mathcal{F}_{k-1}]\,.$$

Taking now $k > n$ and considering the C.E. of both sides given $\mathcal{F}_n \subseteq \mathcal{F}_{k-1}$, the preceding identity yields by the tower property that

$$\mathbf{E}[X_k Y_k|\mathcal{F}_n] - \mathbf{E}[X_{k-1}Y_{k-1}|\mathcal{F}_n] = \mathbf{E}[(X_k - X_{k-1})(Y_k - Y_{k-1})|\mathcal{F}_n]\,.$$

Summing over $k = n+1, \ldots, \ell$ provides the second claimed identity.

(c). Considering the second identity of part (b) with $D_k = X_k - X_{k-1} = Y_k - Y_{k-1}$ we have (by linearity of the C.E.), that for all $\ell \geq n \geq 0$,

$$Z_n := \mathbf{E}[\sum_{k=n+1}^{\ell} D_k^2|\mathcal{F}_n] = \mathbf{E}[X_\ell^2|\mathcal{F}_n] - X_n^2 \leq C^2\,, \qquad (*)$$

where the inequality in (*) is due to the assumed bound $X_\ell^2 \leq C^2$ (and monotonicity of the C.E.). Next, applying the tower property and taking out the known $D_n^2 \in m\mathcal{F}_n$, it follows from (*) that

$$\sum_{n=1}^{\ell} \mathbf{E}[\sum_{k=n+1}^{\ell} D_n^2 D_k^2] = \sum_{n=1}^{\ell} \mathbf{E}[D_n^2 Z_n] \leq C^2 \mathbf{E}[\sum_{k=1}^{\ell} D_k^2]\,. \qquad (**)$$

Further, considering the expectation of Z_0 we deduce from (*), by yet another application of the tower property, that

$$\mathbf{E}[\sum_{k=1}^{\ell} D_k^2] = \mathbf{E}[Z_0] \le C^2 . \qquad (***)$$

Next note that by assumption for any $k \ge 1$,

$$D_k^2 \le (|X_k| + |X_{k-1}|)^2 \le (2C)^2 = 4C^2 ,$$

hence from (***) we get that

$$\mathbf{E}[\sum_{k=1}^{\ell} D_k^4] \le 4C^2 \mathbf{E}[\sum_{k=1}^{\ell} D_k^2] \le 4C^4 .$$

Similarly, combining (**) and (***) we deduce that

$$2 \sum_{n=1}^{\ell} \mathbf{E}[\sum_{k=n+1}^{\ell} D_n^2 D_k^2] \le 2C^4 ,$$

whereby, upon adding the latter two inequalities we conclude that, as claimed,

$$\mathbf{E}\Big[(\sum_{k=1}^{\ell} D_k^2)^2\Big] = \mathbf{E}\Big[\sum_{k=1}^{\ell} D_k^4\Big] + 2\sum_{n=1}^{\ell} \mathbf{E}[\sum_{k=n+1}^{\ell} D_n^2 D_k^2] \le 6C^4 .$$

Exercise [5.1.15]

1. Since τ is an $\mathcal{F}_n$-stopping time, the event $\{\tau > n\}$ is in $\mathcal{F}_n$ for any n. Hence, the hypothesis of the exercise results for $n = (k-1)r$ with

$$\begin{aligned} \mathbf{P}(\tau > kr) &= \mathbf{P}(\tau > kr \text{ and } \tau > n) \\ &= \mathbf{E}[I_{\tau > n} \mathbf{P}(\tau > n + r | \mathcal{F}_n)] \le (1-\varepsilon)\mathbf{P}(\tau > (k-1)r) \end{aligned}$$

   and the stated geometric bound follows by induction on k.

2. The geometric bound on $\mathbf{P}(\tau > kr)$ implies that τ/r has finite expectation (see part (a) of Lemma 1.4.31), hence so does τ.

Exercise [5.1.26]

We start with the integrability of $f(S_n)$. To this end, recall that $|S_n| \le n$ so by the hint we know that $f(S_n) \ge \inf_{|y| \le n+|x|} f(y)$ is bounded below. You can easily check that if $f(\cdot)$ is super-harmonic, so is $f(\cdot) + c$ for any $c \in \mathbb{R}$ and hence assume hereafter without loss of generality that $f(y) \ge 0$ whenever $|y| \le n + |x|$. Then, since $S_k = S_{k-1} + \xi_k$ with ξ_k uniform on $B(0,1)$ and independent of S_{k-1}, we have by Fubini's theorem that

$$\begin{aligned} \mathbf{E} f(S_k) &= \mathbf{E}[\frac{1}{|B(0,1)|} \int_{B(0,1)} f(S_{k-1} + z) dz] \\ &= \mathbf{E}[\frac{1}{|B(0,1)|} \int_{B(S_{k-1},1)} f(y) dy] \le \mathbf{E} f(S_{k-1}) \end{aligned}$$

since $f(\cdot)$ is super-harmonic. Iterating this inequality over $k = n, \ldots, 1$ we deduce that $\mathbf{E} f(S_n) \le f(x)$ is finite, i.e. $f(S_n)$ is integrable.

Obviously, S_n is adapted to the filtration $\mathcal{F}_n = \sigma(\xi_k, k \le n)$ and hence so is $f(S_n)$. Further, as we have seen already, the R.C.P.D. of S_{n+1} given $\mathcal{F}_n$ is precisely the uniform measure on the ball $B(S_n, 1)$. Hence,

$$\mathbf{E}[f(S_{n+1})|\mathcal{F}_n] = \frac{1}{|B(0,1)|}\int_{B(S_n,1)} f(y)dy \le f(S_n)$$

by the fact that $f(\cdot)$ is a super-harmonic function.

Finally, recall that if $f(S_n)$ is a sup-MG with respect to some filtration $\mathcal{F}_n$ then it is also a sup-MG (with respect to its canonical filtration), see the remark following Definition 5.1.4.

Exercise [5.1.35]

(a). Since τ is an $\mathcal{F}_n$-stopping-time, clearly $\Omega \cap \{\omega : \tau(\omega) \le n\} = \{\omega : \tau(\omega) \le n\} \in \mathcal{F}_n$ for all n, hence $\Omega \in \mathcal{F}_\tau$.
Next, if $A \in \mathcal{F}_\tau$ then for each n both the event $A \cap \{\omega : \tau(\omega) \le n\}$ and its complement are in the σ-algebra $\mathcal{F}_n$. Consequently, with τ an $\mathcal{F}_n$-stopping time we deduce that

$$A^c \cap \{\omega : \tau(\omega) \le n\} = \{\omega : \tau(\omega) \le n\} \cap \Big(A \cap \{\omega : \tau(\omega) \le n\}\Big)^c \in \mathcal{F}_n\,.$$

This applies for all n, so by definition $A^c \in \mathcal{F}_\tau$.
Finally, consider events $\{A_\ell\}_{\ell=1}^\infty$ such that each $A_i \in \mathcal{F}_\tau$, namely $A_i \cap \{\omega : \tau(\omega) \le n\} \in \mathcal{F}_n$ for each i and n. Then, for any n,

$$\Big(\bigcup_\ell A_\ell\Big) \bigcap \{\omega : \tau(\omega) \le n\} = \bigcup_\ell \Big(A_\ell \bigcap \{\omega : \tau(\omega) \le n\}\Big) \in \mathcal{F}_n\,.$$

That is $\cup_\ell A_\ell \in \mathcal{F}_\tau$, completing the proof that $\mathcal{F}_\tau$ is a σ-algebra.
We finish this part by considering non-random $\tau(\omega) = m$, in which case $A \cap \{\omega : \tau(\omega) \le n\} = A$ for any $A \in \mathcal{F}_\infty$ and integer $n \ge m$, whereas $A \cap \{\omega : \tau(\omega) \le n\} = \emptyset$ in case $n < m$. Thus, the requirement that $A \cap \{\omega : \tau(\omega) \le n\} \in \mathcal{F}_n$ for all $n \ge 0$ is equivalent to $A \in \mathcal{F}_n$ for all $n \ge m$, i.e. to $A \in \mathcal{F}_m$. That is, in this case $\mathcal{F}_\tau = \mathcal{F}_m$, as claimed.
(b). To prove that $X_\tau \in m\mathcal{F}_\tau$, fix a Borel set B and note that for any finite n,

$$\begin{aligned}\{X_\tau \in B\} \cap \{\omega : \tau(\omega) \le n\} &= \bigcup_{\ell=1}^n \Big(\{X_\tau \in B\} \cap \{\omega : \tau(\omega) = \ell\}\Big) \\ &= \bigcup_{\ell=1}^n \Big(\{X_\ell \in B\} \cap \{\omega : \tau(\omega) = \ell\}\Big).\end{aligned}$$

Since $\{X_\ell\}$ is adapted to the filtration $\{\mathcal{F}_\ell\}$ with respect to which τ is a stopping time, we deduce from the latter identity that $\{X_\tau \in B$ and $\tau \le n\} \in \mathcal{F}_n$ for any finite n, so it only remains to show that $\{X_\tau \in B\} \in \mathcal{F}_\infty$. To see this, note that since $\{X_\tau \in B, \tau \le n\} \uparrow \{X_\tau \in B, \tau < \infty\}$, the latter set is in $\mathcal{F}_\infty$, whereas by definition

$$\{X_\tau \in B, \tau = \infty\} = \{X_\infty \in B\} \bigcap \{\tau < \infty\}^c$$

is the intersection of two elements of $\mathcal{F}_\infty$, hence also in $\mathcal{F}_\infty$.
Next note that $\{\tau \le \ell \wedge n\} \in m\mathcal{F}_n$ for any two finite integers $\ell \ge 0$, $n \ge 0$. Consequently the π-system of sets $\{\tau \le \ell\}$, $\ell \ge 1$, is contained in $\mathcal{F}_\tau$, from which it follows that the σ-algebra $\sigma(\tau)$ generated by this π-system, is also contained in $\mathcal{F}_\tau$.
Finally, $X_k I_{\{\tau=k\}} = X_\tau I_{\{\tau=k\}}$, so with $X_\tau \in m\mathcal{F}_\tau$ and $\tau \in m\mathcal{F}_\tau$, we conclude that $X_k I_{\{\tau=k\}} \in m\mathcal{F}_\tau$.
(c). Considering part (b) for $X_k := \mathbf{E}[Y_k|\mathcal{F}_k] \in m\mathcal{F}_k$, we have that $\mathbf{E}[Y_k|\mathcal{F}_k] I_{\{\tau=k\}} \in m\mathcal{F}_\tau$, and so it suffices to show that for any $A \in \mathcal{F}_\tau$,

$$\mathbf{E}\Big[\mathbf{E}[Y_\tau|\mathcal{F}_\tau] I_{\{\tau=k\}} I_A\Big] = \mathbf{E}\Big[\mathbf{E}[Y_k|\mathcal{F}_k] I_{\{\tau=k\}} I_A\Big]. \qquad (*)$$

Since $A \cap \{\tau = k\} \in \mathcal{F}_\tau$ (by part (b)), by definition of the C.E. the LHS of (*) equals to $\mathbf{E}[Y_\tau I_{\{\tau=k\}} I_A]$. As $A \in \mathcal{F}_\tau$, we have that $A \cap \{\tau = k\} \in \mathcal{F}_k$, so again by definition of the C.E. the RHS of (*) equals to $\mathbf{E}[Y_k I_{\{\tau=k\}} I_A]$. Clearly, $Y_k I_{\{\tau=k\}} = Y_\tau I_{\{\tau=k\}}$ so by the preceding we conclude that (*) holds for any $A \in \mathcal{F}_\tau$.
(d). Since $\theta \le \tau$, we have that $\{\tau \le n\} \subseteq \{\theta \le n\}$ for any non-random n. Hence,

$$A \cap \{\tau \le n\} = \Big(A \cap \{\theta \le n\}\Big) \cap \{\tau \le n\},$$

for any $A \in \mathcal{F}_\theta$, for which we have by definition that $A \cap \{\theta \le n\} \in \mathcal{F}_n$. Further, $\{\tau \le n\} \in \mathcal{F}_n$, so the same applies for the LHS of the preceding identity. That is, $A \cap \{\tau \le n\} \in \mathcal{F}_n$. This holds for any non-random $n \ge 0$, which by definition of $\mathcal{F}_\tau$ amounts to $A \in \mathcal{F}_\tau$. In conclusion, any set $A \in \mathcal{F}_\theta$ is also in $\mathcal{F}_\tau$, as claimed.

Exercise [5.1.13]

For any n the event $\{\tau \le n\} = \bigcup_{k=0}^n \{X_k \in B\}$ is clearly in $\mathcal{F}_n = \sigma(X_k, k \le n)$. Hence, τ is a stopping time with respect to this filtration. In contrast, $\{\theta \le n\} = \bigcap_{k>n} \{X_k \notin B\}$ which for independent, non-degenerate random variables X_k is certainly not in $\sigma(X_k, k \le n)$.

Exercise [5.1.14]

(a). Clearly, if $Z \overset{d}{=} -Z$ then

$$\mathbf{P}(Z \ge 0) \ge \mathbf{P}(Z > 0) + \frac{1}{2}\mathbf{P}(Z = 0) = \frac{1}{2}\,.$$

This applies for $Z = S_n - S_k = \sum_{i=k+1}^n \xi_i$ since $\xi_i \overset{d}{=} -\xi_i$ are independent random variables. Indeed, the characteristic function $\Phi_{\xi_i}(\theta) = \Phi_{-\xi_i}(\theta) = \Phi_{\xi_i}(-\theta)$ is then real valued by part (b) of Proposition 3.3.2, hence by Lemma 3.3.8 the characteristic function of $S_n - S_k$ is also real valued, and applying once more part (b) of Proposition 3.3.2 we conclude that $S_n - S_k \overset{d}{=} -(S_n - S_k)$.
(b). If $S_n > x > 0 = S_0$ then necessarily $1 \le \tau \le n$. By definition $x - S_\tau$ is negative, hence

$$\begin{aligned}\mathbf{P}(S_n > x) &= \mathbf{P}(1 \le \tau \le n, S_n - S_\tau > x - S_\tau)\\ &\ge \mathbf{P}(1 \le \tau \le n, S_n - S_\tau \ge 0) = \sum_{k=1}^n \mathbf{P}(\tau = k, S_n - S_k \ge 0)\end{aligned}$$

since the events $\{\tau = k, S_n - S_k \ge 0\}$ are disjoint.
Further, $\{\xi_i\}$ are independent variables, so for any $1 \le k \le n$ the event $\{S_n - S_k \ge 0\} \in \sigma(\xi_i, i > k)$ is independent of $\{\tau = k\}$ and by part (a)

$$\frac{1}{2}\mathbf{P}(\tau = k) \le \mathbf{P}(\tau = k)\mathbf{P}(S_n - S_k \ge 0) = \mathbf{P}(\tau = k, S_n - S_k \ge 0).$$

(c). With $S_0 = 0 < x$, we have from part (b) that

$$\mathbf{P}\left(\max_{k=1}^n S_k > x\right) = \mathbf{P}(\tau \le n) = \sum_{k=1}^n \mathbf{P}(\tau = k) \le 2\mathbf{P}(S_n > x).$$

(d). Here we set $\tau = \inf\{k \ge 0 : S_k \ge x\}$ and note that for integer $x > 0 = S_0$ and increments $\xi_i \in \{-1, 1\}$, necessarily $S_\tau = x$. Thus, for each $k = 0, 1, \ldots, n$,

$$\{S_n \ge x, \tau = k\} = \{S_n \ge x, \tau = k, S_k = x\} = \{S_n - S_k \ge 0, \tau = k\}\,.$$

Further, since $\{\tau = k\} \in \mathcal{F}_k^{\mathbf{S}}$ is independent of $S_n - S_k = \sum_{i=k+1}^n \xi_i$ and the event $\{S_n \geq x\}$ implies that $\{\tau \leq n\}$, we have that

$$\mathbf{P}(S_n \geq x) = \sum_{k=0}^n \mathbf{P}\Big(S_n \geq x,\, \tau = k\Big) = \sum_{k=0}^n \mathbf{P}(\tau = k)\mathbf{P}(S_n - S_k \geq 0)\,.$$

Recall that for symmetric SRW $S_n - S_k \stackrel{\mathrm{d}}{=} -(S_n - S_k)$ is integer valued (see solution of part (a)), hence

$$\mathbf{P}(S_n - S_k \geq 1) = \mathbf{P}(S_n - S_k \leq -1) = 1 - \mathbf{P}(S_n - S_k \geq 0)\,,$$

and by the same argument as before,

$$\mathbf{P}(S_n \geq x + 1) = \sum_{k=0}^n \mathbf{P}(\tau = k)\mathbf{P}(S_n - S_k \geq 1) = \sum_{k=0}^n \mathbf{P}(\tau = k) - \mathbf{P}(S_n \geq x)\,.$$

Since S_n is integer valued, we deduce from the latter identity that

$$\begin{aligned} 2\mathbf{P}(S_n \geq x) - \mathbf{P}(S_n = x) &= \mathbf{P}(S_n \geq x) + \mathbf{P}(S_n \geq x + 1) \\ &= \mathbf{P}(\tau \leq n) = \mathbf{P}(\max_{k=1}^n S_k \geq x) \qquad (*) \end{aligned}$$

as claimed.
With Z_n denoting the number of *strict* sign changes within the sequence $\{S_0 = 0, S_1, \ldots, S_n\}$, we proceed to show that $Z_{2n+1} \stackrel{\mathrm{d}}{=} (|S_{2n+1}| - 1)/2$, namely, that for any integer $r \geq 1$,

$$\mathbf{P}(Z_{2n+1} \geq r) = \mathbf{P}(|S_{2n+1}| \geq 2r + 1)$$

(this trivially extends to $r = 0$ since $|S_{2n+1}| \geq 1$). To this end, recall that S_{2n} is always an even integer and $S_{2n+1} = S_{2n} + \xi_{2n+1}$ has a symmetric law, hence

$$\begin{aligned} &\mathbf{P}(|S_{2n+1}| \geq 2r + 1) = 2\mathbf{P}(S_{2n+1} \geq 2r + 1) \\ = \;& 2[\mathbf{P}(S_{2n+1} \geq 2r + 1, \xi_{2n+1} = 1) + \mathbf{P}(S_{2n+1} \geq 2r + 1, \xi_{2n+1} = -1)] \\ = \;& 2\mathbf{P}(S_{2n} \geq 2r)\mathbf{P}(\xi_{2n+1} = 1) + 2\mathbf{P}(S_{2n} \geq 2r + 2)\mathbf{P}(\xi_{2n+1} = -1) \\ = \;& \mathbf{P}(S_{2n} \geq 2r) + \mathbf{P}(S_{2n} \geq 2r + 1) = \mathbf{P}(\max_{j=1}^{2n} S_j \geq 2r) \qquad (**) \end{aligned}$$

where the last identity is due to (*). Further, for any random walk $\{S_j, j = 1, \ldots, 2n\} \stackrel{\mathrm{d}}{=} \{S_{k+1} - S_1, k = 1, \ldots, 2n\}$, with the latter independent of S_1. Thus, for $r \geq 1$,

$$\begin{aligned} \mathbf{P}(\max_{j=1}^{2n} S_j \geq 2r) &= \mathbf{P}(\max_{k=1}^{2n} S_{k+1} - S_1 \geq 2r | S_1 = -1) \\ &= \mathbf{P}(\max_{k=1}^{2n+1} S_k \geq 2r - 1 | S_1 = -1) \qquad (***) \end{aligned}$$

We also note that Z_{2n+1} is invariant with respect to a global sign change of the path of the symmetric SRW, hence independent of $S_1 \in \{-1, 1\}$. Therefore, upon combining the identities (**) and (***), it suffices to show that for any $r \geq 1$,

$$\mathbf{P}(Z_{2n+1} \geq r | S_1 = -1) = \mathbf{P}(\max_{k=1}^{2n+1} S_k \geq 2r - 1 | S_1 = -1)\,. \qquad (****)$$

Next, let $T_0 = 1$ and for $j \geq 1$ define the stopping times $T_{2j-1} := \inf\{k \geq T_{2j-2} : S_k = 1\}$, and $T_{2j} := \inf\{k \geq T_{2j-1} : S_k = -1\}$, so in case $S_1 = -1$, the strict sign changes of the SRW are recorded at times $\{T_j, j \geq 1\}$, with

$$\{Z_{2n+1} \geq r\} = \{T_r \leq 2n + 1\} = \{\sum_{j=1}^r \tau_j \leq 2n\}\,,$$

where $\tau_j := T_j - T_{j-1}$, $j \geq 1$. Further, in this case $S_{T_j} = (-1)^{j+1}$ for all $j \geq 0$ and it is not hard to check that the random vectors $\underline{X}_j := \{\tau_j, \xi_{T_{j-1}+\ell}, \ell = 1, \dots, \tau_j\}$ are independent of each other (for example, iteratively apply part (a) of Exercise 5.1.38), with $\{\underline{X}_{2j-1}\}$ identically distributed (in j), each having the law of increments of the SRW starting at -1 and run up to its first hitting time of $+1$, whereas $\{\underline{X}_{2j}\}$ are likewise identically distributed (in j), as such increments for the SRW starting at $+1$ and run up to its first hitting time of -1. In particular $\underline{X}_2 \stackrel{\mathrm{d}}{=} -\underline{X}_1$ (namely, having the same law for these vectors length, while sign reversing their other coordinates). Since increments of the symmetric SRW are uniformly distributed, the path $k \mapsto S'_k$ induced by $\{(-1)^{j-1}\underline{X}_j, j = 1, \dots, r\}$ has the same joint law as the original SRW $\{S_k : k \leq T_r\}$ induced from $\{\underline{X}_j, j = 1, \dots, r\}$. However, in terms of the path $k \mapsto S'_k$ we have that $T_j = \inf\{k \geq T_{j-1} : S'_k = 2j - 1\}$ for $j \geq 1$. That is, the event $T_r \leq 2n + 1$ corresponds now to

$$\max_{k=1}^{2n+1} S'_k \geq 2r - 1 \,,$$

from which (****) follows.

Exercise [5.1.37]

Theorem 5.1.32 tells us that if $(X_n, \mathcal{F}_n)$ is a MG then so is $(X_{n \wedge \tau}, \mathcal{F}_n)$ for any $\mathcal{F}_n$-stopping time τ. Hence, $\{X_n\}$ is also a local martingale.

Conversely, suppose $\{X_n\}$ is an integrable local martingale and let $\tau_k \uparrow \infty$ denote the $\mathcal{F}_n$-stopping times such that $(X_{n \wedge \tau_k}, \mathcal{F}_n)$ is a MG. Then, by definition

$$X_{(n+1)\wedge\tau_k} - X_{n\wedge\tau_k} = (X_{n+1} - X_n) I_{\tau_k > n} \,,$$

and taking out the $\mathcal{F}_n$-known $I_{\{\tau_k > n\}}$ we deduce that

$$0 = \mathbf{E}[X_{(n+1)\wedge\tau_k} - X_{n\wedge\tau_k} | \mathcal{F}_n] = \mathbf{E}[X_{n+1} - X_n | \mathcal{F}_n] I_{\{\tau_k > n\}} \,.$$

Taking $k \uparrow \infty$ we further know that $I_{\{\tau_k > n\}} \to 1$ and conclude that $\mathbf{E}[X_{n+1} - X_n | \mathcal{F}_n] = 0$ a.s. for all n. That is, $(X_n, \mathcal{F}_n)$ is a MG.