Stat 310B/Math 230B Theory of Probability

Homework 2 Solutions

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Exercise [4.3.11]

In the proof of Theorem 4.1.2 we use the Radon-Nikodym theorem only for showing the existence of C.E. in case $X \in L^1(\Omega, \mathcal{F}, \mathbf{P})$ and $X \geq 0$.

So, assuming that X is non-negative and integrable we proceed to provide the existence of its C.E. without invoking the Radon-Nikodym theorem. To this end, let $X_n = \min(X, n)$, which is bounded, hence in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Applying Theorem 4.3.10 as in the notes, we have $Y_n \in L^2(\Omega, \mathcal{G}, \mathbf{P})$ such that $\mathbf{E}((X_n - Y_n)Z) = 0$ for all $Z \in L^2(\Omega, \mathcal{G}, \mathbf{P})$. In particular, setting $Z = I_A$ for an arbitrary $A \in \mathcal{G}$, we deduce that $Y_n = \mathbf{E}[X_n|\mathcal{G}]$. Since $n \mapsto X_n$ is a non-decreasing sequence of non-negative random variables, by the monotonicity and positivity of the C.E. the same applies for $n \mapsto Y_n$. Consequently, its limit $Y = \sup_n Y_n$ exists as a non-negative $\overline{\mathbb{R}}$-valued element of $m\mathcal{G}$ (see Theorem 1.2.22). Recall that $\mathbf{E}Y_n = \mathbf{E}X_n \leq \mathbf{E}X$ for all n, so by monotone convergence $\mathbf{E}Y \leq \mathbf{E}X$ is finite and in particular $Y \in L^1(\Omega, \mathcal{G}, \mathbf{P})$ is $\mathbb{R}$-valued. By monotone convergence of $Y_n I_A$ to $Y I_A$ and $X_n I_A$ to $X I_A$ it further holds that for all $A \in \mathcal{G}$

$$\mathbf{E}[Y I_A] = \lim_{n\to\infty} \mathbf{E}[Y_n I_A] = \lim_{n\to\infty} \mathbf{E}[X_n I_A] = \mathbf{E}[X I_A]\,.$$

Thus, Y satisfies (4.1.1) and we conclude that the C.E. $\mathbf{E}[X|\mathcal{G}] = Y$ exists.

Exercise [4.4.4]

1. By Definition 4.4.2, for any R.C.P.D. two conditions have to be met. As $g(\mathbb{S}) \subseteq \mathbb{T}$, we have that $\mathbf{E}[I_{\{Y\in\mathbb{T}\}}|\mathcal{G}] = 1$ and thus taking $\widehat{\mathbf{P}}_{Y|\mathcal{G}}(\mathbb{T}, \omega) = 1$ for all $\omega \in \Omega$ will be a version of the relevant C.E. For the same reason, this choice does not violate the requirement that the set function $\widehat{\mathbf{P}}_{Y|\mathcal{G}}(\cdot, \omega)$ is a probability measure for each $\omega \in \Omega$.

2. If $X(\omega) \in A$ then clearly $Y(\omega) = g(X(\omega)) \in g(A)$. Conversely, if $Y(\omega) \in g(A)$ then $X(\omega) = g^{-1}(Y(\omega)) \in g^{-1}(g(A)) = A$ because the mapping g is one to one. Furthermore, setting $B = g(A)$ we have that $A = h(B) \in SS$ for the one to one $h = g^{-1} : \mathbb{T} \mapsto \mathbb{S}$, and since h is also a measurable mapping it follows that $B = h^{-1}(A) \in \mathcal{T} \subseteq \mathcal{B}$.

3. For any $A \in SS$ we have that

   $$\widehat{\mathbf{Q}}(A, \cdot) = \widehat{\mathbf{P}}_{Y|\mathcal{G}}(g(A), \cdot) = \mathbf{E}[I_{\{Y\in g(A)\}}|\mathcal{G}] = \mathbf{E}[I_{\{X\in A\}}|\mathcal{G}]$$

   as we have seen in part (b) that $I_{\{Y\in g(A)\}} = I_{\{X\in A\}}$.

   Also, $\widehat{\mathbf{Q}}(\cdot, \omega)$ is a probability measure on $(\mathbb{S}, SS)$ for any fixed ω. Indeed, first by part (b) for any $A \in SS$, $g(A) \in \mathcal{T}$ so by the non-negativity of the R.C.P.D. $\widehat{\mathbf{P}}_{Y|\mathcal{G}}(\cdot, \omega)$,

   $$\widehat{\mathbf{Q}}(A, \omega) = \widehat{\mathbf{P}}_{Y|\mathcal{G}}(g(A), \omega) \geq \widehat{\mathbf{P}}_{Y|\mathcal{G}}(\emptyset, \omega) = 0 = \widehat{\mathbf{Q}}(\emptyset, \omega)\,.$$

   Second, since g is one to one, for any disjoint $A_n \in SS$ we also have that $g(A_n) \in \mathcal{T}$ are disjoint, hence by countable additivity of the R.C.P.D. $\widehat{\mathbf{P}}_{Y|\mathcal{G}}(\cdot, \omega)$,

   $$\widehat{\mathbf{Q}}(\cup_n A_n, \omega) = \widehat{\mathbf{P}}_{Y|\mathcal{G}}(\cup_n g(A_n), \omega) = \sum_n \widehat{\mathbf{P}}_{Y|\mathcal{G}}(g(A_n), \omega) = \sum_n \widehat{\mathbf{Q}}(A_n, \omega)\,.$$

   Finally, g is onto hence $g(\mathbb{S}) = \mathbb{T}$ and so $\widehat{\mathbf{Q}}(\mathbb{S}, \omega) = \widehat{\mathbf{P}}_{Y|\mathcal{G}}(\mathbb{T}, \omega) = 1$ by part (a).

   In conclusion, by Definition 4.4.2, $\widehat{\mathbf{Q}}(\cdot, \cdot)$ is the R.C.P.D. of X given $\mathcal{G}$.

Exercise [4.4.5]

Let g be the one to one and onto mapping associated with the $\mathcal{B}$-isomorphism. Then, by Proposition 4.4.3, for the random variable $g(X)$ and the σ-algebra $\sigma(g(Y))$, there exists an R.C.P.D. $\widehat{\mathbf{P}}_{g(X)|\sigma(g(Y))}$.

We claim that $\sigma(Y) = \sigma(g(Y))$. To show this, note first that any $A \in \sigma(Y)$ is of the form $A = Y^{-1}(S)$ for some $S \in SS$. While solving part (b) of Exercise 4.4.4 we noted that $S = g^{-1}(B)$ with $B = g(S) \in \mathcal{B}$, hence $A = (g(Y))^{-1}(B)$ is in $\sigma(g(Y))$. Conversely, since $(g(Y))^{-1}(B) = Y^{-1}(g^{-1}(B))$ for any $B \in \mathcal{B}$, if $A \in \sigma(g(Y))$ then $A = Y^{-1}(g^{-1}(B))$ for some $B \in \mathcal{B}$. By measurability of g we have that $S = g^{-1}(B) \in SS$ and hence $A = Y^{-1}(S) \in \sigma(Y)$.

Furthermore, by Definition 4.4.2, we know that $\widehat{\mathbf{P}}_{g(X)|\sigma(g(Y))}(B, \cdot)$ is a $\sigma(g(Y))$ measurable function for each $B \in \mathcal{B}$ fixed. Therefore, by Theorem 1.2.26 there exists a Borel function $f(B, \cdot) : \mathbb{T} \mapsto [0, 1]$ such that $\widehat{\mathbf{P}}_{g(X)|\sigma(g(Y))} = f(B, g(Y(\omega)))$. Now let $\widehat{\mathbf{P}}_{X|Y}(y, A) = h(y, A)$ for the $[0, 1]$-valued $h(y, A) = f(g(A), g(y))$. Note that per fixed $A \in SS$ the set $g(A)$ is in $\mathcal{B}$ and $h(\cdot, A)$ being the composition of the measurable f and g is measurable on $(\mathbb{S}, SS)$. Further, by part (b) of Exercise 4.4.4 and since $\sigma(g(Y)) = \sigma(Y)$,

$$\begin{aligned}\widehat{\mathbf{P}}_{X|Y}(Y(\omega), A) &= f(g(A), g(Y(\omega))) = \widehat{\mathbf{P}}_{g(X)|\sigma(g(Y))}(g(A), \cdot) \\ &= \mathbf{E}[I_{\{g(X)\in g(A)\}}|\sigma(g(Y))] = \mathbf{E}[I_{\{X\in A\}}|\sigma(Y)]\,,\end{aligned}$$

hence establishing part (a). For part (b), note that for any fixed $\omega \in \Omega$, the set function

$$\widehat{\mathbf{P}}_{X|Y}(Y(\omega), \cdot) = \widehat{\mathbf{P}}_{g(X)|\sigma(g(Y))}(g(\cdot), g(Y(\omega)))$$

is a probability measure on (S, SS) in view of part (c) of Exercise 4.4.4.