Stat 218 Introduction to Stochastic Processes

Problem Set 3

Date: 6/5/2007

Let $\{X(t) = (X_1(t), X_2(t)) \,:\, t \ge 0\}$ be a two-dimensional Brownian motion with initial condition $X(0) = x^0 = (x^0_1, x^0_2)$. Introduce the process $\{(R(t), \Theta(t)) \,:\, t \ge 0\}$ in polar coordinates, by letting (for $\Theta(t) \in (-\pi, \pi]$),

$$R(t) \equiv \sqrt{X_1(t)^2 + X_2(t)^2}\,, \qquad \Theta(t) = \arctan\{X_2(t)/X_1(t)\}\,. \tag{1}$$

1. For $t \ge 0$, $h > 0$, Let $\Delta R \equiv R(t+h) - R(t)$ and $\Delta\Theta \equiv \Theta(t+h) - \Theta(t)$. Compute the first two moments of these quantities and show that, for $r > 0$,

$$\mathbb{E}\{\Delta R |\, R(t) = r,\, \Theta(t) = \theta\} = \frac{1}{r}\, h + o(h)\,, \qquad \mathbb{E}\{(\Delta R)^2 |\, R(t) = r,\, \Theta(t) = \theta\} = h + o(h)\,, \tag{2}$$

$$\mathbb{E}\{\Delta \Theta |\, R(t) = r,\, \Theta(t) = \theta\} = 0\,, \qquad \mathbb{E}\{(\Delta \Theta)^2 |\, R(t) = r,\, \Theta(t) = \theta\} = \frac{1}{r^2}\, h + o(h)\,, \tag{3}$$

2. Consider the process $\{\Theta(t);\, t \ge 0\}$. Is this a Markov process? If yes, find an expression for the transition probability. If no, justify your claim.

3. Consider the process $\{R(t);\, t \ge 0\}$. Is this a Markov process? If yes, find an expression for the transition probability. If no, justify your claim.

4. Let $p_t(r|u)$ the density of $R(t_0 + t)$ given $R(t_0) = u$. Show that it satisfies the following backward diffusion equation

$$\partial_t p_t(r|u) = \frac{1}{2u} \partial_u p_t(r|u) + \frac{1}{2} \partial_u^2 p_t(r|u)\,. \tag{4}$$

It might be convenient to recall the definition of the 0-th Bessel function

$$I_0(\alpha) \equiv \frac{1}{2\pi} \int_0^{2\pi} e^{\alpha \cos\theta}\, \mathrm{d}\theta\,. \tag{5}$$